THE THEORY OF FUNCTIONS
OF A
REAL VARIABLE
AND THE
THEORY OF FOURIER'S SERIES

BY

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PREFACE

IN this volume those parts of the subject, as presented in the first edition, published in 1907, have been treated which were not dealt with in the volume published in 1921. Almost the whole of the matter has been re-written, and much new matter has been added which is largely the fruit of investigations that have been carried out by various Mathematicians in the intervening time.

In Chapter I, on numerical sequences, a greatly extended account of the theory of convergence of numerical series is given, together with a fairly full account of the theories of conventional summation with which the names of Cesàro, Hölder, and M. Riesz are associated.

Chapter II contains a systematic account of the theories of convergence and oscillation of sequences and series of which the terms are functions of one or more variables; and Chapter III contains the application of these theories to the special, but important, case of power-series.

In Chapter IV an account is given of the theorem of Weierstrass relating to the representation of continuous functions by sequences of polynomials; of the theory of convergence of sequences on the average; and of F. Riesz' classification of summable functions. The proof of the fundamental result of Baire, relating to the representation of a function as the limit of a sequence of continuous functions, is obtained by a method which is due to de la Vallée Poussin, but with some modification and extension.

Chapter V is devoted to those parts of the theory of integration which were not dealt with in Volume I. Considerable space has been allotted here to a discussion of various theories of integration, due to W. H. Young, Tonelli, and Perron. A short account is also given of the conventional summation of integrals.

Chapter VI contains an account of the construction, by various methods, of functions which exhibit assigned peculiarities, and in particular, of non-differentiable continuous functions.

A special feature of the volume consists of the prominence given to what I have called the General Convergence Theorem, together with its developments and consequences. This Theorem is treated very fully in Chapter VII, with a view to the applications of it to the theories of Fourier's series and integrals contained in the later chapters.

The large amount of matter contained in Chapter VIII, on Trigonometrical Series, gives ample evidence of the recent activity of Mathematicians in the investigation of properties of Fourier's series and of the

coefficients in the series. Most of the recent progress in this subject has been due to the exploitation of the theory of Lebesgue integration and to the application to Fourier's series of various conventional methods of summation. Although the remarkable history of the theory of these series covers a period of upwards of a century and a half, there still remains for solution at least one fundamentally important question which has hitherto baffled all attempts at settlement. In this chapter, mainly from considerations of space, I have given references, without proofs, in the case of some results that have been quite recently published.

The importance of the representation of functions by Fourier's integrals, together with the interesting modern theory of Fourier transforms, is such that I have devoted Chapter IX entirely to this subject.

Chapter X has been added on the representation of functions by series of normal orthogonal functions, not only on account of the intrinsic importance of the subject, but also because the processes which have been employed in various recent investigations in this domain afford excellent illustrations of ideas and methods which have been developed earlier in this work.

By far the greater part of the proofs of the volume were read in slip by Prof. G. H. Hardy, F.R.S., to whom I desire to express my gratitude for many important criticisms and suggestions, the adoption of which has done much to improve the presentation of the subject.

My thanks are also due to the Officials and Readers of the University Press for the courtesy they have shewn me, and the trouble they have taken, in connection with the heavy work of printing the volume.

E. W. HOBSON.

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CHAPTER I

SEQUENCES AND SERIES OF NUMBERS

1. Let us consider a set of numbers $a_1, a_2, a_3, \dots a_n, \dots$, such that the number a_n is defined for each value of n by means of a norm, consisting of a prescribed rule or set of rules. Let the numbers

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_n, \dots$$

be denoted by $s_1, s_2, s_3, \dots s_n, \dots$; and let us consider the aggregate $(s_1, s_2, \dots s_n, \dots)$. If this aggregate form a convergent sequence, in accordance with the definition given in I, § 23, it has (see I, § 30) a limit s_ω, or s, which is said to be the *limiting sum*, or simply the *sum*, of the infinite series $a_1 + a_2 + \dots + a_n + \dots$; in which case the series is said to be convergent.

The condition that the sequence $(s_1, s_2, s_3, \dots s_n, \dots)$ may be convergent is that, corresponding to each arbitrarily chosen positive number ϵ, a value of n can be so determined that $| s_{n+m} - s_n | < \epsilon$, for $m = 1, 2, 3, \dots$. This is then the necessary and sufficient condition that the infinite series $a_1 + a_2 + \dots + a_n + \dots$ may be convergent.

The difference $s_{n+m} - s_n \equiv a_{n+1} + a_{n+2} + \dots + a_{n+m}$ is called a *partial remainder* of the infinite series, and it may be denoted by $R_{n,m}$. Thus the condition of convergence of the infinite series may be stated as follows:

The necessary and sufficient condition that the series $a_1 + a_2 + \dots + a_n + \dots$, or $\sum\limits_{n=1} a_n$, may be convergent is that, corresponding to each arbitrarily chosen positive number ϵ, a value of n can be so determined that all the partial remainders $R_{n,1}, R_{n,2}, \dots R_{n,m}, \dots$ are numerically less than ϵ.

Since $R_{n-1,1} = a_n$, it is seen to be a necessary, but not a sufficient, condition for the convergence of the series that $| a_n |$ be arbitrarily small, when n is sufficiently great. This condition may be written in the form

$$\lim_{n\sim\infty} a_n = 0.$$

If the series $a_1 + a_2 + \dots + a_n + \dots$ be convergent, then, for any value of n, the series $a_{n+1} + a_{n+2} + \dots$ is also convergent, and has, in the sense defined above, a limiting sum which may be denoted by R_n. This limiting sum is called the *remainder after n terms* of the original convergent series; thus $s = s_n + R_n$. Whether the series be convergent or not, s_n may be called the nth partial sum of the series $\sum\limits_{n=1} a_n$.

It is clear that, the given series being convergent, the sequence $(R_1, R_2, \dots R_n, \dots)$ is also convergent, and that its limit is zero. That this

may be the case has sometimes been stated to be the necessary and sufficient condition for the convergence of the series; such a statement of the condition is, however, circular, because the existence of the numbers R_n cannot be assumed unless the given series is already known to be convergent.

It is important to observe that the number s has not been defined as the sum of the infinite series $a_1 + a_2 + \ldots + a_n + \ldots$; for that would have implied the completion of an indefinitely great number of operations of addition; but conversely, the limiting sum, or simply the sum, of the infinite series has been defined to be that number s which was itself defined, as in I, § 30, by means of a convergent sequence.

NON-CONVERGENT ARITHMETIC SERIES.

2. The partial sums $s_1, s_2, \ldots s_n, \ldots$ of a series $a_1 + a_2 + \ldots + a_n + \ldots$ may be represented in the usual manner by an enumerable set of points G, on a straight line. The set G has a derivative G', which is a closed set, in the ordinary sense of the term, in case G' is bounded. If G' is unbounded, it is closed in the extended sense (I, §§ 53, 55), when one of the improper points $+\infty$, $-\infty$ is regarded as belonging to the set, or when both these points belong to the set.

The following cases may arise:

(1) The derivative G' may consist of a single proper point s. In this case the series is *convergent*, and all the points of G, with the possible exception of a finite number of them, lie in the interval $(s - \delta, s + \delta)$; where δ is an arbitrarily chosen positive number. The points of G then all lie between two fixed points A and B; or $|s_n|$ is bounded.

(2) The set G' may consist of one of the improper points $+\infty$, $-\infty$. In this case $|s_n|$ has no upper boundary, and the series is said to be *divergent*. If N be an arbitrarily large positive number, all the numbers s_n, except possibly a finite number of them, are of the same sign, and numerically exceed N. An example of a divergent series is the series

$$1/1 + 1/2 + \ldots + 1/n + \ldots.$$

For this series we have $R_{n,m} = 1/(n+1) + 1/(n+2) + \ldots + 1/(n+m)$, and thus $R_{n,m} > m/(n+m)$. However great n may be chosen, we have $R_{n,n} > 1/2$; which is inconsistent with the condition for convergence of the series. As the sequence $\{s_n\}$ is monotone and increasing, it can have no upper limit, except the improper point $+\infty$; and thus the series is divergent.

(3) The set G' may be a (bounded) closed set, which contains a finite, or an indefinitely great, number of points. If U and L be the upper and lower boundaries of G', the series is said to be *an oscillating series* with

U and L as the upper and lower *limits of indeterminancy** of the sum of the series. The numbers U and L may also be spoken of as *the upper and the lower sums* of the series respectively, and they may be denoted by $\bar{s}$, $\underline{s}$.

It is always possible to determine a sequence $(s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$ of partial sums, where $n_1 < n_2 < n_3 < \ldots$, which converges to the point U, and another such sequence which converges to L, or to any other point of G' which may be chosen. It thus appears that, by introducing a suitable system of bracketing the terms of an oscillating series, according to some norm, and amalgamating the terms in each bracket, the series may be converted into a convergent one, of which the limiting sum is any chosen point of G', including either limit of indeterminancy. The set G' may be non-dense in the interval (L, U), or it may consist of all the points of that closed interval; or it may consist of a closed set of the most general type, as described in I, § 80.

The oscillating series $1 - 1 + 1 - 1 + 1 - \ldots$ has 1 and 0 for its upper and lower limits of indeterminancy; and G' consists of these two points. Again, let

$$s_1 = 1/2,\ s_2 = 1/3,\ s_3 = 1/4,\ s_4 = 2/3,\ s_5 = 1/5,\ s_6 = 2/4,\ s_7 = 1/6, \ldots,$$

and generally

$$s_{m(m+1)+1} = 1/(2m+2),\ s_{m(m+1)+2} = 2/(2m+1), \ldots,\qquad s_{(m+1)^2} = (m+1)/(m+2),$$

$$s_{(m+1)^2+1} = 1/(2m+3),\ s_{(m+1)^2+2} = 2/(2m+2), \ldots,\qquad s_{(m+1)(m+2)} = (m+1)/(m+3),$$

where $m = 0, 1, 2, 3, \ldots$, and where only those numbers are taken which are less than unity.

It follows that the series

$$\frac{1}{2} - \frac{1}{2.3} - \frac{1}{3.4} + \frac{5}{3.4} - \frac{7}{3.5} + \ldots$$

has 1 and 0 for the upper and lower limits of indeterminancy. The set G consists of all the rational numbers between 0 and 1; so that G' consists of all the points of the closed interval $(0, 1)$. By introducing a properly chosen system of brackets, and amalgamating the terms in each bracket, the series may be converted into one converging to a limiting sum which is any prescribed number in the interval $(0, 1)$.

(4) The set G' may consist of two or more points, amongst which there is at least one improper point, $+\infty$ or $-\infty$. In such cases the series is also said to be an oscillating series; one or both of the limits of indeterminancy being infinite. Such a series may be converted into a divergent

* This term is due to Du Bois-Reymond; see his *Antrittsprogramm*, p. 3.

series, by introduction of a properly defined system of brackets; or on the other hand it may be converted into a series which converges to any proper point of G', provided such a point exists.

It should be observed that, by some writers, all series which are not convergent are spoken of as divergent, but the term non-convergent will here be employed in that sense, as including both divergent and oscillating series.

From a certain point of view there exist but two classes of series, those which oscillate and those which do not oscillate. The latter class includes both convergent and divergent series. A divergent series may be regarded as one which converges to one of the improper numbers $+\infty$ and $-\infty$, and this is a certain justification for classing convergent and divergent series together, as distinguished from oscillating series. If $(s_1, s_2, \ldots s_n, \ldots)$ denote a sequence of numbers, let us consider the corresponding sequence $(\bar{s}_1, \bar{s}_2, \ldots \bar{s}_n, \ldots)$, where $\bar{s}_n$ is defined by

$$\bar{s}_n = \frac{s_n}{1 + |s_n|}.$$

It is clear that all the numbers $\bar{s}_n$ lie within the interval $(-1, 1)$, and the improper numbers $+\infty$, $-\infty$ may be taken to correspond to the numbers $1, -1$ respectively. If the sequence $\{s_n\}$ is convergent, it is easily seen that the sequence $\{\bar{s}_n\}$ is also convergent; but if $\{s_n\}$ diverges to $+\infty$, or to $-\infty$, the corresponding sequence $\{\bar{s}_n\}$ converges to 1, or to -1. If $\{s_n\}$ is an oscillating sequence, so also is $\{\bar{s}_n\}$. Thus the classification of sequences into oscillating and non-oscillating sequences is invariant for the transformation $s_n = \dfrac{s_n}{1 + |s_n|}$.

3. A series* may be constructed which oscillates between infinite limits of indeterminancy, but which, by introducing a suitable system of brackets, in accordance with a norm, may be converted into a series which converges to any prescribed number whatever, or which diverges to ∞, or to $-\infty$.

If $x' = \dfrac{2x - 1}{\sqrt{x(1-x)}}$, where the positive sign is ascribed to the radical, the points x, of the interval $(0, 1)$, have a $(1, 1)$ correspondence with the points x', of the unbounded interval $(-\infty, \infty)$. It is easily seen that a set of points $\{x\}$, in the interval $(0, 1)$, corresponds to a set $\{x'\}$, in the interval $(-\infty, \infty)$, the relation of order being conserved in the correspondence. Further, a limiting point of the one set corresponds to a limiting point of the other set. The rational points of the interval $(0, 1)$, of x, correspond to a set of points x', everywhere dense in $(-\infty, \infty)$. This method of correspondence may be applied to the series obtained in (3),

* See Hobson, *Proc. Lond. Math. Soc.* (2), vol. III (1904), p. 50.

which oscillates between the limits of indeterminancy 0, 1, and which can be made, by introducing suitable brackets, to converge to any prescribed number in the interval (0, 1). We find that

$$s_1' = 0,\ s_2' = -1/\sqrt{2},\ s_3' = -2/\sqrt{3},\ s_4' = 1/\sqrt{2},\ s_5' = -3/2,$$
$$s_6' = 0,\ s_7' = -4/\sqrt{5},\ s_8' = -1/\sqrt{6},\ s_9' = 2/\sqrt{3},\ s_{10}' = -5/\sqrt{6},$$

and generally

$$s'_{m(m+1)+1} = -2m/\sqrt{2m+1},\ s'_{m(m+1)+2} = -(2m-3)/\sqrt{2(2m-1)}, \ldots,$$
$$s'_{(m+1)^2} = m/\sqrt{m+1}, \qquad s'_{(m+1)^2+1} = -(2m+1)/\sqrt{2m+2}, \ldots,$$
$$s'_{(m+1)(m+2)} = (m-1)/\sqrt{2(m+1)}.$$

Therefore the series

$$-\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{2}}\right) - \left(\frac{3}{2} + \frac{1}{\sqrt{2}}\right) + \frac{3}{2} - \frac{4}{\sqrt{5}} + \ldots$$

has the required character. It may be converted into a series which converges to any assigned number whatever, or may diverge, by suitably bracketing the terms together, in accordance with a norm: the terms in each bracket being amalgamated.

An oscillating series which has the two limits of indeterminancy $+\infty$, $-\infty$, and for which G' has no proper points, may be constructed, for example, by taking $s_{2n-1} = n$, $s_{2n} = -n$. Thus the series

$$1 - 2 + 3 - 4 + \ldots + (2n-1) - 2n + \ldots$$

oscillates, with $+\infty$, $-\infty$ as limits of indeterminancy. The series

$$1 + (-2+3) + (-4+5) + \ldots,$$
$$(1-2) + (3-4) + (5-6) + \ldots,$$

are both divergent. The set G' contains no proper points; and thus the series cannot be converted by bracketing into a convergent series.

4. From the time, in the seventeenth century, when infinite series were first employed, until far into the nineteenth century, such series were freely used, with but little enquiry as to whether they were convergent or not. It was generally held, as for example by Lagrange, that the convergence of a_n to zero, as n is indefinitely increased, is sufficient to ensure the convergence of the series Σa_n, although it had been established by J. Bernoulli that the series $\sum\limits_{n=1} \frac{1}{n}$ is divergent. The first writer who completely emancipated himself from the uncritical extension of the operation of arithmetic addition to the case in which the number of such operations is indefinitely great was Bolzano, who gave* the necessary and sufficient condition for the convergence of a series in the form given in § 1, that $|s_{n+m} - s_n|$ must be arbitrarily small for all values of m, provided n is

* *Rein analytischer Beweis* ..., Prag 1817; this is reproduced in Oswald's *Klassiker der exacten Wissensch.* No. 153, p. 21.

sufficiently large. It was, however, owing to the writings of Cauchy* and Abel†, since Bolzano's work remained almost unknown for a long time, that the modern theory of the convergence and divergence of series gradually attained acceptance by mathematicians. An interesting account has been given by Burkhardt‡ of the history of attempts, made even in the nineteenth century, to justify the employment of non-convergent series in calculations. In recent times, various rigorous methods have been devised, by which, in accordance with strict definitions, such series may be employed. An account of some of these methods will be given later.

THE O–o NOTATION.

5. Let $\psi(n)$ denote a function, defined for the values 1, 2, 3, ... of the variable n, and such that $\psi(n) > 0$, for all the values of n. If $\phi(n)$ denote a function of n, such that $\frac{|\phi(n)|}{\psi(n)}$ is less than some positive number K, independent of n, we may write $\phi(n) = O\{\psi(n)\}$; but if $\lim\limits_{n\sim\infty} \frac{\phi(n)}{\psi(n)} = 0$, we write $\phi(n) = o\{\psi(n)\}$.

Thus, for example, $a_n = O(1)$ means that $|a_n|$ is bounded; and $a_n = o(1)$ means that $\lim\limits_{n\sim\infty} a_n = 0$. Again $a_n = O(n)$ means that $\frac{|a_n|}{n}$ is bounded; and $a_n = o(n)$ means that $\lim\limits_{n\sim\infty} \frac{a_n}{n} = 0$; $a_n = O(n^{-k})$ means that $n^k a_n$ is bounded; and $a_n = o(n^{-k})$ means that $\lim\limits_{n\sim\infty} (n^k a_n) = 0$.

It is easily seen that

$$O\{\psi_1(n)\} . O\{\psi_2(n)\} = O\{\psi_1(n)\,\psi_2(n)\},$$
$$O\{\psi_1(n)\} . o\{\psi_2(n)\} = o\{\psi_1(n)\,\psi_2(n)\},$$
$$o\{\psi_1(n)\} . o\{\psi_2(n)\} = o\{\psi_1(n)\,\psi_2(n)\}.$$

The same notation may be applied to the case of functions of a variable x which varies continuously in a field $a \leqq x < \infty$, or in a field $a \leqq x < A$. Thus $\phi(x) = O\{\psi(x)\}$ denotes that $\frac{\phi(x)}{\psi(x)}$ is bounded for all values of x in the field; and $\phi(x) = o\{\psi(x)\}$ denotes that $\lim\limits_{x\sim\infty} \frac{\phi(x)}{\psi(x)} = 0$, or $\lim\limits_{x\sim A} \frac{\phi(x)}{\psi(x)} = 0$, as the case may be; the function $\psi(x)$ is, as before, assumed to be positive throughout the given field of the variable.

The O–o notation was first employed systematically by Landau§, although, as stated by him, the symbol O was employed earlier by

* For his definition of the condition of convergence, see *Cours d'Analyse Alg.* (1821), p. 125.

† See his memoir on the binomial theorem, *Crelle's Journal*, vol. I (1826), p. 313; also, for a more exact formulation, see *Œuvres*, 2nd ed. vol. II, p. 197.

‡ *Math. Annalen*, vol. LXX (1911), p. 169. Reiff's *Gesch. der unendlichen Reihen*, Tübingen (1889) may also be consulted.

§ *Vertheilung der Primzahlen*, vol. I, pp. 31, 59–62; also vol. II, p. 883.

Bachmann*. It has recently come into general use in investigations connected with series and integrals.

A GENERAL PROPERTY OF SEQUENCES.

6. It will be proved that:

If $\lim_{n\sim\infty} a_n = 0$, *then* s_n, *the partial sum of the series* $a_1 + a_2 + \dots + a_n \dots$, *has the property* $s_n = o(n)$; *i.e.* $\lim_{n\sim\infty} \frac{s_n}{n} = 0$. *Also, if* $a_n = O(1)$, *then* $s_n = O(n)$.

It is convenient to establish the following general theorem of which the above theorem is a particular case:

If $\{\beta_n\}$ *denotes a monotone increasing sequence of positive numbers, such that* β_n *increases indefinitely with* n, *and if* $\{\alpha_n\}$ *be any sequence of numbers, then*

$$\overline{\lim_{n\sim\infty}} \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n} \geqq \overline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq \underline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq \underline{\lim_{n\sim\infty}} \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n};$$

and in particular, if† $\lim_{n\sim\infty} \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$ *has a definite value, then* $\lim_{n\sim\infty} \frac{\alpha_n}{\beta_n}$ *also has the same definite value. If* $\frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$ *diverges to* $+\infty$, *or to* $-\infty$, $\frac{\alpha_n}{\beta_n}$ *also diverges to* ∞, *or to* $-\infty$.

The first theorem is obtained by taking $\beta_n = n$, $\alpha_n = s_n$. To prove the general theorem, let U and L denote the upper and lower limits of

$$\frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

An integer n_η may be determined, such that, if η be an arbitrarily chosen positive number,

$$U + \eta > \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n} > L - \eta, \text{ for } n \geqq n_\eta.$$

We find, by taking $n = n_\eta, n_\eta + 1, \dots n_\eta + m - 1$, successively,

$$(U + \eta)(\beta_{n_\eta + m} - \beta_{n_\eta}) > \alpha_{n_\eta + m} - \alpha_{n_\eta} > (L - \eta)(\beta_{n_\eta + m} - \beta_{n_\eta});$$

or $$(U + \eta)\left(1 - \frac{\beta_{n_\eta}}{\beta_{n_\eta + m}}\right) > \frac{\alpha_{n_\eta + m}}{\beta_{n_\eta + m}} - \frac{\alpha_{n_\eta}}{\beta_{n_\eta + m}} > (L - \eta)\left(1 - \frac{\beta_{n_\eta}}{\beta_{n_\eta + m}}\right).$$

Keeping n_η fixed, and letting m increase indefinitely, we thus have

$$U + \eta \geqq \overline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq L - \eta;$$

or, since η is arbitrary, $U \geqq \overline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq \underline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq L$.

In case $\frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$ diverges to ∞, if N be an arbitrarily chosen positive number, n_1 may be so chosen that $\frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n} > N$, for $n \geqq n_1$; we have then

$$\alpha_{n_1+m} - \alpha_{n_1} > N(\beta_{n_1+m} - \beta_{n_1}), \quad \text{or} \quad \frac{\alpha_{n_1+m}}{\beta_{n_1+m}} - \frac{\alpha_{n_1}}{\beta_{n_1+m}} > N\left(1 - \frac{\beta_{n_1}}{\beta_{n_1+m}}\right).$$

* *Analytische Zahlentheorie*, vol. II, p. 401.

† See Stolz, *Vorlesungen über allgemeine Arithmetik*, Leipzig (1885), p. 174.

Letting m increase indefinitely, we have

$$\overline{\lim_{n\sim\infty}} \frac{\alpha_n}{\beta_n} \geqq N;$$

and since N is arbitrary, it follows that $\lim\limits_{n\sim\infty} \frac{\alpha_n}{\beta_n} = \infty$. The case in which $\frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$ diverges to $-\infty$ may be treated in a similar manner.

EXAMPLES.

(1) If $\sum\limits_1^n a_n$ converges, as $n \sim \infty$, to a definite number, then

$$a_1 + 2a_2 + 3a_3 + \dots + na_n = o(n).$$

Let $\alpha_n = s_1 + s_2 + \dots + s_n$, $\beta_n = n + 1$, then

$$\lim_{n\sim\infty} \frac{s_1 + s_2 + \dots + s_n}{n+1} = \lim_{n\sim\infty} s_n,$$

since the last limit exists. This may be written in the form

$$\lim_{n\sim\infty} \left[s_n - \frac{a_1 + 2a_2 + \dots + na_n}{n+1} \right] = \lim_{n\sim\infty} s_n,$$

whence it follows that $$\lim_{n\sim\infty} \frac{a_1 + 2a_2 + \dots + na_n}{n+1} = 0.$$

(2) If $\lim\limits_{n\sim\infty} na_n$ exists, then $\lim\limits_{n\sim\infty} \frac{a_1 + 2a_2 + 3a_3 + \dots + na_n}{n} = \lim\limits_{n\sim\infty} na_n$. To prove this let $\alpha_n = a_1 + 2a_2 + \dots + na_n$, $\beta_n = n$. In particular, if $na_n = o(1)$, then

$$a_1 + 2a_2 + \dots + na_n = o(n).$$

(3)* If $\{M_n\}$ be a monotone increasing sequence of positive numbers, which diverges, then, if $\sum\limits_{n=1} a_n$ is convergent, $\lim\limits_{n\sim\infty} \frac{M_1 a_1 + M_2 a_2 + \dots + M_n a_n}{M_n} = 0$. This is the generalization of Ex. (1).

The relation $\frac{M_1 a_1 + M_2 a_2 + \dots + M_n a_n}{M_n} = o(1)$ is satisfied if $\frac{M_n}{M_n - M_{n-1}} a_n = o(1)$. This is the generalization of Ex. (2).

(4)† If $p > 0$, and $\lim\limits_{n\sim\infty} \frac{a_n}{n^{p-1}}$ has a definite value, then $\lim\limits_{n\sim\infty} \frac{s_n}{n^p} = \frac{1}{p} \lim\limits_{n\sim\infty} \frac{a_n}{n^{p-1}}$.

(5)† If $\lim\limits_{n\sim\infty} na_n$ has a definite value, then $\lim\limits_{n\sim\infty} \frac{s_n}{\log n} = \lim\limits_{n\sim\infty} na_n$.

(6)† The two conditions $a_1 + 2a_2 + 3a_3 + \dots + na_n = o(n)$, $\lim\limits_{n\sim\infty} \frac{s_1 + s_2 + \dots + s_n}{n} = s$ are sufficient to ensure that the series $a_1 + a_2 + \dots$ should converge to s. Each of these conditions is necessary, and the two together are sufficient.

(7)‡ If $\{\beta_n\}$ denote a monotone sequence of decreasing numbers which converges to 0 as $n \sim \infty$, and if $\lim\limits_{n\sim\infty} \frac{\alpha_n - \alpha_{n+1}}{\beta_n - \beta_{n+1}}$ exists, then $\lim\limits_{n\sim\infty} \frac{\alpha_n}{\beta_n}$ also exists and has the same value. This may be obtained from the general theorem by changing α_n, β_n into $1/\alpha_n$, $1/\beta_n$ respectively.

* See Pringsheim, *Sitzungsber. Münch. Akad.* vol. XXX (1900), pp. 44–46.

† *Ibid.* vol. XXXI, pp. 507, 524, 531.

‡ See, for an independent proof, Bromwich's *Theory of Infinite Series*, p. 377.

CONVERGENCE AND DIVERGENCE OF SERIES WITH POSITIVE TERMS.

7. If all the terms of a series, with the possible exception of a finite number of them, be positive or zero, it is clear that the sequence $(s_1, s_2, \ldots s_n, \ldots)$ is monotone non-diminishing, from and after some fixed value of n. It follows that the series is either convergent or divergent, but cannot oscillate. Moreover the convergence or divergence of the series is unaffected by the removal of a finite set of the numbers $s_1, s_2, \ldots$; and this set may be so chosen that the partial sums corresponding to the negative terms are all removed. Thus there is no loss of generality in considering only series in which all the terms are positive or zero.

If $a_1 + a_2 + \ldots + a_n + \ldots$ be such a series, it is clear that the sequence $s_1, s_2, \ldots s_n, \ldots$ of partial sums is monotone non-diminishing, and therefore either converges to a definite limit s, the sum of the series, or is divergent. We may thus state that:

The necessary and sufficient condition that a series $a_1 + a_2 + \ldots + a_n + \ldots$, *of which all the terms are* $\geqq 0$, *should be convergent is that a positive number* K *exists, such that* $s_n < K$, *for all values of* n.

8. The following property is possessed by a convergent series of which all the terms are positive. The expression positive will be taken to include zero.

A series such that all its terms are terms of a convergent series

$$a_1 + a_2 + \ldots + a_n + \ldots,$$

all the terms of which are positive, is also convergent.

If $s'_{n'}$ be a partial sum of the second series, n can be so determined that all the terms in $s'_{n'}$ are contained in the terms of s_n; then $s'_{n'} \leq s_n < K$, where K is a fixed positive number; since $s'_{n'} < K$, and $s'_{n'}$ cannot diminish as n increases, it follows that $s'_{n'}$ has a definite limit as $n' \sim \infty$; therefore the second series is convergent.

If a second series be obtained by rearranging, in accordance with any prescribed norm, the order of the terms of a convergent series

$$a_1 + a_2 + \ldots + a_n + \ldots,$$

all the terms of which are positive, then the second series converges to the same sum as the first. It is assumed that the new series is of the same type ω, *as the original one* (see § 29).

This theorem may be expressed by the statement that a convergent series of positive terms is *unconditionally convergent*. Let s'_n denote the nth partial sum of the second series. If ϵ be a prescribed positive number, n_1 may be taken so great that $s - s_{n_1} < \epsilon$. An integer n_2 can be so chosen that s'_{n_2} contains all the terms of s_{n_1}; therefore $s'_n \geqq s_{n_1}$, if $n \geqq n_2$.

We have now, $s'_n > s - \epsilon$, if $n \geqq n_2$. For any value of n the terms of s'_n are all contained in s_m, if m is sufficiently large; and therefore $s'_n < s$, for all values of n. Since s'_n is in the interval $(s - \epsilon, s)$ if $n \geqq n_2$, and ϵ is arbitrary, it follows that $\lim s'_n = s$.

9. Let two series

$$a_1 + a_2 + \dots + a_n + \dots, \quad b_1 + b_2 + \dots + b_n + \dots,$$

in each of which all the terms are positive (> 0), be considered, and let their nth partial sums be denoted by s_n, s'_n respectively. If the series are both convergent, we have $\lim_{n\sim\infty} R_n = 0$, $\lim_{n\sim\infty} R'_n = 0$. In case $0 < \overline{\lim_{n\sim\infty}} \frac{R'_n}{R_n} < \infty$, the second series may be said to *converge as rapidly as the first*; and in case $\lim_{n\sim\infty} \frac{R'_n}{R_n} = 0$, the second series may be said to converge *more rapidly* than the first. The first series may then also be said to converge more slowly than the second. If the series are both divergent, then, in case $0 < \lim_{n\sim\infty} \frac{s'_n}{s_n} < \infty$, the second series may be said to *diverge as slowly as* the first; and in case $\lim_{n\sim\infty} \frac{s'_n}{s_n} = 0$, the second series may be said to *diverge more slowly* than the first; and also the first series may be said to *diverge more rapidly* than the second.

If two convergent series $\sum_{n=1} c_n$, $\sum_{n=1} c'_n$, *for both of which the terms are positive* (> 0), *be such that* $\lim_{n\sim\infty} \frac{c_n}{c'_n} = 0$, *then the first series converges more rapidly than the second.*

If ϵ be any prescribed positive number, then $c_n < \epsilon c'_n$, provided n is greater than some fixed integer n_ϵ. It follows that $R_{n_\epsilon, m} < \epsilon R'_{n_\epsilon, m}$, for all positive integral values of m. Consequently we have $R_{n_\epsilon} \leqq \epsilon R'_{n_\epsilon}$; and thus $\frac{R_n}{R'_n} \leqq \epsilon$, for all values of $n \geqq n_\epsilon$. Since ϵ is arbitrary, it follows that $\lim_{n\sim\infty} \frac{R_n}{R'_n} = 0$; from which the result follows.

If two divergent series $\sum_{n=1} d_n$, $\sum_{n=1} d'_n$, *both consisting of positive terms* (> 0), *be such that* $\lim_{n\sim\infty} \frac{d_n}{d'_n} = 0$, *then the first series diverges more slowly than the second.*

If $\frac{d_n}{d'_n} < \epsilon$, for $n > n_\epsilon$, we have $s_n \dot{-} s_{n_\epsilon} < \epsilon\,(s'_n - s'_{n_\epsilon})$; therefore

$$\frac{s_n}{s'_n} < \epsilon + \frac{s_{n_\epsilon} - \epsilon s'_{n_\epsilon}}{s'_n},$$

from which it follows that $\overline{\lim_{n\sim\infty}} \frac{s_n}{s'_n} \leqq \epsilon$. Since ϵ is arbitrary we must have $\lim_{n\sim\infty} \frac{s_n}{s'_n} = 0$, and thus the result has been established.

10. *If the series* $a_1 + a_2 + \dots + a_n + \dots$, *all the terms of which are positive, be convergent, so also is the series* $k_1 a_1 + k_2 a_2 + \dots + k_n a_n + \dots$; *where* $k_1, k_2, \dots k_n, \dots$ *are positive numbers, all of which are less than some positive number* K, *independent of* n.

For it is clear that $R'_{n,m} < KR_{n,m}$; where $R'_{n,m}$ denotes a partial remainder of the second series. It follows that, for all sufficiently large values of n, $R'_{n,m} < \epsilon$, where ϵ is an arbitrarily small positive number. The condition of convergence of the second series is thus satisfied.

The series $\sum\limits_{n=1} a_n$ being taken to be convergent, so that $a_n = R_{n-1} - R_n$, let us consider the series $\sum\limits_{n=1} a'_n$, where $a'_n = R^\rho_{n-1} - R^\rho_n$, and $a'_1 = s^\rho - R^\rho_1$; ρ being a fixed positive number. We find at once that $s'_n = s^\rho - R^\rho_n$, and therefore $\lim\limits_{n\sim\infty} s'_n = s^\rho$. The series $\sum\limits_{n=1}^{\infty} a'_n$ is accordingly convergent; and since $\dfrac{R'_n}{R_n} = R_n^{\rho-1}$, we see that, in case $0 < \rho < 1$, R'_n/R_n increases indefinitely as n increases, and thus the convergence of the second series is slower than that of the first. The following theorem has accordingly been established:

Having given a convergent series of positive numbers, another such series can be determined which converges more slowly than the given one.

We have

$$a'_n = \frac{R^\rho_{n-1} - R^\rho_n}{R_{n-1} - R_n} a_n = \frac{1 - \left(\dfrac{R_n}{R_{n-1}}\right)^\rho}{1 - \dfrac{R_n}{R_{n-1}}} \cdot a_n R^{\rho-1}_{n-1};$$

and since $R_n/R_{n-1} < 1$, for each value of n, we have $\overline{\lim\limits_{n\sim\infty}} \dfrac{R_n}{R_{n-1}} \leq 1$. In case $\overline{\lim\limits_{n\sim\infty}} \dfrac{R_n}{R_{n-1}} = 1$, the corresponding limit of $\dfrac{1 - (R_n/R_{n-1})^\rho}{1 - (R_n/R_{n-1})}$ is ρ; and in any case $\dfrac{1 - (R_n/R_{n-1})}{1 - (R_n/R_{n-1})^\rho}$ is less than some positive number K, independent of n. Thus since $a_n R^{\rho-1}_{n-1} < Ka'_n$, we see that the series $\sum\limits_{n=1} a_n R^{\rho-1}_{n-1}$ is convergent. We thus obtain the following theorem:

If $\sum\limits_{n=1} a_n$ is a convergent series of which the terms are positive, the series $\sum\limits_{n=1} a_n R^{\rho-1}_{n-1}$ is also convergent, for every positive value of ρ. When $\rho < 1$, its convergence is slower than that of $\sum\limits_{n=1} a_n$.

It was first established by Abel* that, if $\sum\limits_{n=1} d_n$ be a divergent series of positive terms, a sequence $\{k_n\}$ of positive numbers, increasing indefinitely with n, can be so determined that the series $\sum\limits_{n=1} d_n/k_n$ is also divergent. The corresponding result for convergent series, here stated, was established† by Du Bois-Reymond. The special theorem that this result is realized by $k_n = R^{\rho-1}_{n-1}$, where $0 < \rho < 1$, is due‡ to Pringsheim.

* *Crelle's Journal*, vol. III (1828), p. 81; also *Œuvres*, vol. I, p. 198 (2nd edition).

† *Crelle's Journal*, vol. LXXVI (1873), p. 85.

‡ *Math. Annalen*, vol. XXXV (1890), pp. 329, 330.

It is clear that a sequence of series may be formed, commencing with the convergent series $\sum_{n=1} a_n$, all of which are convergent, and such that the convergence of each one of them is slower than that of the preceding series. This idea is due to Du Bois-Reymond; Pringsheim has indicated (*loc. cit*) a general method of forming such a sequence of series.

If a sequence of convergent series with positive terms $\sum_{n=1} a_n^{(p)}$, *where* $p = 1, 2, 3, \ldots$, *be such that, for each value of* n, *the sequence*

$$a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, \ldots$$

is monotone increasing, then a series can be formed which converges more slowly than any of the series of the sequence.

A theorem practically equivalent to this has been given* by Hadamard. To establish the theorem, let $R_n^{(p)}$ denote the nth remainder of the series $\sum_{n=1} a_n^{(p)}$. Let the integer n_2 be the smallest integer such that $R_{n_2}^{(2)} \leqq \frac{1}{2}$; and let n_3 be the smallest integer which is $> n_2$ and also such that $R_{n_3}^{(3)} \leqq \frac{1}{2^2}$, and that also $R_{n_3}^{(3)} - R_{n_3}^{(1)} < R_{n_2}^{(2)} - R_{n_2}^{(1)}$.

Let $b_n = a_n^{(1)}$, for $n \leqq n_2$; the difference $R_{n_2}^{(2)} - R_{n_3}^{(3)}$, which is greater than $R_{n_2}^{(1)} - R_{n_3}^{(1)}$, or than $a_{n_2+1}^{(1)} + \ldots + a_{n_3}^{(1)}$, can be divided into $n_3 - n_2$ parts $b_{n_2+1}, b_{n_2+2}, \ldots b_{n_3}$, greater respectively than $a_{n_2+1}^{(1)}, a_{n_2+2}^{(1)}, \ldots a_{n_3}^{(1)}$. We proceed to determine the integers $n_4, n_5, \ldots$ successively, in a similar manner. In general, if n_{p-1} is determined, n_p is the smallest integer ($> n_{p-1}$) which satisfies the conditions

$$R_{n_p}^{(p)} \leqq \frac{1}{2^{p-1}}, \quad R_{n_p}^{(p)} - R_{n_p}^{(p-2)} < R_{n_{p-1}}^{(p-1)} - R_{n_{p-1}}^{(p-2)}.$$

The difference $R_{n_{p-1}}^{(p-1)} - R_{n_p}^{(p)}$ which is greater than $R_{n_{p-1}}^{(p-2)} - R_{n_p}^{(p-2)}$, can be divided into $n_p - n_{p-1}$ parts $b_{n_{p-1}+1}, b_{n_{p-1}+2}, \ldots b_{n_p}$, greater respectively than $a_{n_{p-1}+1}^{(p-2)}, a_{n_{p-1}+2}^{(p-2)}, \ldots a_{n_p}^{(p-2)}$. Proceeding indefinitely in this manner, the terms of a series $\sum_{n=1} b_n$ are defined; and this series is convergent, since it is equivalent to

$$s_{n_2}^{(1)} + (R_{n_2}^{(2)} - R_{n_3}^{(3)}) + (R_{n_3}^{(3)} - R_{n_4}^{(4)}) + \ldots,$$

in which $R_{n_p}^{(p)}$ converges to zero, as $p \sim \infty$. Moreover the terms of the series $\sum_{n=1} b_n$ are, for $n > n_{p-1}$, greater than those of the series $\sum_{n=1} a_n^{(p-2)}$.

A series has thus been constructed which converges at least as slowly as any of the given series.

If the series $\sum_{n=1} b_n$ does not converge more slowly than the series

* *Acta Math.* vol. XVIII (1894), p. 328.

$\sum\limits_{n=1} a_n^{(p)}$, for every value of p, we may form a series which converges more slowly than $\sum\limits_{n=1} b_n$; and this new series will converge more slowly than $\sum\limits_{n=1} a_n^{(p)}$, whatever value p may have.

11. *If the series $d_1 + d_2 + \ldots + d_n + \ldots$ be divergent, so also is the series $k_1 d_1 + k_2 d_2 + \ldots + k_n d_n + \ldots$; where $k_1, k_2, \ldots k_n, \ldots$ are positive numbers all of which exceed some positive number K, independent of n.*

For, s_n, s'_n denoting the partial sums of the two series, we have $s'_n > K s_n$, and thus, if s_n increases indefinitely with n, so also does s'_n.

The divergent series $\sum\limits_{n=1} d_n$, all the terms of which are positive, is such that d_n can be expressed in the form $M_{n+1} - M_n$, where $\{M_n\}$ is a monotone increasing sequence of positive numbers without upper limit. Conversely every series of the form $\sum\limits_{n=1} (M_{n+1} - M_n)$ is divergent.

We have only to take $M_n = s_{n-1}$, to prove the first part of this theorem. To prove the converse, we observe that $M_{n+1} - M_1$ is the partial sum of the series, and this increases indefinitely with n.

If the series $\sum\limits_{n=1} d_n$ is divergent, then d_n can be expressed in the form $d_n = \dfrac{M_{n+1} - M_n}{M_n}$. Conversely every series for which the general term has the form $\dfrac{M_{n+1} - M_n}{M_n}$ is divergent. The numbers $\{M_n\}$ are taken to be those of a positive monotone sequence without upper limit.

Let $\{M_n\}$ be defined by the relations $M_{n+1} = (1 + d_n) M_n$; then

$$M_{n+1} = M_1 (1 + d_1)(1 + d_2) \ldots (1 + d_n) > M_1 (1 + s_n);$$

hence if s_n increases indefinitely with n, so also does M_{n+1}, and therefore the sequence $\{M_n\}$ satisfies the prescribed condition.

To prove the converse, we observe that, if n_1 be any fixed value of n,

$$\sum_{n=n_1}^{n_1+m} \frac{M_{n+1} - M_n}{M_n} > \frac{M_{n_1+m+1} - M_{n_1}}{M_{n_1+m}} > \tfrac{1}{2},$$

provided m has sufficiently large values. The series therefore cannot converge, since it has partial remainders greater than $\frac{1}{2}$, however large n_1 may be. The nearer M_{n+1}/M_n is to unity, the smaller is d_n.

In case $d_n < 1$, for all values of n, let $d_n = \dfrac{M_{n+1} - M_n}{M_{n+1}}$; then

$$\begin{aligned} M_{n+1} = M_n/(1 - d_n) &= M_1/(1 - d_1)(1 - d_2) \ldots (1 - d_n) \\ &> M_1 (1 + d_1)(1 + d_2) \ldots (1 + d_n), \end{aligned}$$

or $M_{n+1} > M_1 (1 + s_n)$; it follows that M_{n+1} increases indefinitely with n, if $\sum\limits_{n=1} d_n$ is divergent. It is easily seen, as before, that a series of which the general term is of the form $\dfrac{M_{n+1} - M_n}{M_{n+1}}$ is divergent.

The following theorem has now been established:

If $\{M_n\}$ be a monotone increasing sequence of positive numbers without upper limit, the series of which the general term is $(M_{n+1} - M_n)/M_{n+1}$ is divergent. Conversely, the terms d_n of any divergent series such that $d_n < 1$ can be expressed in the form $(M_{n+1} - M_n)/M_{n+1}$.

As there is complete latitude in the choice of a particular sequence $\{M_n\}$, there is a corresponding variety in the nature of the divergent series formed from it.

12. *If the series $\sum\limits_{n=1} d_n$ is divergent, and s_n denote its nth partial sum, then the series $\sum\limits_{n=2} \frac{d_n}{s_{n-1}}$, $\sum\limits_{n=1} \frac{d_n}{s_n}$ are both divergent.*

The first part of this theorem was first established by Abel*, and the second part by Dini†. In order to prove it, we may take $s_{n-1} = M_n$, $d_n = M_{n+1} - M_n$; and the results are then equivalent to the foregoing theorems.

Since the ratio either of $\frac{d_n}{s_{n-1}}$, or of $\frac{d_n}{s_n}$, to d_n converges to zero, as $n \sim \infty$, it follows from § 9 that the series $\sum\limits_{n=2} \frac{d_n}{s_{n-1}}$, $\sum\limits_{n=1} \frac{d_n}{s_n}$ both diverge more slowly than the series $\sum\limits_{n=1} d_n$.

The following result, due essentially to Abel, has now been established:

Having given a divergent series of which the terms are positive, another divergent series can be defined which diverges more slowly than the given one.

It was also shewn by Abel (*loc. cit.*) that the series $\sum\limits_{n=1} \frac{d_n}{s_n^{1+\lambda}}$ converges, provided $\lambda > 0$, the series $\sum\limits_{n=1} d_n$ being as before divergent. For

$$\frac{1}{s_{n-1}^{\lambda}} - \frac{1}{s_n^{\lambda}} = (s_n - d_n)^{-\lambda} - s_n^{-\lambda} > \frac{\lambda d_n}{s_n^{\lambda+1}};$$

and it follows that
$$\sum_{n=2}^{n=n_1} \frac{d_n}{s_n^{\lambda+1}} < \frac{1}{\lambda}\left\{\frac{1}{a_1^{\lambda}} - \frac{1}{s_{n_1}^{\lambda}}\right\}.$$

Since s_{n_1} has the limit ∞, as n_1 is independently increased, the convergence of the series $\sum\limits_{n=1} \frac{d_n}{s_n^{\lambda+1}}$ holds good. In comparing this result with the divergence theorem above, it should be observed that the series $\sum\limits_{n=1} \frac{d_n}{s_{n-1}^{1+\lambda}}$, for $\lambda > 0$, is not necessarily convergent. This is seen from a consideration of the case in which $s_{n-1}^{1+\lambda} = s_n$, $d_1 > 1$.

* *Crelle's Journal*, vol. III (1828), p. 80: also *Œuvres*, vol. II, p. 198, 2nd ed.

† *Annali dell' Univ. Tosc.* vol. IX, p. 8. Also separately, *Sulle serie a termini positivi*, Pisa, 1867, Tipografia Nistri.

EXAMPLE.

Consider the divergent series $1 + 1 + 1 + \ldots$; we have then $s_n = n$. It follows from Abel's first theorem that $\Sigma \frac{1}{n}$ is divergent, and from the second theorem that $\Sigma \frac{1}{n^{1+\lambda}}$ is convergent, provided $\lambda > 0$.

From Dini's theorem it is deducible that $\Sigma \dfrac{\frac{1}{n}}{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}$ is a divergent series.

It is clear that, by continuation of the process of forming from a given divergent series one which diverges more slowly, an endless sequence of divergent series can be obtained, each of which diverges more slowly than the preceding one.

The following theorem is the analogue of the theorem of § 10.

If a sequence of divergent series with positive terms,* $\sum\limits_{n=1} d_n^{(p)}$, *where* $p = 1, 2, 3, \ldots$, *be such that, for each value of* n, *the sequence* $d_n^{(1)}, d_n^{(2)}, \ldots$ *is monotone decreasing, then a divergent series can be formed which diverges more slowly than any of the series of the sequence.*

The proof of the theorem is precisely similar to that of the corresponding theorem for convergent series, given in § 10.

CRITERIA OF CONVERGENCE AND DIVERGENCE OF SERIES WITH POSITIVE TERMS.

13. Much attention has been devoted by mathematicians to the problem of obtaining criteria sufficient to decide the question as to whether a series of prescribed form converges or diverges. These tests, as regards a series $\sum\limits_{n=1} a_n$, of which all the terms are positive, are usually obtained by comparing the series with other series which are known to be either convergent or to be divergent. Such tests, formed by comparison with other series, fall in the main under two heads, first those in which the general term a_n is alone involved in the criteria, and secondly, those in which the criteria have reference to the form of the ratio a_{n+1}/a_n. These tests may be referred to as of the first and second kinds respectively. All such tests provide sufficient, but not necessary, conditions for the convergence or divergence of series; no test can be given which will be decisive as regards every series that can be defined; thus the necessary and sufficient condition of convergence, which, for a series of positive terms, is that s_n should be bounded, cannot in the general case be reduced to any equivalent form which is of simpler application. Various sets of criteria of convergence were given during the first half of the nineteenth century, the most important of which will be given below; and more general theories of such criteria were given by Dini†, and Du Bois-Reymond‡. The most complete

* Hadamard, *Acta Math.* vol. XVIII (1894), p. 326.

† *Sulle serie a termini positivi*, Pisa, 1867. ‡ *Crelle's Journal*, vol. LXXVI (1873), p. 61.

general theory of such criteria, in which all the known criteria are obtained from a unified point of view, is that given by Pringsheim*.

The two simplest tests of the first and second kinds respectively were given by Cauchy†, and may be stated in the following somewhat generalized forms:

(1) *A series* $\sum_{n=1} a_n$, *of which all the terms are positive, is convergent if* $\overline{\lim}_{n\sim\infty} a_n^{\frac{1}{n}} < 1$, *and is divergent if* $\overline{\lim}_{n\sim\infty} a_n^{\frac{1}{n}} > 1$.

It will be observed that the only case in which this test fails to distinguish between convergence and divergence is when $\overline{\lim}_{n\sim\infty} a_n^{\frac{1}{n}} = 1$.

To establish the test, let $\overline{\lim}_{n\sim\infty} a_n^{\frac{1}{n}} = k$. If $k < 1$, let ρ be a number between k and 1, then, for all sufficiently large values of n, we have $a_n^{\frac{1}{n}} < \rho$, and thus $R_{n,m}$ is, for a sufficiently large value of n, less than $\rho^{n+1} + \rho^{n+2} + \ldots + \rho^{n+m}$, or than $\frac{\rho^{n+1}}{1-\rho}$; and this for $m = 1, 2, 3, \ldots$. It is clear that, for a sufficiently large value of n, $\frac{\rho^{n+1}}{1-\rho} < \epsilon$, where ϵ is arbitrarily chosen, and then $R_{n,m} < \epsilon$; thus the condition of convergence is satisfied.

If $k > 1$, let ρ be a number between 1 and k; if $\overline{\lim}_{n\sim\infty} a_n^{\frac{1}{n}} = k$, there are an indefinitely great set of values of n for which $a_n^{\frac{1}{n}} > \rho$, or for which $a_n > \rho^n > 1$. The condition $\lim_{n\sim\infty} a_n = 0$ not being satisfied, the series is divergent.

(2) *A series* $\sum_{n=1} a_n$, *of which all the terms are positive, is convergent if* $\overline{\lim}_{n\sim\infty} \frac{a_{n+1}}{a_n} < 1$, *and it is divergent if* $\underline{\lim}_{n\sim\infty} \frac{a_{n+1}}{a_n} > 1$.

Cauchy himself considered only the case in which $\lim_{n\sim\infty} \frac{a_{n+1}}{a_n}$ exists. In that case the criterion fails only when the limit has the value 1, when some more effective test is required. In the general case, the test fails when both the inequalities $\overline{\lim}_{n\sim\infty} \frac{a_{n+1}}{a_n} \geqq 1$, $\underline{\lim}_{n\sim\infty} \frac{a_{n+1}}{a_n} \leqq 1$ are satisfied, that is, when 1 is in the interval of indeterminancy of $\lim_{n\sim\infty} \frac{a_{n+1}}{a_n}$. First, let $\overline{\lim}_{n\sim\infty} \frac{a_{n+1}}{a_n} = k < 1$, and let ρ be a number between k and 1, then, for all sufficiently large values of n, we have $a_{n+1} < \rho a_n$, and thus $R_{n,m} < a_n(\rho + \rho^2 + \ldots + \rho^m) < \frac{\rho a_n}{1-\rho}$.

* *Math. Annalen*, vol. xxxv (1890), p. 297 and vol. xxxix (1891), p. 125.

† See *Cours d'Analyse Alg.* (1821), pp. 133, 134; also *Résum. analyt.* p. 150.

Since $a_{n+m} < \rho^m a_n$, we see that $\lim_{m\sim\infty} a_{n+m} = 0$, and thus $\lim_{n\sim\infty} a_n = 0$. If now n be sufficiently large, we have $R_{n,m} < \epsilon$, for all values of m; and thus the series is convergent.

If $\lim_{n\sim\infty} \frac{a_{n+1}}{a_n} = k > 1$, let ρ be a number between 1 and k, then $a_{n+1} > \rho a_n$, for all sufficiently large values of n. Hence $a_{n+m} > \rho^m a_n$, and thus a_{n+m} increases indefinitely with m. Since the condition $\lim_{n\sim\infty} a_n = 0$ is not satisfied, the series is divergent.

14. When the above tests fail, other tests must be applied; one of the simplest of these is that known as Cauchy's condensation test, which may be stated as follows:

If, in the series $\sum_{n=1} a_n$, *all the terms of which are positive, and such that* $a_n \geqq a_{n+1}$, *for all values of* n *(at least from and after some fixed value of* n*), then the two series* $\sum_{n=1} a_n$, $\sum_{n=1} 2^n a_{2^n}$ *are both convergent or both divergent.*

To prove the theorem, we observe that $a_{2^n} + a_{2^n+1} + \ldots + a_{2^{n+1}-1}$ is less than $2^n a_{2^n}$, and greater than $2^n a_{2^{n+1}}$. It follows that $s_{2^{n+1}-1}$ is less than $a_1 + \sum_1^n 2^n a_{2^n}$ and greater than $a_1 + \frac{1}{2}\sum_2^{n+1} 2^n a_{2^n}$. From this we see that, in case $\sum_1^n 2^n a_{2^n}$ is convergent, $s_{2^{n+1}-1}$ converges to a fixed limit, as $n \sim \infty$, and therefore the series $\sum_{n=1} a_n$ is convergent. Conversely, if $\sum_{n=1} a_n$ is convergent, $s_{2^{n+1}-1}$ converges to a definite limit, and hence $\sum_2^{n+1} 2^n a_{2^n}$ converges.

For example, let $a_n = \frac{1}{n}$; then $2^n a_{2^n} = 1$, from which it follows that the series $\sum \frac{1}{n}$ is divergent.

If the series Σa_n, all the terms of which are positive, be such that $a_n \geqq a_{n+1}$, for all values of n, a continuous monotone non-increasing function $f(x)$ may be defined for the infinite interval $(1, \infty)$ such that $f(n) = a_n$, for all integral values of n. A precise form of $f(x)$ will often be suggested by the form of a_n, or it may be defined by

$$f(x) = a_{n+1}(x - n) - a_n(x - n - 1),$$

in the interval (a_n, a_{n+1}).

Let us consider the function $F(x) = \int_1^x f(x)\,dx$; we have

$$F(n) = \sum_{r=1}^{r=n-1} \int_r^{r+1} f(x)\,dx;$$

and this is not less than $a_2 + a_3 + \ldots + a_n$, and not greater than

$$a_1 + a_2 + \ldots + a_{n-1}.$$

We thus have
$$s_n - a_1 \leqq F(n) \leqq s_{n-1}.$$

If the series is convergent, $\lim_{n\sim\infty} s_{n-1}$ is finite, and consequently $\lim_{n\sim\infty}\int_1^n f(x)\,dx$ exists. Since $\int_1^n f(x)\,dx \leqq \int_1^h f(x)\,dx \leqq \int_1^{n+1} f(x)\,dx$, where $n \leqq h \leqq n+1$, it follows that $\int_1^\infty f(x)\,dx = \lim_{n\sim\infty}\int_1^n f(x)\,dx$. Conversely, since $\lim_{n\sim\infty} s_n \leqq a_1 + \lim_{n\sim\infty} F(n)$, the existence of the integral involves the convergence of the series.

When the series is divergent, since $s_n - \int_1^n f(x)\,dx \leqq a_1$, and since $s_n - \int_1^n f(x)\,dx \geqq s_{n+1} - \int_1^{n+1} f(x)\,dx$, it follows that $\lim_{n\sim\infty}\left\{s_n - \int_1^n f(x)\,dx\right\}$ exists, and is between 0 and a_1.

We have established the following theorem:

If a_n is positive for all values of n, and $a_n \geqq a_{n+1}$, and if $f(x)$ be a continuous monotone non-increasing function, defined for the interval $(1, \infty)$, and such that $f(n) = a_n$, for all integral values of n, then the series $\sum\limits_{n-1} a_n$ and the integral $\int_1^\infty f(x)\,dx$ are either both convergent, or both divergent; and in the latter case $s_n - \int_1^n f(x)\,dx$ converges to a number between 0 and a_1.

EXAMPLES.

(1) Let $f(x) = \frac{1}{x}$; then since $\int_1^n \frac{dx}{x}$ is divergent, as $n \sim \infty$, the series $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} + \dots$ is divergent. Also $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log_e n$ converges, as $n \sim \infty$, to a definite number C, between 0 and 1. The number C is known as Mascheroni's constant, and also as Euler's constant.

(2) Let $f(x) = \frac{1}{x^p}$, then $\int_1^n \frac{dx}{x^p} = \frac{n^{1-p} - 1}{1-p}$. This is convergent, if $p > 1$, and divergent if $p < 1$. When $p > 1$, the sum of the series is between $\frac{1}{p-1}$ and $\frac{p}{p-1}$.

(3) Let $f(x) = \frac{1}{x(\log x)^p}$, then $\int_2^n \frac{dx}{x(\log x)^p} = \frac{1}{(\log 2)^{p-1}(p-1)} - \frac{1}{(\log n)^{p-1}(p-1)}$

except that, when $p = 1$, $\int_2^n \frac{dx}{x\log x} = \log\left(\frac{\log n}{\log 2}\right)$.

It follows that the series $\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \dots$

converges when $p > 1$, and diverges when $p \leqq 1$.

(4) Let
$$f(x) = \frac{1}{x\log x \,.\, \log\log x \dots (\log\log\dots\log x)^p},$$
then
$$\int_2^n f(x)\,dx = \frac{1}{(\log\log\dots\log 2)^{p-1}(p-1)} - \frac{1}{(\log\log\dots\log n)^{p-1}(p-1)}$$

except that, when $p = 1$, $\int_2^n f(x)\,dx = \log\left(\frac{\log\log\ldots\log n}{\log\log\ldots\log 2}\right)$.

Therefore the series $\sum_{n=2} \frac{1}{n\log n.\log\log n\ldots(\log\log\ldots\log n)^p}$ converges when $p > 1$, and diverges when $p \leq 1$.

15. If the series $\sum_{n=1} c_n$ is convergent, it is a sufficient condition for the convergence of $\sum_{n=1} a_n$ that $\frac{a_n}{c_n}$ should be less than some fixed positive number K independent of n. For, if s_n, s_n' denote the partial sums of the series Σa_n, Σc_n respectively, we have $s_n < K s_n'$; and thus, if s_n' has a definite limit, so also has s_n. This criterion may be stated in the following form:

If $\sum_{n=1} a_n$, $\sum_{n=1} c_n$ be two series with positive terms, and the latter be convergent, it is a sufficient condition for the convergence of $\sum_{n=1} a_n$ that $\overline{\lim_{n\sim\infty}}\frac{a_n}{c_n}$ should not be infinite.

Again, if $\frac{a_n}{d_n}$ is greater than some positive number L, independent of n, and the series $\sum_{n=1} d_n$ is divergent, so also is $\sum_{n=1} a_n$. For $s_n > L s_n'$; and thus if s_n' increases indefinitely with n, so also does s_n. This theorem may be stated as follows:

If Σa_n, Σd_n be two series with positive terms, and the latter be divergent, then it is sufficient for the divergence of $\sum_{n=1} a_n$ that $\underline{\lim_{n\sim\infty}}\frac{a_n}{d_n}$ should be greater than zero.

It has been shewn, in § 11, that every divergent series can be expressed in the form $\sum_{n=1}\frac{M_{n+1}-M_n}{M_n}$. The following theorem gives the corresponding result for a convergent series:

Every convergent series $\sum_{n=1} c_n$ is such that c_n can be expressed in the form $\frac{M_{n+1}-M_n}{M_{n+1}M_n}$; and conversely a series of which the terms have the latter form is convergent.

For let $M_n^{-1} = \sum_n^\infty c_n$, then $c_n = M_n^{-1} - M_{n+1}^{-1} = \frac{M_{n+1}-M_n}{M_{n+1}M_n}$.

Conversely, we have $s_n = \sum_1^n c_n = M_1^{-1} - M_{n+1}^{-1}$; and thus s_n converges to M_1^{-1}.

This theorem and the corresponding theorem for divergent series, given in § 11, may be employed to express the conditions in the two first theorems above in the following form:

The series $\sum_{n=1} a_n$ *is convergent if a monotone increasing sequence of positive numbers* $\{M_n\}$ *exists, without upper limit, such that* $\overline{\lim}\, a_n \dfrac{M_{n+1} M_n}{M_{n+1} - M_n} < \infty$.
The series is divergent, if $\{M_n\}$ *can be so determined that* $\underline{\lim}\, a_n \dfrac{M_n}{M_{n+1} - M_n} > 0$.

If $\rho > 0$, the sequence $\{M_n^\rho\}$ has the same essential characteristic as the sequence $\{M_n\}$, that it is monotone increasing, and such that M_n^ρ has no upper limit as $n \sim \infty$.

If $\dfrac{M_{n+1}^\rho - M_n^\rho}{M_{n+1}^\rho M_n^\rho}$ be denoted by $c_{\rho, n}$, the series $\Sigma c_{\rho, n}$ is convergent, and in case $\rho < 1$, it converges more slowly than the series $\Sigma c_{1, n}$ or Σc_n. We may write $c_{\rho, n}$ in the form $\dfrac{M_{n+1} - M_n}{M_{n+1} M_n^\rho} \dfrac{1 - \lambda_n^\rho}{1 - \lambda_n}$, where λ_n denotes $\dfrac{M_n}{M_{n+1}}$. Since $\dfrac{1 - \lambda_n^\rho}{1 - \lambda_n}$ has a finite lower limit, as $n \sim \infty$, it follows that the series of which the general term is $\dfrac{M_{n+1} - M_n}{M_{n+1} M_n^\rho}$ is also convergent. We thus have the theorem:

The series of which the general term is $\dfrac{M_{n+1} - M_n}{M_{n+1} M_n^\rho}$, *where* ρ *is any positive number, is convergent, and converges the more slowly, the more slowly* M_n^ρ *increases as* n *is indefinitely increased.*

Employing this result, we may now state the following general criteria, equivalent to forms due to Pringsheim*:

If $\underline{\lim}\, a_n \dfrac{M_n}{M_{n+1} - M_n} > 0$, *or in case* $a_n < 1$ *for all values of* n,

$$\underline{\lim}\, a_n \frac{M_{n+1}}{M_{n+1} - M_n} > 0,$$

then the series Σa_n *is divergent.*

If $\overline{\lim}\, a_n \dfrac{M_{n+1} M_n^\rho}{M_{n+1} - M_n} < \infty$; *where* ρ *is a fixed positive number, then* Σa_n *is convergent.*

The numbers M_n are subject to no condition except that M_n increases indefinitely with n, the sequence $\{M_n\}$ being monotone. It is clear that the criteria will be the more efficient the more slowly M_n increases as n increases. Commencing with a given sequence $\{M_n\}$, by substitution for M_n of ever more slowly increasing numbers, there can be obtained a succession of criteria of continually increasing delicacy.

The criteria may be somewhat simplified if it be assumed that $\{M_n\}$ is such that $\dfrac{M_{n+1}}{M_n}$ is less than some fixed positive number independent of n.

* *Math. Annalen*, vol. xxxv (1890), p. 337.

The criterion of convergence then takes the form $\overline{\lim}\, a_n \dfrac{M_n^{\rho+1}}{M_{n+1}-M_n} < \infty$.

16. An important series of criteria may be obtained by substituting successively $\log_e M_n$, $\log_e \log_e M_n$, ... for M_n in the sequence $\{M_n\}$. It is convenient to denote $\log_e z$, $\log_e \log_e z$, ... $\log_e \log_e \dots \log_e z$ by

$$\log^{(1)} z,\ \log^{(2)} z,\ \dots \log^{(m)} z,$$

z by $\log^{(0)}(z)$, and the product $z \log^{(1)} z \log^{(2)} z \dots \log^{(m)} z$ by $L_m(z)$; also $z = L_0(z)$. It can easily be shewn that the two functions $z - 1 - \log z$, $\log z - 1 + \frac{1}{z}$ are both positive for all values of z in the indefinite interval $(0, \infty)$; except that they both vanish when $z = 1$. Thus we have

$$\log_e z \leqq z - 1, \quad \log_e z \geqq 1 - \frac{1}{z}, \quad \text{for } 0 < z < \infty.$$

Let $z = \dfrac{M_{n+1}}{M_n}$; we then have $\log M_{n+1} - \log M_n \leqq \dfrac{M_{n+1} - M_n}{M_n}$ and $\geqq \dfrac{M_{n+1} - M_n}{M_{n+1}}$. Again, assuming that n is so large that $\log M_n > 0$, we have

$$\log^{(2)} M_{n+1} - \log^{(2)} M_n \leqq \frac{\log M_{n+1} - \log M_n}{\log M_n} \leqq \frac{M_{n+1} - M_n}{M_n \log M_n},$$

and $$\log^{(2)} M_{n+1} - \log^{(2)} M_n \geqq \frac{\log M_{n+1} - \log M_n}{\log M_{n+1}} \geqq \frac{M_{n+1} - M_n}{M_{n+1} \log M_{n+1}}.$$

Proceeding in this manner, we find that

$$\log^{(m)} M_{n+1} - \log^{(m)} M_n \leqq \frac{M_{n+1} - M_n}{L_{m-1}(M_n)},$$

$$\log^{(m)} M_{n+1} - \log^{(m)} M_n \geqq \frac{M_{n+1} - M_n}{L_{m-1}(M_{n+1})},$$

the number n being taken to be sufficiently large.

If it be assumed that $\dfrac{M_{n+1}}{M_n}$ is less than a fixed positive number, independent of n, which is equivalent to the assumption that $\overline{\lim\limits_{n\sim\infty}} \dfrac{M_{n+1}}{M_n}$ is finite, we see that $\lim\limits_{n\sim\infty} \dfrac{\log M_{n+1}}{\log M_n} = 1$, and generally that $\lim\limits_{n\sim\infty} \dfrac{\log^{(k)} M_{n+1}}{\log^{(k)} M_n} = 1$.

It then follows from the two inequalities obtained above that

$$\log^{(m)} M_{n+1} - \log^{(m)} M_n \text{ and } \frac{M_{n+1} - M_n}{L_{m-1}(M_n)}$$

are in a ratio which lies in an interval $(k_m, 1)$, where $0 < k_m < 1$.

In accordance with the theorems of § 15, we have the criteria that, if

$$\underline{\lim}\, a_n \frac{\log^{(m)} M_n}{\log^{(m)} M_{n+1} - \log^{(m)} M_n} > 0,$$

the series Σa_n is divergent; and if

$$\overline{\lim}\, a_n \frac{\log^{(m)} M_{n+1} (\log^{(m)} M_n)^\rho}{\log^{(m)} M_{n+1} - \log^{(m)} M_n} < \infty,$$

then Σa_n is convergent.

Employing the restriction that $\overline{\lim\limits_{n\sim\infty}} \frac{M_{n+1}}{M_n}$ is finite, we now obtain the following criteria*.

If $\underline{\lim\limits_{n\sim\infty}}\, a_n \frac{L_m(M_n)}{M_{n+1} - M_n} > 0$, *the series* Σa_n *is divergent; and if, where* $\rho > 0$,

$$\overline{\lim_{n\sim\infty}}\, a_n \frac{(\log^{(m)} M_n)^\rho L_m(M_n)}{M_{n+1} - M_n} < \infty,$$

the series is convergent, whatever integral value m may have; it being assumed that the sequence $\{M_n\}$ *is such that* $\overline{\lim\limits_{n\sim\infty}} \frac{M_{n+1}}{M_n}$ *is finite.*

If we take $M_n = n$, we obtain, as a special case, a series of criteria which were first given explicitly by De Morgan†, although the essentials are to be found in a posthumous memoir of Abel. The particular case in which $m = 1$ had been given by Cauchy, and the full criteria were re-discovered by Bertrand. The corresponding criteria for integrals were first given fully by Bonnet‡.

If $\underline{\lim\limits_{n\sim\infty}}\, na_n > 0$, Σa_n *is divergent, and in general, if* $\underline{\lim\limits_{n\sim\infty}}\, a_n L_m(n) > 0$, Σa_n *is divergent. If* $\overline{\lim\limits_{n\sim\infty}}\, a_n n^{\rho+1} < \infty$, Σa_n *is convergent, and in general, if* $\overline{\lim\limits_{n\sim\infty}}\, a_n L_m(n) (\log^{(m)} n)^\rho < \infty$, Σa_n *is convergent; the number* ρ *being positive.*

17. *If* $\sum\limits_{n=1} d_n$ *be a divergent series, and* $f(x) = O(x^{-1-\delta})$, *where* $f(x) \geqq 0$, *and* $\delta > 0$, *then the series* $\sum\limits_{n=1} f(s_n) d_n$ *is convergent. In particular, the series* $\sum\limits_{n=1} e^{-\rho s_n} d_n$ *is convergent if* $\rho > 0$, *and it is divergent if* $\rho \leqq 0$.

Since $f(s_n) d_n < \frac{K d_n}{s_n^{1+\delta}}$, where K is some fixed positive number, and the series $\sum\limits_{n=1} \frac{d_n}{s_n^{1+\delta}}$ has been shewn in § 12 to be convergent, it follows that the series $\sum\limits_{n=1} f(s_n) d_n$ is convergent. If $f(x) = e^{-\rho x}$, $(\rho > 0)$, it is easily seen that $x^{1+\delta} e^{-\rho x}$ has, for positive values of x, a finite maximum.

In case $\overline{\lim\limits_{n\sim\infty}}\, d_n$ is finite, d_n is less, for all values of n, than some fixed number D. The terms of the series $\sum\limits_{n=m} d_n f(s_{n-1})$ are less than the corre-

* See Pringsheim, *Math. Annalen*, vol. XXXV (1890), p. 339.

† *Differential and Integral Calculus* (1839), p. 326.

‡ *Liouville's Journal*, vol. VIII (1843), p. 78.

sponding terms of $2^{1+\delta} K \sum_{n=m} \frac{d_n}{s_n^{1+\delta}}$, where m is such that $s_{m-1} > D$; and therefore the series $\sum_{n=1} d_n f(s_{n-1})$ is convergent. It has thus been shewn that:

If $\sum_{n=1} d_n$ be a divergent series, and $d_n = O(1)$, and $f(x) = O(x^{-1-\delta})$, where $f(x) \geqq 0$, and $\delta > 0$, then the series $\sum_{n=2} f(s_{n-1}) d_n$ is convergent. In particular, the series $\sum_{n=2} d_n e^{-\rho s_{n-1}}$ is convergent, if $\rho > 0$.

The following theorem has been given by Littlewood*. It can easily be proved by the method employed in § 14:

If $\sum_{n=1} d_n$ is divergent, and $d_n = O(1)$, and if $f(x)$ be a continuous positive decreasing function of x, then $\sum_{n=1} d_n f(s_n)$ converges or diverges with $\int_1^\infty f(x)\,dx$.

The series Σd_n being divergent, the series $\frac{\log s_n - \log s_{n-1}}{\log s_{n-1}}$ is also divergent, in accordance with the theorem of § 11.

Now $\log s_n - \log s_{n-1} \leqq \frac{s_n - s_{n-1}}{s_{n-1}}$, and it thus follows that the series $\sum \frac{d_n}{s_{n-1} \log s_{n-1}}$, or $\sum \frac{d_n}{L(s_{n-1})}$, is divergent. It is easily seen that it diverges more slowly than Σd_n.

Similarly we see that

$$\log^{(2)} s_n - \log^{(2)} s_{n-1} \leqq \frac{\log s_n - \log s_{n-1}}{\log s_{n-1}} \leqq \frac{s_n - s_{n-1}}{L(s_{n-1})},$$

and since $\sum \frac{\log^{(2)} s_n - \log^{(2)} s_{n-1}}{\log^{(2)} s_{n-1}}$ is divergent, it follows that $\sum \frac{d_n}{L^{(2)}(s_{n-1})}$ is divergent. Proceeding in this manner it is seen that *the series $\sum \frac{d_n}{L^{(m)}(s_{n-1})}$ is divergent, for $m = 0, 1, 2, 3, \ldots$, provided Σd_n is divergent.*

18. Writing M_{n+1} for s_n, we see from the first theorem of § 17, that for any monotone increasing sequence $\{M_n\}$, the series

$$\Sigma (M_{n+1} - M_n) e^{-\rho M_{n+1}}$$

is convergent if $\rho > 0$, and divergent if $\rho \leqq 0$.

Employing the criteria of § 15, we now see, by substituting for c_n or d_n the value $(M_{n+1} - M_n) e^{-\rho M_{n+1}}$, that *if $\overline{\lim_{n\sim\infty}} \frac{a_n}{M_{n+1} - M_n} e^{\rho M_{n+1}} < \infty$, $\rho > 0$, the series Σa_n is convergent; and if $\underline{\lim_{n\sim\infty}} \frac{a_n}{M_{n+1} - M_n} e^{\rho M_{n+1}} > 0$, $\rho \leqq 0$, the series Σa_n is divergent.*

* *Messenger of Math.*, vol. XXXIX (1910), p. 191.

If $\frac{a_n}{M_{n+1}-M_n}e^{\rho M_{n+1}} < K$, for all values of n, we see that

$$\log\frac{a_n}{M_{n+1}-M_n}+\rho M_{n+1}$$

is less than a fixed number, and thus that $\frac{1}{M_{n+1}}\log\frac{a_n}{M_{n+1}-M_n}+\rho$ has for its upper limit, as $n\sim\infty$, a number $\leqq 0$. Similarly, if

$$\frac{a_n}{M_{n+1}-M_n}e^{\rho M_{n+1}} > k,$$

we see that $\frac{1}{M_{n+1}}\log\frac{a_n}{M_{n+1}-M_n}+\rho$ has for its lower limit a number which is $\geqq 0$. We thus obtain the following criteria:

If $\underline{\lim}_{n\sim\infty}\frac{1}{M_{n+1}}\log\frac{M_{n+1}-M_n}{a_n}>0$, *the series* Σa_n *is convergent; and if* $\overline{\lim}_{n\sim\infty}\frac{1}{M_{n+1}}\log\frac{M_{n+1}-M_n}{a_n}<0$, *the series is divergent.*

If we substitute $\log^{(m+1)}M_n$ for M_n in these criteria, and assume that $\frac{M_{n+1}}{M_n}=O(1)$, we obtain* the following scale of criteria:

If $\underline{\lim}_{n\sim\infty}\frac{1}{\log^{(m+1)}M_n}\log\left(\frac{M_{n+1}-M_n}{L_m(M_n)\,a_n}\right)>0$, *the series* Σa_n *is convergent; and if* $\overline{\lim}_{n\sim\infty}\frac{1}{\log^{(m+1)}M_n}\log\left(\frac{M_{n+1}-M_n}{L_m(M_n)\,a_n}\right)<0$, *the series is divergent.*

If we take $M_n=n$, we have the following scale of criteria:

If $\underline{\lim}_{n\sim\infty}\frac{1}{n}\log\left(\frac{1}{a_n}\right)>0$, *and generally if* $\underline{\lim}_{n\sim\infty}\frac{1}{\log^{(m+1)}n}\log\frac{1}{L_m(n)\,a_n}>0$, *the series is convergent; and if* $\overline{\lim}_{n\sim\infty}\frac{1}{n}\log\frac{1}{a_n}<0$, *and generally if*

$$\overline{\lim}_{n\sim\infty}\frac{1}{\log^{(m+1)}n}\log\frac{1}{L_m(n)\,a_n}<0,$$

the series is divergent.

The first of the criteria of this scale are equivalent to the following criteria due to Cauchy†:

If $\overline{\lim}_{n\sim\infty}a_n^{\frac{1}{n}}<1$, *the series* Σa_n *is convergent; and if* $\underline{\lim}_{n\sim\infty}a_n^{\frac{1}{n}}>1$, *the series* Σa_n *is divergent.*

It has however been shewn in § 13, that this last condition may be replaced by the less stringent condition $\overline{\lim}_{n\sim\infty}a_n^{\frac{1}{n}}>1$.

The criteria given by the case $m=0$, that if $\underline{\lim}_{n\sim\infty}\frac{1}{\log n}\log\frac{1}{na_n}>0$, the

* See Dini, *Annali dell' Univ. Tosc.* vol. IX, p. 11.

† *Cours d'Analyse Alg.* (1821), p. 133.

series is convergent, and that if $\overline{\lim_{n\sim\infty}} \frac{1}{\log n} \log \frac{1}{na_n} < 0$, the series is divergent, were given by Cauchy*; the whole scale was given by Bertrand†. An equivalent scale of criteria was also given by de Morgan‡.

19. It has already been observed that the criteria which have been obtained for the convergence or the divergence of a series Σa_n with positive terms yield sufficient conditions for such convergence or divergence, but not necessary conditions. It can be shewn that there exists no set of positive numbers $\lambda_1, \lambda_2, \ldots \lambda_n, \ldots$ for which $\lim_{n\sim\infty} \lambda_n = 0$, such that the condition $\overline{\lim_{n\sim\infty}} \frac{a_n}{\lambda_n} < \infty$ is a necessary condition for the convergence of the series Σa_n. On the contrary, a convergent series Σa_n can always be constructed such that $\overline{\lim_{n\sim\infty}} \frac{a_n}{\lambda_n} = \infty$, when the sequence $\{\lambda_n\}$ has been prescribed. An increasing sequence of integers $n_1, n_2, \ldots n_m, \ldots$ can be so chosen that $\lambda_{n_1} \leqq \frac{1}{2^2}, \lambda_{n_2} \leqq \frac{1}{2^6}, \ldots \lambda_{n_m} \leqq \frac{1}{2^{4m-2}}, \ldots$.

Now let $a_n = \frac{1}{2^{2n}}$, for all values of n which do not belong to the sequence $\{n_m\}$; and let $a_{n_m} = \frac{1}{2^{2m-1}}$, for $m = 1, 2, 3, \ldots$.

The series Σa_n consists of the terms of the convergent series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots,$$

in a different order, and is therefore convergent. But $\frac{a_{n_m}}{\lambda_{n_m}} \geqq 2^{2m-1}$, and thus $\overline{\lim_{n\sim\infty}} \frac{a_n}{\lambda_n} = \infty$.

In particular it has been shewn that there exists no set of positive numbers $\{\lambda_n\}$ satisfying the condition $\lim_{n\sim\infty} \lambda_n = 0$, such that $\lim_{n\sim\infty} \frac{a_n}{\lambda_n} = 0$ is a necessary§ condition for the convergence of the series Σa_n. It can, however, be shewn that it is a necessary condition for the convergence of Σa_n that $\underline{\lim_{n\sim\infty}} \frac{a_n}{\lambda_n} = 0$, provided the sequence $\{\lambda_n\}$ be properly chosen. For if $\underline{\lim_{n\sim\infty}} \frac{a_n}{\lambda_n} > 0$, k can be so chosen that $a_n > k\lambda_n$, for all sufficiently large values of n, and thus, if the sequence $\{\lambda_n\}$ be so chosen that $\Sigma\lambda_n$ is divergent, the series Σa_n is also divergent. The incorrect statement has been made

* *Cours d'Analyse Alg.* (1821), p. 137.

† *Liouville's Journal* (1), vol. VII (1842), p. 37. See also Paucker, *Crelle's Journal*, vol. XLII (1851), p. 138.

‡ *Differential and Integral Calculus*, p. 326.

§ This is contrary to an assumption made by Du Bois-Reymond; on this point see Pringsheim, *Math. Annalen*, vol. XXXV (1890), p. 346.

by Dini* and others, that if $\Sigma\lambda_n$ is divergent, it is a necessary condition for the convergence of Σa_n that $\lim\limits_{n\sim\infty} \frac{a_n}{\lambda_n} = 0$. This statement only becomes correct when the additional condition is added that $\lim\limits_{n\sim\infty} \frac{a_n}{\lambda_n}$ has a unique value.

It can be shewn in a similar manner that there exists no set $\{\lambda_n\}$ of monotone increasing numbers such that the condition $\underline{\lim}_{n\sim\infty} \frac{a_n}{\lambda_n} > 0$ is necessary for the divergence of Σa_n. In fact, if $\{\lambda_n\}$ be prescribed and be such that $\Sigma\lambda_n$ is divergent, a series Σa_n can be so determined as to be divergent and also such that $\underline{\lim}_{n\sim\infty} \frac{a_n}{\lambda_n} = 0$.

20. Let $a_1 + a_2 + \ldots + a_n + \ldots$, $b_1 + b_2 + \ldots + b_n + \ldots$ denote two series of which the terms are positive. Let it be assumed that $\frac{a_{n+1}}{a_n} \geqq \frac{b_{n+1}}{b_n}$, for all values of n that are $\geqq m$. We find, by giving n the values

$$m, m + 1, \ldots, n - 1,$$

that $\frac{a_n}{a_m} \geqq \frac{b_n}{b_m}$, or $a_n \geqq kb_n$, for $n \geqq m$, where k denotes the number $\frac{a_m}{b_m}$. From this it follows that, if Σb_n is divergent, so also is Σa_n.

Similarly, if it be assumed that $\frac{a_{n+1}}{a_n} \leqq \frac{b_{n+1}}{b_n}$, for $n \geqq m$, we see that $a_n \leqq kb_n$; and thus that, if Σb_n be convergent, so also is Σa_n.

Taking Σc_n, Σd_n to denote a convergent series and a divergent series respectively, sufficient conditions for the convergence or divergence of the series Σa_n may be expressed in the forms

$$\text{(A)} \quad \begin{cases} \overline{\lim}_{n\sim\infty} P_n \left(\dfrac{a_n}{d_n} - \dfrac{a_{n+1}}{d_{n+1}}\right) < 0, \text{ for divergence;} \\ \underline{\lim}_{n\sim\infty} P_n \left(\dfrac{a_n}{c_n} - \dfrac{a_{n+1}}{c_{n+1}}\right) > 0, \text{ for convergence;} \end{cases}$$

where $\{P_n\}$ denotes any arbitrary sequence of positive numbers.

To shew that these conditions are sufficient, assume that the first is satisfied, we have then $P_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right) < -\eta$, where η is some positive number, provided $n \geqq$ some number m. It then follows that $\frac{a_{n+1}}{a_n} \geqq \frac{d_{n+1}}{d_n}$, for $n \geqq m$; and thus that Σa_n is divergent. The second condition can be similarly shewn to be sufficient.

These criteria are spoken of by Pringsheim† as the general type of criteria of the second kind, since they are reducible by writing $\frac{P_n}{a_n}$, or $\frac{P_n}{a_{n+1}}$, for P_n, to a form in which only the ratio of a_{n+1} to a_n is involved.

* *Loc. cit.* p. 12. See Pringsheim, *loc. cit.* p. 343. † *Loc. cit.* p. 359.

It should be observed that from the relation $a_n \geqq kb_n$ or from the relation $a_n \leqq kb_n$, it cannot be deduced that $\frac{a_{n+1}}{a_n} \geqq \frac{b_{n+1}}{b_n}$, or $\frac{a_{n+1}}{a_n} \leqq \frac{b_{n+1}}{b_n}$. Consequently it does not necessarily follow that, when one of the criteria of the first kind is satisfied, the corresponding criterion of the second kind is also satisfied. If Σa_n is a convergent series, the limit of $\Sigma \frac{a_{n+1}}{a_n}$ does not necessarily exist, but may oscillate in any manner. In fact, if we have a prescribed convergent series Σa_n, we may by an alteration of the order of the terms, which alteration does not affect the convergence of the series, ensure that the limit of $\frac{a_{n+1}}{a_n}$ oscillates in any prescribed manner.

If we take $P_n = \frac{1}{a_{n+1}}$, the above criteria become

$$(\mathrm{B}) \quad \begin{cases} \overline{\lim_{n\sim\infty}} \left(\frac{1}{d_n}\frac{a_n}{a_{n+1}} - \frac{1}{d_{n+1}}\right) < 0, \text{ for divergence;} \\ \underline{\lim_{n\sim\infty}} \left(\frac{1}{c_n}\frac{a_n}{a_{n+1}} - \frac{1}{c_{n+1}}\right) > 0, \text{ for convergence.} \end{cases}$$

The second of these criteria (A), (B) can be reduced to a different form by utilizing the theorem that $\Sigma e^{-\rho s_n} d_n$ is a convergent series when $\rho > 0$. Thus, let $c_n = e^{-\rho s_n} d_n$, the second criterion (A) then becomes

$$\underline{\lim_{n\sim\infty}} P_n \left(\frac{a_n}{d_n} e^{\rho s_n} - \frac{a_{n+1}}{d_{n+1}} e^{\rho s_{n+1}}\right) > 0.$$

This may be written in the form

$$\underline{\lim_{n\sim\infty}} \left[P_n e^{\rho s_n} \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right) - P_n e^{\rho s_n} \frac{a_{n+1}}{d_{n+1}} \left(e^{\rho(s_{n+1}-s_n)} - 1\right)\right] > 0.$$

If we choose P_n to be such that $P_n a_{n+1} e^{\rho s_n} = 1$, the criterion becomes

$$\underline{\lim_{n\sim\infty}} \left[\frac{a_n}{a_{n+1}}\frac{1}{d_n} - \frac{1}{d_{n+1}} - \frac{e^{\rho d_{n+1}} - 1}{d_{n+1}}\right] > 0.$$

Assuming that $\overline{\lim_{n\sim\infty}}\, d_n < \infty$, we have $\frac{e^{\rho d_{n+1}} - 1}{d_{n+1}} \leqq \frac{\rho}{1 - \rho d_{n+1}}$, since $e^z \leqq \frac{1}{1-z}$; it follows that, by choosing ρ sufficiently small, $\overline{\lim_{n\sim\infty}} \frac{e^{\rho d_{n+1}} - 1}{d_{n+1}}$ may be made as small as we please. Consequently the criterion becomes

$$\underline{\lim_{n\sim\infty}} \left(\frac{a_n}{a_{n+1}}\frac{1}{d_n} - \frac{1}{d_{n+1}}\right) > 0,$$

provided $\overline{\lim_{n\sim\infty}}\, d_n < \infty$. This restriction on d_n may be removed. For, assuming that the condition $\underline{\lim_{n\sim\infty}} \left(\frac{a_n}{a_{n+1}}\frac{1}{d_n} - \frac{1}{d_{n+1}}\right) > 0$ is satisfied for a set of values of d_n such that $\overline{\lim_{n\sim\infty}}\, d_n = \infty$, a positive number λ can be chosen

so small that $$\underline{\lim}_{n\sim\infty}\left(\frac{a_n}{a_{n+1}}\frac{1}{d_n}-\frac{1}{d_{n+1}}-\lambda\right)>0,$$
and therefore such that
$$\underline{\lim}_{n\sim\infty}\left\{\frac{a_n}{a_{n+1}}\left(\frac{1}{d_n}+\lambda\right)-\left(\frac{1}{d_{n+1}}+\lambda\right)\right\}>0.$$

Let $\dfrac{1}{d'_n}=\dfrac{1}{d_n}+\lambda$, thus $\overline{\lim}\, d'_n \leqq \dfrac{1}{\lambda}$; then by hypothesis the condition $\underline{\lim}\left(\dfrac{a_n}{a_{n+1}}\dfrac{1}{d'_n}-\dfrac{1}{d'_{n+1}}\right)>0$ is satisfied. In case the series $\Sigma d'_n$ converges, this condition falls under the second condition (B), and is sufficient for the convergence of Σa_n. In case $\Sigma d'_n$ diverges, since $\overline{\lim}_{n\sim\infty} d'_n<\infty$, it falls under the preceding case.

It has now been shewn that the condition
$$\underline{\lim}_{n\sim\infty}\left(\frac{a_n}{a_{n+1}}\frac{1}{d_n}-\frac{1}{d_{n+1}}\right)>0$$
is sufficient for the convergence of Σa_n, where Σd_n is any divergent series. Combining this condition with the second criterion (B), namely that $\underline{\lim}_{n\sim\infty}\left(\dfrac{a_n}{a_{n+1}}\dfrac{1}{c_n}-\dfrac{1}{c_{n+1}}\right)>0$, we see that *a sufficient condition of convergence of the series Σa_n is that $\underline{\lim}_{n\sim\infty}\left(\dfrac{a_n}{a_{n+1}}\phi(n)-\phi(n+1)\right)>0$, where $\{\phi(n)\}$ is any assigned sequence of positive numbers.* For the series $\Sigma\, 1/\phi(n)$ is either convergent or divergent, and in either case the criterion is sufficient.

This criterion, which may easily be proved directly, was first obtained by Kummer*, who however added the unnecessary condition that $\phi(n)$ must be such that $\lim_{n\sim\infty} a_n\phi(n)=0$. That this latter restriction is unnecessary was shewn by Dini and by Du Bois-Reymond.

Companion criteria of convergence and divergence may now be stated as follows:
$$\underline{\lim}_{n\sim\infty}\left(\frac{a_n}{a_{n+1}}\frac{1}{d_n}-\frac{1}{d_{n+1}}\right)>0,\ \text{for convergence},$$
$$\overline{\lim}_{n\sim\infty}\left(\frac{a_n}{a_{n+1}}\frac{1}{d_n}-\frac{1}{d_{n+1}}\right)<0,\ \text{for divergence}.$$

The particular case in which $d_n=1$ gives the criteria of d'Alembert and Cauchy which were obtained by comparing the series Σa_n with a geometric series. Thus we have
$$\begin{cases}\underline{\lim}_{n\sim\infty}\left(\dfrac{a_n}{a_{n+1}}-1\right)>0, \text{ equivalent to } \overline{\lim}_{n\sim\infty}\dfrac{a_{n+1}}{a_n}<1, \text{ for convergence},\\[2ex] \overline{\lim}_{n\sim\infty}\left(\dfrac{a_n}{a_{n+1}}-1\right)<0, \text{ equivalent to } \underline{\lim}_{n\sim\infty}\dfrac{a_{n+1}}{a_n}>1, \text{ for divergence}.\end{cases}$$

* *Crelle's Journal*, vol. XIII (1835), p. 171. A direct proof of this criterion has been given by Stolz, *Vorlesungen über allg. Arith.* vol. I, p. 259. The criterion was re-discovered by Jensen.

If $d_n = 1/n$, we have Raabe's criteria*

$$\begin{cases} \varliminf_{n\sim\infty} n\left(\dfrac{a_n}{a_{n+1}} - 1\right) > 1, \text{ for convergence,} \\ \varlimsup_{n\sim\infty} n\left(\dfrac{a_n}{a_{n+1}} - 1\right) < 1, \text{ for divergence.} \end{cases}$$

In general, let $d_n = 1/L_m(n)$; we obtain then Bertrand's† logarithmic scale of criteria

$$\begin{cases} \varliminf_{n\sim\infty} \left\{L_m(n)\dfrac{a_n}{a_{n+1}} - L_m(n+1)\right\} > 0, \text{ for convergence,} \\ \varlimsup_{n\sim\infty} \left\{L_m(n)\dfrac{a_n}{a_{n+1}} - L_m(n+1)\right\} < 0, \text{ for divergence.} \end{cases} \quad m = 1, 2, 3, \ldots.$$

It is easily seen that all these criteria fail in case $\lim\limits_{n\sim\infty} \dfrac{a_n}{a_{n+1}}$ oscillates between limits one of which is greater than unity, and the other less than unity.

21. If $\dfrac{a_n}{a_{n+1}}$ has the form $1 + \dfrac{A}{n} + O\left(\dfrac{1}{n^{1+\lambda}}\right)$, where $\lambda > 0$, we have

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = A + O\left(\frac{1}{n^\lambda}\right),$$

and thus $\lim\limits_{n\sim\infty} n\left(\dfrac{a_n}{a_{n+1}} - 1\right) = A$. Therefore, in accordance with Raabe's test, the series Σa_n is convergent if $A > 1$, and divergent if $A < 1$. In the case $A = 1$, we can apply Bertrand's test for $m = 1$, $L_1(n) = n \log n$.

We have

$$n \log n \,.\, \frac{a_n}{a_{n+1}} - (n+1)\log(n+1) = (n+1)\log\frac{n}{n+1} + O\left(\frac{1}{n^\lambda}\right)\log n.$$

Now $\lim\limits_{n\sim\infty} O\left(\dfrac{1}{n^\lambda}\right)\log n = 0$, since $\lim\limits_{n\sim\infty} \dfrac{\log n}{n^\lambda} = 0$. Moreover

$$\lim_{n\sim\infty} (n+1)\log\frac{n}{n+1}$$

has the value -1. It follows that in this case the series is divergent. The following rule has thus been established:

If $\dfrac{a_n}{a_{n+1}}$ *has the form* $1 + \dfrac{A}{n} + O\left(\dfrac{1}{n^{1+\lambda}}\right)$, *where* $\lambda > 0$, *the series* Σa_n *is convergent if* $A > 1$, *and it is divergent if* $A \leqq 1$.

For example, consider the series

$$1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)} + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\ldots\beta(\beta+1)(\beta+n-1)}{1\,.\,2\ldots n\,.\,\gamma(\gamma+1)\ldots(\gamma+n-1)}.$$

* Baumgartner and Ettinghausen's *Zeitschr. f. Math. u. Physik*, vol. x. See also Duhamel, *Liouville's Journal*, vol. IV (1839), p. 214 and vol. VI, p. 85.

† *Liouville's Journal*, vol. VII (1842), p. 42. See also Bonnet, *ibid.* vol. VIII (1843), p. 89, and Paucker, *Crelle's Journal* (1851), vol. XLII, p. 143.

We have

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = 1 + \frac{n(\gamma+1-\alpha-\beta)-\alpha\beta+\gamma}{(\alpha+n)(\beta+n)}$$
$$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + O\left(\frac{1}{n^2}\right);$$

hence the series is convergent if $\gamma - \alpha - \beta > 0$, and it diverges if

$$\gamma - \alpha - \beta \leqq 0.$$

If
$$\frac{a_n}{a_{n+1}} = \frac{n^m + \alpha n^{m-1} + \beta n^{m-2} + \dots}{n^m + A n^{m-1} + B n^{m-2} + \dots},$$

we have $\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha - A}{n} + O\left(\frac{1}{n^2}\right)$; and thus the series is convergent if $\alpha - A > 1$, and is divergent if $\alpha - A \leqq 1$. This criterion* was given by Gauss.

22. Employing the theorem that if Σd_n is divergent, $\Sigma d_n e^{-\rho s_n}$ is convergent, for $\rho > 0$, we may in the criterion $(B)_2$, of § 20, write $d_n e^{-\rho s_n}$ for c_n. It is thus a sufficient condition for the convergence of Σa_n that

$$\varliminf_{n\sim\infty} \left(\frac{a_n}{a_{n+1}} \frac{e^{\rho s_n}}{d_n} - \frac{e^{\rho s_{n+1}}}{d_{n+1}}\right) > 0,$$

which is equivalent to the condition that $\frac{a_n}{a_{n+1}} \frac{e^{\rho s_n}}{d_n} - \frac{e^{\rho s_{n+1}}}{d_{n+1}} >$ a positive number p, for all values of $n \geqq$ a fixed number n_1. From this it now follows that

$$\frac{a_n}{a_{n+1}} \frac{d_{n+1}}{d_n} > e^{\rho d_{n+1}} + p d_{n+1} e^{-\rho s_n}, \text{ for } n \geqq n_1,$$

or
$$\frac{1}{d_{n+1}} \log\left(\frac{a_n}{a_{n+1}} \frac{d_{n+1}}{d_n}\right) > \rho + \frac{1}{d_{n+1}} \log_e (1 + p d_{n+1} e^{-\rho s_{n+1}}) > \rho,$$

for $n \geqq n'$, where ρ is some fixed positive number. This condition is certainly satisfied if $\varliminf_{n\sim\infty} \frac{1}{d_{n+1}} \log \frac{a_n\, d_{n+1}}{a_{n+1}\, d_n} > 0$, provided ρ be properly chosen.

Hence *a sufficient condition of convergence of* Σa_n *is that*

$$\varliminf_{n\sim\infty} \frac{1}{d_{n+1}} \log \frac{a_n\, d_{n+1}}{a_{n+1}\, d_n} > 0.$$

In a similar manner it can be shewn that *a sufficient condition of divergence of the series* Σa_n *is that*

$$\varlimsup_{n\sim\infty} \frac{1}{d_{n+1}} \log \frac{a_n\, d_{n+1}}{a_{n+1}\, d_n} < 0.$$

If it be assumed that $\varlimsup_{n\sim\infty} d_n$ is finite, in the whole of the foregoing

* *Opera*, vol. III, p. 139.

investigation $e^{-\rho s_{n-1}}$ may be substituted for $e^{-\rho s_n}$. We thus obtain the criteria

$$\begin{cases} \underline{\lim}_{n\sim\infty} \dfrac{1}{d_n} \log \dfrac{a_n d_{n+1}}{a_{n+1} d_n} > 0, \textit{ sufficient for convergence,} \\ \overline{\lim}_{n\sim\infty} \dfrac{1}{d_n} \log \dfrac{a_n d_{n+1}}{a_{n+1} d_n} < 0, \textit{ sufficient for divergence,} \end{cases}$$

provided $\overline{\lim}_{n\sim\infty} d_n$ *is finite.*

Since the divergence of the series $\Sigma \dfrac{d_n}{L^{(m)}(s_{n-1})}$ is a necessary consequence of the divergence of Σd_n, we can replace d_n by $\dfrac{d_n}{L^{(m)}(s_{n-1})}$ in the above criteria. They then give rise to the scale of criteria

$$\begin{cases} \underline{\lim}_{n\sim\infty} \dfrac{1}{d_n} L^{(m)}(s_{n-1}) \log \dfrac{a_n}{a_{n+1}} \dfrac{d_{n+1}}{d_n} \cdot \dfrac{L^{(m)}(s_{n-1})}{L^{(m)}(s_n)} > 0, \textit{ sufficient for convergence,} \\ \overline{\lim}_{n\sim\infty} \dfrac{1}{d_n} L^{(m)}(s_{n-1}) \log \dfrac{a_n}{a_{n+1}} \dfrac{d_{n+1}}{d_n} \cdot \dfrac{L^{(m)}(s_{n-1})}{L^{(m)}(s_n)} < 0, \textit{ sufficient for divergence,} \end{cases}$$

where $m = 0, 1, 2, 3, \ldots$.

In case we let $d_n = 1$, we have the following scale of criteria:

$$\underline{\lim}_{n\sim\infty} \log \frac{a_n}{a_{n+1}} > 0,$$

$$\underline{\lim}_{n\sim\infty} L^{(m)}(n-1) \log \frac{a_n L^{(m)}(n-1)}{a_{n+1} L^{(m)}(n)} > 0, \quad m = 0, 1, 2, \ldots,$$

sufficient for convergence;

$$\overline{\lim}_{n\sim\infty} \log \frac{a_n}{a_{n+1}} < 0,$$

$$\overline{\lim}_{n\sim\infty} L^{(m)}(n-1) \log \frac{a_n L^{(m)}(n-1)}{a_{n+1} L^{(m)}(n)} < 0, \quad m = 0, 1, 2, \ldots,$$

sufficient for divergence.

The criteria corresponding to $m = 0$ are

$$\underline{\lim}_{n\sim\infty} (n-1) \log \frac{a_n}{a_{n+1}} \frac{(n-1)}{n} > 0,$$

$$\overline{\lim}_{n\sim\infty} (n-1) \log \frac{a_n}{a_{n+1}} \frac{(n-1)}{n} < 0.$$

Since $$\lim_{n\sim\infty} n \log \frac{n}{n+1} = -\lim_{n\sim\infty} \log \left(1 + \frac{1}{n}\right)^{-n} = -1,$$

these criteria become

$$\underline{\lim}_{n\sim\infty} n \log \frac{a_n}{a_{n+1}} > 1, \text{ for convergence,}$$

$$\overline{\lim}_{n\sim\infty} n \log \frac{a_n}{a_{n+1}} < 1, \text{ for divergence,}$$

a criterion which was given by Schlömilch.

23. Let $M(x)$, $m(x)$ denote monotone functions, positive in the infinite interval $(1, \infty)$, of x, which both diverge to ∞, as $x \sim \infty$, and are such that $M(x) > m(x)$. It will be assumed that $f(x)$ is monotone non-increasing, and positive.

Let

$$\phi(x) = \int_{M(x)}^{M(x+h)} f(x)\,dx \Big/ \int_{m(x)}^{m(x+h)} f(x)\,dx \equiv \frac{F\{M(x+h)\} - F\{M(x)\}}{F\{m(x+h)\} - F\{m(x)\}},$$

where h is an arbitrarily chosen positive number. It will be shewn that:

The series $\sum_1 f(n)$ *converges if* $\overline{\lim_{x\sim\infty}}\, \phi(x) < 1$, *and diverges if* $\underline{\lim_{x\sim\infty}}\, \phi(x) > 1$.

First assume that $\underline{\lim_{x\sim\infty}}\, \phi(x) > 1$; then $\phi(x) \geqq 1 + \epsilon$, for $x \geqq \xi$, where ϵ is some positive number, so that

$$F\{M(x+h)\} - F\{M(x)\} \geqq (1+\epsilon)[F\{m(x+h)\} - F\{m(x)\}],$$

from which it follows that

$$F\{M(\xi+nh)\} - F\{M(\xi)\} \geqq (1+\epsilon)[F\{m(\xi+nh)\} - F\{m(\xi)\}],$$

for all positive integral values of n.

We have now

$$\frac{F\{M(\xi+nh)\} - F\{M(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}} \geqq (1+\epsilon)\left\{1 + \frac{F\{M(\xi)\} - F\{m(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}}\right\}$$
$$\geqq 1+\epsilon,$$

provided n be chosen so large that $F\{m(\xi+nh)\} > F\{M(\xi)\}$. From this it follows that

$$\underline{\lim_{n\sim\infty}}\, \frac{F\{M(\xi+nh)\} - F\{M(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}} \geqq 1+\epsilon.$$

If $\int_1^\infty f(x)\,dx$, or $F(\infty)$, were finite, the limit of the expression on the left hand side would have the unique value 1, hence it follows from the inequality that $F(\infty) = \infty$, and thus that $\sum_1 f(n)$ diverges (see § 14).

Next assume that $\overline{\lim_{x\sim\infty}}\, \phi(x) < 1$; then $\phi(x) \leqq 1 - \epsilon$ for $x \geqq \xi$, and for some positive number ϵ.

We find as before that

$$F\{M(\xi+nh)\} - F\{M(\xi)\} \leqq (1-\epsilon)[F\{m(\xi+nh)\} - F\{m(\xi)\}],$$

and hence that

$$\frac{F\{M(\xi+nh)\} - F\{M(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}} \leqq (1-\epsilon)\left\{1 + \frac{F\{M(\xi)\} - F\{m(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}}\right\}.$$

If now $F(\infty) = \infty$, we find from this inequality that

$$\overline{\lim_{n\sim\infty}}\, \frac{F\{M(\xi+nh)\} - F\{M(\xi)\}}{F\{m(\xi+nh)\} - F\{M(\xi)\}} \leqq 1-\epsilon,$$

which is impossible, since $F\{M(\xi+nh)\} > F\{m(\xi+nh)\}$. It thus follows that $F(\infty)$ must be finite, and therefore that $\sum_{n=1} f(n)$ converges.

Since $\dfrac{\{M(x+h)-M(x)\}\,f\{M(x+h)\}}{\{m(x+h)-m(x)\}\,f\{m(x)\}} < \dfrac{F\{M(x+h)\}-F\{M(x)\}}{F\{m(x+h)\}-F\{m(x)\}}$

$$< \frac{\{M(x+h)-M(x)\}\,f\{M(x)\}}{\{m(x+h)-m(x)\}\,f\{m(x+h)\}},$$

we see that the criteria can be reduced to the form

$$\begin{cases} \underline{\lim}_{x\sim\infty} \dfrac{\{M(x+h)-M(x)\}\,f\{M(x+h)\}}{\{m(x+h)-m(x)\}\,f\{m(x)\}} > 1, \text{ for divergence,} \\ \overline{\lim}_{x\sim\infty} \dfrac{\{M(x+h)-M(x)\}\,f\{M(x)\}}{\{m(x+h)-m(x)\}\,f\{m(x+h)\}} < 1, \text{ for convergence,} \end{cases}$$

where $M(x) > m(x)$.

A special pair of criteria can be obtained by taking $m(x) = x$ in the first criterion, and $m(x) = x - h$ in the second; we then have

$$\underline{\lim}_{x\sim\infty} \frac{\{M(x+h)-M(x)\}\,f\{M(x+h)\}}{hf(x)} > 1, \text{ for divergence, where } M(x) > x,$$

$$\overline{\lim}_{x\sim\infty} \frac{\{M(x+h)-M(x)\}\,f\{M(x)\}}{hf(x)} < 1, \text{ for convergence, where } M(x) > x - h.$$

In particular, if $h = 1$, we have

$$\begin{cases} \underline{\lim}_{x\sim\infty} \dfrac{\{M(x+1)-M(x)\}\,f\{M(x+1)\}}{f(x)} > 1, \text{ for divergence, when } M(x) > x, \\ \overline{\lim}_{x\sim\infty} \dfrac{\{M(x+1)-M(x)\}\,f\{M(x)\}}{f(x)} < 1, \text{ for convergence, when } M(x) > x - 1. \end{cases}$$

These criteria were given by Kohn*.

If $M(x)$, $m(x)$ have definite differential coefficients $M'(x)$, $m'(x)$, $\dfrac{M(x+h)-M(x)}{hM'(x)}$ and $\dfrac{m(x+h)-m(x)}{hm'(x)}$ differ arbitrarily little from unity, if h be taken small enough. In this manner we can obtain the following criteria:

$$\begin{cases} \underline{\lim}_{x\sim\infty} \dfrac{M'(x)\,f\{M(x)\}}{f(x)} > 1, \text{ when } M(x) > x, \text{ for divergence,} \\ \overline{\lim}_{x\sim\infty} \dfrac{M'(x)\,f\{M(x)\}}{f(x)} < 1, \text{ for convergence.} \end{cases}$$

The companion conditions obtained by taking $M(x) = x$, $m(x) < x$ are

$$\begin{cases} \underline{\lim}_{x\sim\infty} \dfrac{f(x)}{m'(x)\,f\{m(x)\}} > 1, \text{ when } m(x) < x, \text{ for divergence,} \\ \overline{\lim}_{x\sim\infty} \dfrac{f(x)}{m'(x)\,f\{m(x)\}} < 1, \text{ for convergence.} \end{cases}$$

These criteria are due to Ermakoff†.

* *Archiv der Math.* vol. LXVII (1882), p. 63.

† *Bulletin d. Sc. Mat.* (1), vol. II (1871), p. 250; (2), vol. VII (1883), p. 142.

Taking $M(x) = e^x$, $m(x) = \log x$, we have the special criteria

$$\overline{\lim_{x\sim\infty}} \frac{e^x f(e^x)}{f(x)} > 1, \text{ for divergence; } \underline{\lim_{x\sim\infty}} \frac{e^x f(e^x)}{f(x)} < 1, \text{ for convergence;}$$

$$\overline{\lim_{x\sim\infty}} \frac{x f(x)}{f(\log x)} > 1, \text{ for divergence; } \underline{\lim_{x\sim\infty}} \frac{x f(x)}{f(\log x)} < 1, \text{ for convergence.}$$

THE CONVERGENCE OF SERIES IN GENERAL.

24. We proceed to consider the case in which the terms of a series Σa_n are not all of the same sign. The simplest case is that in which the positive and negative signs occur alternatively. In this case the following criterion is frequently applicable to decide the question of the convergence of the series:

If the terms of a series $u_1 - u_2 + u_3 - u_4 + \ldots$ *be of alternate signs, and if* $u_n \geqq u_{n+1}$, *for every value of* n, *it is necessary and sufficient for the convergence of the series that* $\lim_{n\sim\infty} u_n = 0$. *When this last condition is not satisfied the series oscillates between limits, both of which are in the interval* $(0, u_1)$, *where we may assume* u_1 *to be positive.*

Since $|s_n - s_{n-1}| = |u_n|$, it is necessary for the convergence of s_n that $\lim_{n\sim\infty} u_n = 0$. Again we have

$$|s_{n+m} - s_n| = |u_{n+1} - u_{n+2} + \ldots + (-1)^{m+1} u_{n+m}| \leqq |u_{n+1}|.$$

If $\lim_{n\sim\infty} u_n = 0$, for all sufficiently large values of n we have $|s_{n+m} - s_n| < \epsilon$, an arbitrarily chosen positive number, for $m = 1, 2, 3, \ldots$. Thus the condition of convergence is satisfied.

If $\lim_{n\sim\infty} |u_n|$ is not zero, it is seen that $0 < s_{2n} < u_1$, and that s_{2n} does not diminish as n increases; therefore s_{2n} has a definite limit, in the interval $(0, u_1)$. Similarly, we see that s_{2n+1} never increases as n increases, and that it lies in the interval $(0, u_1)$; thus s_{2n+1} has a definite limit in the interval $(0, u_1)$. The limits of s_{2n}, s_{2n+1} both lie in the interval $(0, u_1)$ and differ from one another.

For series in which the signs may be distributed in any manner the following theorem is of importance. It was first established by Catalan* and Dedekind†, and depends essentially upon a lemma due to Abel‡.

This Lemma consists of the identity

$$k_1 u_1 + k_2 u_2 + \ldots + k_n u_n = \sum_{r=1}^{r=n-1} (k_r - k_{r+1}) s_r + k_n s_n,$$

where s_r denotes the rth partial sum of the series $\sum_{n=1} u_n$. It has a rôle in

* *Traité élémentaire des Series* (1860), p. 32.

† See his edition of Dirichlet's *Vorl. ü. Zahlentheorie*, 3rd ed. p. 255.

‡ For a history of the theorem see Pringsheim, *Math. Annalen*, vol. xxv (1885), p. 423.

the theory of infinite series similar to that of integration by parts in the Integral Calculus.

If the series $u_1 + u_2 + \ldots + u_n + \ldots$ *be either convergent or oscillating between finite limits, and* $\{k_n\}$ *be a sequence of numbers such that* $\lim_{n\sim\infty} k_n = 0$, *and that the series* $\sum_{n=1} | k_n - k_{n+1} |$ *is convergent, then the series*

$$k_1 u_1 + k_2 u_2 + \ldots + k_n u_n + \ldots$$

is convergent. In particular it is sufficient that $\{k_n\}$ *form a monotone non-increasing sequence and that* $\lim_{n\sim\infty} k_n = 0$.

A partial remainder of the series is expressed by

$$\begin{aligned} k_{n+1}u_{n+1} + k_{n+2}u_{n+2} + \ldots + k_{n+m}u_{n+m} \\ &= k_{n+1}(s_{n+1} - s_n) + k_{n+2}(s_{n+2} - s_{n+1}) + \ldots + k_{n+m}(s_{n+m} - s_{n+m-1}) \\ &= -k_{n+1}s_n + (k_{n+1} - k_{n+2})s_{n+1} + (k_{n+2} - k_{n+3})s_{n+2} + \ldots \\ &\qquad + (k_{n+m-1} - k_{n+m})s_{n+m-1} + k_{n+m}s_{n+m}. \end{aligned}$$

If the series $u_1 + u_2 + \ldots$ is either convergent, or oscillating between finite limits, we have $| s_n | < A$, for all values of n, where A is some fixed positive number. The integer n can be chosen so large that $| k_{n+m} | < \epsilon$, where $m = 1, 2, 3, \ldots$.

If the series $\sum_{n=1} | k_n - k_{n+1} |$ is convergent, n may be chosen so large that $| k_{n+1} - k_{n+2} | + \ldots + | k_{n+m-1} - k_{n+m} | < \epsilon$, for $m = 2, 3, \ldots$. It follows that, if n be sufficiently large,

$$| k_{n+1} u_{n+1} + \ldots + k_{n+m} u_{n+m} | < 3A\epsilon,$$

for the values 1, 2, 3, ... of m. Since ϵ is arbitrary, it follows that the series $k_1 u_1 + k_2 u_2 + \ldots$ is convergent.

The following theorem may also be established:

If the series $u_1 + u_2 + \ldots + u_n + \ldots$ *be convergent, and* $\{k_n\}$ *is a sequence of numbers, such that* $| k_n |$ *is less than a fixed positive number* K, *for all values of* n, *and is also such that the series* $\sum_{n=1} | k_n - k_{n+1} |$ *is convergent, then the series* $k_1 u_1 + k_2 u_2 + \ldots$ *is convergent. In particular it is sufficient that* $\{k_n\}$ *should be a non-diminishing sequence of numbers with a finite upper limit.*

We find, as before, by writing $s - s_{n+m} = R_{n+m}$,

$$\begin{aligned} k_{n+1}u_{n+1} + \ldots + k_{n+m}u_{n+m} \\ &= k_{n+1}R_n - (k_{n+1} - k_{n+2})R_{n+1} - (k_{n+2} - k_{n+3})R_{n+2} - \ldots \\ &\qquad - (k_{n+m-1} - k_{n+m})R_{n+m-1} - k_{n+m}R_{n+m}. \end{aligned}$$

If n be taken so large that $| k_{n+1} - k_{n+2} | + \ldots + | k_{n+m-1} - k_{n+m} | < \epsilon$, for $m = 2, 3, \ldots$, and also so large that $| R_{n+m} | < \epsilon$, for $m = 0, 1, 2, \ldots$, we have

$$| k_{n+1}u_{n+1} + \ldots + k_{n+m}u_{n+m} | < 3K\epsilon,$$

for $m = 1, 2, 3, \ldots$. Since $3K\epsilon$ is arbitrarily small, the condition for convergence of the series is satisfied.

EXAMPLE.

Since $\sum_{r=1}^{n} \sin r\theta = \sin\frac{n+1}{2}\theta \sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}$, we have, for any value of θ which is not zero or a multiple of 2π, $\left|\sum_{r=1}^{n}\sin n\theta\right| < \left|\operatorname{cosec}\frac{\theta}{2}\right|$, and thus the series oscillates between finite limits. Similarly, if θ is neither zero nor a multiple of 2π, the series $\sum_{r=1}^{n}\cos r\theta$ oscillates between finite limits. The first series is convergent, and the second is divergent, when $\theta = 0$.

It follows from the above theorem that, if $\{k_n\}$ be a sequence of numbers which converges to zero, and is such that $\sum_{n=1} |k_n - k_{n+1}|$ is convergent, the series $\sum_{n=1}^{\infty} k_n \sin n\theta$ is convergent, for any fixed value of θ; and the series $\sum_{n=1}^{\infty} k_n \cos n\theta$ is convergent for any fixed value of θ that is neither zero nor a multiple of 2π. In particular $\{k_n\}$ may be any sequence of non-increasing numbers which converges to zero.

25. It can be shewn that, if the series $|u_1| + |u_2| + \ldots + |u_n| + \ldots$ is convergent, the series $u_1 + u_2 + \ldots + u_n + \ldots$ is also convergent. Let the first n terms of this second series contain n_1 terms with the positive sign, and n_2 terms with the negative sign, and let $\sigma_{n_1}, -\sigma'_{n_2}$ denote their sums; thus $s_n = \sigma_{n_1} - \sigma'_{n_2}$. Now $\sigma_{n_1} + \sigma'_{n_2}$ is the nth partial sum of the series $|u_1| + |u_2| + \ldots$; and, since this series is convergent, $\sigma_{n_1} + \sigma'_{n_2}$ is less than a fixed positive number, whatever value n has. It follows that $\sigma_{n_1}, \sigma'_{n_2}$ are each less than some fixed positive number, however large n_1, n_2 may be.

Since $\{\sigma_{n_1}\}$, $\{\sigma'_{n_2}\}$ are both monotone non-diminishing sequences it follows that $\lim_{n\sim\infty} \sigma_{n_1}$, $\lim_{n\sim\infty} \sigma'_{n_2}$ are both definite numbers; hence $\lim_{n\sim\infty} s_n$ is a definite number, and therefore the series $u_1 + u_2 + \ldots$ is convergent. The limits of $\sigma_{n_1}, \sigma'_{n_2}$ are independent of the orders of the terms in the two series, and of the particular sequences of n_1 and n_2.

If the series $\sum_{n=1} |u_n|$ is convergent, then the series $\sum_{n=1}^{\infty} u_n$ is said to be *absolutely convergent*.

If two series $\Sigma a_n, \Sigma b_n$ are such that $\frac{a_n}{b_n}$ is bounded, then if Σb_n is absolutely convergent, so also is Σa_n.

For, if $\left|\frac{a_n}{b_n}\right| \leq K$, for all values of n, we have $\sum_{m=1}^{m=n} |a_m| \leq K \sum_{m=1}^{m=n} |b_m|$ and thus, if n is indefinitely increased, $\sum_{m=1} |a_m|$ converges to a value which does not exceed K times the sum of the absolutely convergent series $\sum_{m=1}^{\infty} |b_m|$.

It may happen that $\sigma_{n_1} - \sigma'_{n_2}$ has a definite limit, as n is indefinitely increased, but that $\sigma_{n_1}, \sigma'_{n_2}$ increase indefinitely. In that case the series $\sum_{n=1}^{\infty} u_n$ is convergent, but $\sum_{n=1} |u_n|$ is divergent. The series $\sum_{n=1}^{\infty} u_n$ is then said to *converge non-absolutely*.

If a convergent series $\sum_{n=1}^{\infty} u_n$, of which the sum is s, be such that every series which consists of the same terms in a different order converges to the value s, then the series $\sum_{n=\infty}^{\infty} u_n$ is said to be *unconditionally convergent*.

It has been shewn in § 8 that, if all the terms of the series are positive, and the series is convergent, then it is necessarily unconditionally convergent.

For series in general the following theorem will be established:

A series which is absolutely convergent is also unconditionally convergent; and conversely, a series which is unconditionally convergent is also absolutely convergent.

The truth of the theorem is clear from § 8, in case all of the terms have one and the same sign, with the exception of a finite number of them. It will accordingly be assumed that this is not the case. Assuming that the series is absolutely convergent, it has been shewn that $\sigma_{n_1}, \sigma'_{n_2}$ both converge to definite limits, as n is indefinitely increased, and thus n_1 and n_2 are both indefinitely increased. If the given series be rearranged in accordance with any norm, the two series which contain the positive and the negative terms respectively are also rearranged, but as has been shewn in § 8, their limiting sums are not thereby altered, and they converge respectively to $\lim_{n\sim\infty} \sigma_{n_1}$, $-\lim_{n\sim\infty} \sigma'_{n_2}$. It then follows that the rearranged series converges to $\lim_{n\sim\infty} \sigma_{n_1} - \lim_{n\sim\infty} \sigma'_{n_2}$, the same sum as when the terms were in the original order.

Next let it be assumed that the series $\sum_{n=1} u_n$ is unconditionally convergent. It is impossible that one of the two limits $\lim_{n\sim\infty} \sigma_{n_1}$, $\lim_{n\sim\infty} \sigma'_{n_2}$ should be infinite and the other finite, for in that case $\sum_{n=1} u_n$ would be divergent. Thus, unless they are both divergent, the series $\sum_{n=1} |u_n|$ is convergent. Let it be assumed that $\lim_{n\sim\infty} \sigma_{n_1}$, $\lim_{n\sim\infty} \sigma'_{n_2}$ are both ∞; it will then be shewn that $\sum_{n=1} u_n$ cannot be unconditionally convergent.

Corresponding to each number n there are numbers n_1, n_2, both of which increase indefinitely with n. For each value of n_1, let n_1' be the smallest integer such that $\sigma_{n_1'} \geqq 2\sigma_{n_1}$. We then consider the two sequences of numbers $\{n_1'\}$, $\{n_2\}$; a corresponding rearrangement of the terms of the given series can be so made that the first n_1' positive terms of the series

are taken together with the first n_2 negative terms of the series, to make the first $n_1' + n_2$ terms of the new series. The partial sum of the new series is then $\sigma_{n_1'} - \sigma'_{n_2} \geqq 2\sigma_{n_1} - \sigma'_{n_2}$. Since $\sigma_{n_1} - \sigma'_{n_2}$ has a finite limit, and σ_{n_1} diverges, it follows that $\sigma_{n_1'} - \sigma_{n_2}$ diverges. Therefore the series has been so rearranged that it becomes divergent, and this is contrary to the hypothesis that the series is unconditionally convergent. Hence σ_{n_1}, σ'_{n_2} must both be convergent, and therefore the series $\sum\limits_{n=1} | u_n |$ is convergent, or $\sum\limits_{n=1} u_n$ is absolutely convergent.

26. The following theorem, due to Riemann*, will now be established:

The terms of a non-absolutely, or conditionally, convergent series can always be so rearranged in a series of type ω, in accordance with some norm, that either (1) *the new series converges to an arbitrarily assigned sum, or that* (2) *the new series is divergent, or that* (3) *the new series is oscillatory, with arbitrarily assigned limits of indeterminancy. Moreover each such rearrangement may be made in an indefinitely great number of ways.*

Let $k_1, k_2, k_3, \ldots$ be a monotone increasing sequence of positive numbers, defined in accordance with some prescribed law. Take p_1 positive terms of the given series Σu_n, so that $\sigma_{p_1} > k_1$, whilst $\sigma_{p_1-1} \leqq k_1$; next take q_1 negative terms such that $\sigma_{p_1} - \sigma'_{q_1} \leqq k_1$, whilst $\sigma_{p_1} - \sigma'_{q_1-1} > k_1$. Next take $p_2 - p_1$ more positive terms of the given series so that

$$\sigma_{p_2} - \sigma'_{q_1} > k_2,$$

whilst $\sigma_{p_2-1} - \sigma'_{q_1} \leqq k_2$; then take $q_2 - q_1$ more negative terms, so that $\sigma_{p_2} - \sigma'_{q_2} \leqq k_2$, whilst $\sigma_{p_2} - \sigma'_{q_2-1} > k_2$. Proceeding in this manner, we obtain two sequences of integers $\{p_n\}$, $\{q_n\}$ such that $\sigma_{p_n} - \sigma'_{q_n} \leqq k_n$, whilst $\sigma_{p_n} - \sigma'_{q_n-1} > k_n$, and $\sigma_{p_{n-1}} - \sigma_{q_{n-1}} \leqq k_{n-1}$. Denoting the positive terms of the series Σu_n by $a_1, a_2, \ldots a_n, \ldots$, and the negative terms by $b_1, b_2, \ldots b_n, \ldots$, we obtain a rearrangement of the series, which is such that

$$\begin{aligned}(a_1 + a_2 + \ldots + a_{p_1}) - (b_1 + b_2 + \ldots + b_{q_1})\\ + (a_{p_1+1} + \ldots + a_{p_2}) - (b_{q_1+1} + \ldots + b_{q_2}) + \ldots\\ + (a_{p_{n-1}+1} + \ldots + a_{p_n}) - (b_{q_{n-1}+1} + \ldots + b_{q_n})\end{aligned}$$

is $\leqq k_n$, whilst, if we leave out the term b_{q_n}, it is $> k_n$. This sum $\sigma_{p_n} - \sigma'_{q_n}$ accordingly differs from k_n by less than b_{q_n}. If the whole of the last bracket be omitted, the expression then differs from k_n by less than a_{p_n}, in accordance with the mode of determination of a_{p_n}. If n be sufficiently large a_{p_n} and b_{q_n} are both less than an arbitrarily assigned number ϵ. Thus, if some or all of the terms in the last bracket are omitted, the expression then differs from k_n by less than ϵ. If the two last brackets are omitted, the remaining expression differs from k_{n-1} by less than $b_{q_{n-1}}$, and n may be

* *Partielle Differentialgleichungen*, Braunschweig (1869), p. 41.

chosen so large that this and a_{p_n} are both less than ϵ. If the whole of the last bracket and a part of the terms in the last bracket but one are omitted, the remaining expression is between $k_{n-1} - \epsilon$ and $k_n + \epsilon$. It follows that if $p_{n-1} + q_{n-1} \leqq N \leqq p_n + q_n$, the sum s_N, of N terms of the rearranged series, lies between $k_{n-1} - \epsilon$ and $k_n + \epsilon$. Now let the increasing sequence $\{k_n\}$ converge to a positive number k; then $k_{n-1}, k_n, k_{n+1}, \ldots$ all differ from k by less than ϵ, provided n is taken sufficiently large. It follows that s_N differs from k by less than 2ϵ. Since ϵ is arbitrary, it follows that the rearranged series converges to k. Since k may be defined in an indefinitely great number of ways by an increasing sequence $\{k_n\}$, it follows that there are an infinitely great number of such rearrangements of the terms of the given series.

In case the sequence $\{k_n\}$ is divergent, it is clear that the rearranged series will also be divergent.

In case the number k is negative, the method of procedure is essentially the same; we then commence by taking negative terms of the given series, the numbers k_n being taken to be negative, and $\{k_n\}$ to be a diminishing sequence.

To establish (3). Let k, k' be two arbitrarily assigned positive numbers such that $k > k'$. Let k be defined as the limit of a sequence $\{k_n\}$ of increasing positive numbers, and k' as the limit of a similar sequence $\{k'_n\}$; we may suppose that $k_n > k'_n$ for all values of n.

As before, two sequences of increasing integers $\{p_1\}$, $\{q_1\}$ can be so determined that $\sigma_{p_1} > k_1$, $\sigma_{p_1-1} \leqq k_1$; $\sigma_{p_1} - \sigma'_{q_1} \leqq k'_1$, $\sigma_{p_1} - \sigma'_{q_1-1} > k'_1$; and generally so that $\sigma_{p_n} - \sigma'_{q_{n-1}} > k_n$, $\sigma_{p_n-1} - \sigma'_{q_{n-1}} \leqq k_n$, $\sigma_{p_n} - \sigma'_{q_n} \leqq k'_n$, $\sigma_{p_n} - \sigma'_{q_n-1} > k'_n$. In this manner a series

$$(a_1 + \ldots + a_{p_1}) - (b_1 + \ldots + b_{q_1}) + (a_{p_1+1} + \ldots + a_{p_2})$$
$$- (b_{q_1+1} + \ldots + b_{q_2}) + \ldots + (a_{p_{n-1}+1} + \ldots + a_{p_n}) - (b_{q_{n-1}+1} + \ldots + b_{q_n})$$

is formed which differs from k'_n by less than b_{q_n}; if the last bracket is omitted the remainder differs from k_n by less than a_{p_n}. It is now easily seen, as in the previous case, that the series oscillates between k and k'; this rearrangement can be made in an indefinitely great number of ways. In case k and k' are of opposite sign, only a slight modification of the proof is required.

A special case of this general theorem arises when the given series is $a_1 - a_1 + a_2 - a_2 + a_3 - a_3 + \ldots$, where each positive term a_n is followed by a negative term of the same absolute value. Provided $\lim_{n\sim\infty} a_n = 0$, the series is convergent, but if $a_1 + a_2 + a_3 + \ldots$ is divergent, the convergence of the series is non-absolute. It now appears that, from the terms of a divergent series $a_1 + a_2 + \ldots$, where $\lim_{n\sim\infty} a_n = 0$, series can be constructed,

in an indefinite variety of ways, which converge to a prescribed sum, or are divergent, or oscillate between given limits; all such series having the form

$$a_1 + a_2 + \dots + a_{p_1} - a_1 - a_2 - \dots - a_{q_1} + a_{p_1+1} + \dots + a_{p_2} - a_{q_1+1} - a_{q_1+2} - \dots - a_{q_2} + \dots,$$

the numbers $p_1, p_2, \dots; q_1, q_2, \dots$ being determined in the manner explained above.

EXAMPLES.

(1) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n} + \dots$ is non-absolutely convergent, its sum being $\log_e 2$. The series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$, which is obtained by a systematic rearrangement of the terms of the first series, converges to $\frac{3}{2}\log_e 2$.

For we find $\qquad s_{4n} = \sum_{m=1}^{m=n}\left(\frac{1}{4m-3} - \frac{1}{4m-2} + \frac{1}{4m-1} - \frac{1}{4m}\right)$,

and, for the second series, $s'_{3n} = \sum_{m=1}^{m=n}\left(\frac{1}{4m-3} + \frac{1}{4m-1} - \frac{1}{2m}\right)$;

therefore $\qquad s'_{3n} - s_{4n} = \sum_{m=1}^{m=n}\left(\frac{1}{4m-2} - \frac{1}{4m}\right) = \frac{1}{2}s_{2n}$.

Since s_{4n}, s_{2n} both converge to $\log_e 2$, as $n \sim \infty$, it follows that s'_{3n} converges to $\frac{3}{2}\log_e 2$. Since s'_{3n-1}, s'_{3n-2} only differ from s'_{3n} by $\frac{1}{2m}, \frac{1}{4m-1} - \frac{1}{2m}$, it is seen that they have the same limit as s'_{3n}; therefore the second series converges to $\frac{3}{2}\log_e 2$.

(2) By rearranging the terms of the non-absolutely convergent series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n} + \dots,$$

we obtain the series

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots - \frac{1}{n-1} + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} + \dots,$$

$$1 + \frac{1}{2} + \frac{1}{3} - 1 + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{2} + \dots - \frac{1}{n-1} + \frac{1}{3n-2} + \frac{1}{3n-1} + \frac{1}{3n} - \frac{1}{n} + \dots,$$

$$1 - 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{3} + \dots - \frac{1}{n-1} + \frac{1}{(n-1)^2+1} + \frac{1}{(n-1)^2+2} + \dots + \frac{1}{n^2} - \frac{1}{n} + \dots.$$

The first of these series converges to $\log_e 2$, the second to $\log_e 3$, and the third diverges.

CESÀRO'S SUMMATION BY ARITHMETIC MEANS.

27. Denoting by s_n the nth partial sum of the series

$$a_1 + a_2 + \dots + a_n + \dots,$$

and by S_n the arithmetic mean $\frac{s_1 + s_2 + \dots + s_n}{n}$ of the numbers

$$s_1, s_2, \dots s_n, \dots,$$

it has been shewn in § 6, Ex. 1 that, whenever $\lim_{n\sim\infty} s_n$ has a definite value s, so that the given series is convergent, then $\lim_{n\sim\infty} S_n = s$. The latter limit $\lim_{n\sim\infty} S_n$ may, however, exist when the series is not convergent, as is seen, for example in the case of the series $1 - 1 + 1 - 1 + \dots$.

Whenever S_n has a definite limit, as $n \sim \infty$, the given series is said to be *summable by Cesàro's method of arithmetic means*. In view of a development of Cesàro's method of summation, to be considered later, it is also said to be *summable* $(C, 1)$, that is by Cesàro's method, order 1. It may happen that S_n is not convergent but oscillatory; the values $\overline{S}$, $\underline{S}$ of $\overline{\lim_{n\sim\infty}} S_n$, $\underline{\lim_{n\sim\infty}} S_n$ are then said to be the upper and lower sums $(C, 1)$ of the series. In case $\overline{S}$, $\underline{S}$ are both finite numbers, the given series is said to be *bounded* $(C, 1)$.

From the point of view of the theory of sets of points, it may happen that the points $P_1, P_2, \dots P_n, \dots$ which represent the numbers

$$s_1, s_2, \dots s_n, \dots$$

do not converge to a single limiting point, but that the set of points $\overline{P}_1, \overline{P}_2, \dots \overline{P}_n, \dots$, where $\overline{P}_n$ is the centroid of the points $P_1, P_2, \dots P_n$, has a single limiting point $\overline{P}$, which then represents the Cesàro sum S.

We have, since $s_1 + s_2 + \dots + s_n = nS_n$, $s_n = nS_n - (n-1)S_{n-1}$; therefore $a_n = nS_n - 2(n-1)S_{n-1} + (n-2)S_{n-2}$; and hence

$$\frac{a_n}{n} = S_n - 2\left(1 - \frac{1}{n}\right)S_{n-1} + \left(1 - \frac{2}{n}\right)S_{n-2}, \quad \frac{s_n}{n} = S_n - \left(1 - \frac{1}{n}\right)S_{n-1}.$$

In case S_n has a definite limit as $n \sim \infty$, it follows from these last two equalities that $\frac{a_n}{n}$, $\frac{s_n}{n}$ both converge to zero. In case S_n is bounded, but not necessarily convergent, it is seen from the same equalities that $\frac{a_n}{n}$ and $\frac{s_n}{n}$ are both bounded. It has thus been shewn that:

If a series Σa_n be bounded $(C, 1)$, then $a_n = O(n)$, and $s_n = O(n)$. If the series is summable $(C, 1)$, then $a_n = o(n)$, $s_n = o(n)$.

The following theorem may be obtained at once from the general theorem given in § 6. Writing, in that theorem, $\beta_n = n$, $\alpha_n = nS_n$, we have:

If Σa_n is oscillatory, between finite or infinite limits

$$\overline{\lim_{n\sim\infty}} s_n \geqq \overline{\lim_{n\sim\infty}} S_n \geqq \underline{\lim_{n\sim\infty}} S_n \geqq \underline{\lim_{n\sim\infty}} s_n;$$

and in particular, if the series be summable $(C, 1)$, its Cesàro sum lies in the interval formed by the upper and lower sums of the series. If the series is

divergent, then S_n is also divergent, to $+\infty$, or $-\infty$, as the case may be. In case $\lim s_n$ *exists either as a finite number or as $+\infty$ or $-\infty$,* $\lim S_n$, *exists and has the value s.*

28. The following theorem will be established:

If a series $\sum\limits_{n=1} a_n$ is summable $(C, 1)$ the series $\sum\limits_{n=1} \frac{a_n}{n}$ is convergent; and if $\sum\limits_{n=1} a_n$ is bounded $(C, 1)$ the series $\sum\limits_{n=1} \frac{a_n}{n^{1+\delta}}$ is convergent, provided $\delta > 0$; moreover the series $\sum\limits_{n=1} \frac{a_n}{n}$ is then bounded.

The series $\sum\limits_{n=1}^{n=m} \frac{a_n}{n^{1+\delta}}$ may be written in the form

$$\sum_{n=1}^{n=m} \frac{nS_n - 2(n-1)S_{n-1} + (n-2)S_{n-2}}{n^{1+\delta}}, \text{ where } S_{-1} \equiv 0$$

or
$$\sum_{n=1}^{m-2} nS_n \left[\frac{1}{n^{1+\delta}} - \frac{2}{(n+1)^{1+\delta}} + \frac{1}{(n+2)^{1+\delta}}\right]$$
$$+ (m-1) S_{m-1} \left[-\frac{2}{m^{1+\delta}} + \frac{1}{(m-1)^{1+\delta}}\right] + mS_m \cdot \frac{1}{m^{1+\delta}}.$$

The two last terms are equivalent to

$$\frac{S_m}{m^\delta} - \frac{S_{m-1}}{(m-1)^\delta}\left\{2\left(\frac{m-1}{m}\right)^{1+\delta} - 1\right\}.$$

If S_m is bounded, this converges to zero, as $m \sim \infty$, provided $\delta > 0$. If $\delta = 0$, the expression is bounded when S_m is bounded. If S_m has a definite finite limit, and $\delta \geqq 0$, the expression converges to zero, as $m \sim \infty$.

The series $\sum\limits_{n=1}^{m-2} \frac{S_n}{n^\delta}\left[1 - 2\left(1 + \frac{1}{n}\right)^{-(1+\delta)} + \left(1 + \frac{2}{n}\right)^{-(1+\delta)}\right]$
is, since

$$\left(1 + \frac{1}{n}\right)^{-(1+\delta)} = 1 - (1 + \delta + \zeta)\frac{1}{n}, \quad \left(1 + \frac{2}{n}\right)^{-(1+\delta)} = 1 - (1 + \delta + \zeta')\frac{2}{n},$$

where ζ, ζ' are both bounded, and converge to zero, as $n \sim \infty$, numerically less than $\sum\limits_{n=1}^{m-2} |S_n| \frac{2|\zeta'| + 2|\zeta|}{n^{1+\delta}}$.

If $\delta > 0$, and $|S_n|$ is bounded, this is less than a fixed number independent of m.

If $\delta = 0$, the series becomes $\sum\limits_{n=1}^{m-2} \frac{2S_n}{(n+1)(n+2)}$, which is absolutely convergent if S_n is bounded.

It has thus been shewn that, when S_n converges to a finite limit, the series $\sum \frac{a_n}{n}$ is convergent; and that, when S_n is bounded, the series $\sum \frac{a_n}{n^{1+\delta}}$ $(\delta > 0)$ is convergent, and the series $\sum \frac{a_n}{n}$ is bounded.

It should be observed that, although the series $\sum\limits_{n=1} \frac{2S_n}{(n+1)(n+2)}$ is absolutely convergent when S_n is summable $(C, 1)$, it does not follow that the convergence of the series $\sum\limits_{n=1} \frac{a_n}{n}$ is absolute. For, in the transformation, $\frac{a_n}{n}$ is replaced by the sum of three terms, which are then rearranged, and the series may thus become an absolutely convergent series. For example, the series $1 - 1 + 1 - 1 + \ldots$ is convergent $(C, 1)$, but the series $1 - \frac{1}{2} + \frac{1}{3} - \ldots$ is only non-absolutely convergent.

The following is a generalization of the second part of the above theorem, and can be established in the same manner:

If the series $\sum\limits_{n=1} a_n$ is bounded $(C, 1)$, and $\{k_n\}$ be a sequence of numbers such that $k_n = o\left(\frac{1}{n}\right)$, and such that the series $\sum\limits_{n=1} n\,|\,k_n - 2k_{n+1} + k_{n+2}\,|$ is convergent, then the series $\sum\limits_{n=1} k_n a_n$ is convergent.

SERIES OF TRANSFINITE TYPE.

29. If $$s_1, s_2, s_3, \ldots s_n, \ldots s_\omega, s_{\omega+1}, \ldots s_\gamma, \ldots$$
be a set of numbers each one of which is definite, and in which every index that precedes some number β of the second class occurs as a suffix, and if the series
$$u_1 + u_2 + \ldots + u_n + \ldots + u_\omega + u_{\omega+1} + \ldots + u_\gamma + \ldots$$
be formed, where
$$u_1 = s_1,\ u_2 = s_2 - s_1,\ \ldots\ u_n = s_n - s_{n-1}\ \ldots\ s_\omega = \lim s_n,\ u_\omega = s_{\omega+1} - s_\omega,$$
$$u_{\omega+1} = s_{\omega+2} - s_{\omega+1},\ \ldots\ u_\gamma = s_{\gamma+1} - s_\gamma,\ \ldots,$$
in which the indices of u include every number less than β, then the series is said to be a convergent series* of type β. If γ be a limiting number, the limit of a sequence $\{\gamma_n\}$, then s_γ is defined to be $\lim s_{\gamma_n}$. If β be a limiting number, the series has no last term, but if β be a non-limiting number, the last term of the series is
$$u_{[\beta-1]} = s_\beta - s_{[\beta-1]}.$$

An ordinary infinite series
$$u_1 + u_2 + \ldots + u_n + \ldots$$
is of type ω. A series
$$u_1 + u_2 + \ldots + u_n + \ldots + v_1 + v_2 + v_3 + \ldots + v_n + \ldots$$

* Such series have been investigated in a different manner by Hardy, *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 285.

is of type $\omega 2$; and a double series

$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs},$$

$$\begin{array}{l} a_{11} + a_{12} + a_{13} + \dots + a_{1n} + \dots \\ + a_{21} + a_{22} + a_{23} + \dots + a_{2n} + \dots \\ + a_{31} + a_{32} + a_{33} + \dots + a_{3n} + \dots \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ + a_{n1} + a_{n2} + a_{n3} + \dots \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \end{array}$$

is of type ω^2, if it be taken in columns successively, or in rows successively; but it is of type ω if the terms are taken diagonally in the form

$$a_{11} + (a_{12} + a_{21}) + (a_{13} + a_{22} + a_{31}) + \dots .$$

Conversely, a series of any type β is convergent if all the sums

$$s_1, s_2, \dots s_n, \dots s_\omega, \dots s_\gamma, \dots s_\beta$$

be definite numbers.

It is clear that, β being any given number of the second class, any series of the ordinary type ω can have its terms so arranged that the series becomes of type β. For a correspondence can be defined between all the ordinal numbers less than β, and the ordinal numbers of the first class.

Let us now suppose that all the terms of a convergent series

$$u_1 + u_2 + \dots + u_\omega + u_{\omega+1} + \dots + u_\gamma + \dots ,$$

of type β, are positive, and thus that

$$s_1 < s_2 < s_3, \dots < s_\omega, \dots < s_\gamma, \dots < s_\beta .$$

If we represent the numbers

$$s_1, s_2, \dots s_\omega, \dots s_\beta$$

in the usual manner, by points on a straight line, the terms of the series are represented by a set of intervals

$$(0, s_1), (s_1, s_2), \dots (s_\omega, s_{\omega+1}), \dots$$

on the straight line; each interval abuts on the next; and all the points s_α, where α is a limiting number of the second class, are semi-external points of the set of intervals. The end-points and the semi-external points of the set of intervals form an enumerable closed set which has consequently zero content; and it follows, from the theory of the measures of sets of points, that the set of intervals has a measure equal to that of the whole interval $(0, s_\beta)$, which is therefore s_β. Since the measure of an infinite sequence of intervals is equal to the sum of the measures of the intervals, it follows that, if the intervals be arranged in a sequence of type ω, their sum is s_β. The following theorem has thus been established:

If a series of positive numbers be convergent, and of type β, it will also be convergent when arranged in type ω; also the sums will be the same. Conversely, if it be convergent when arranged in type ω, it will also converge to the same sum when arranged in type β.

We may pass to the consideration of series of type β, of which the terms are not necessarily all positive, but of which the convergence is absolute.

An absolutely convergent series of type β is a series which is convergent when each term is replaced by its modulus.

Let us suppose the intervals constructed as before, which represent the terms of the series

$$|u_1| + |u_2| + \dots + |u_\omega| + |u_{\omega+1}| + \dots + |u_\gamma| + \dots .$$

If we choose out from this set of intervals those which correspond to positive terms of the series

$$u_1 + u_2 + \dots + u_\omega + \dots + u_\gamma + \dots,$$

we have a set of intervals which has a definite finite measure; and the same is true of the set of those intervals which correspond to negative terms of the given series. The given series converges to a sum which is the difference of the measures of these two sets of intervals, and this sum is unaffected by the order in which the intervals are taken in either the positive or the negative component. It has thus been shewn that:

If a series be absolutely convergent, and of type β, then the series is convergent, and its sum is independent of the type.

DOUBLE SEQUENCES AND DOUBLE SERIES.

30. A set of numbers $\{s_{mn}\}$, where each of the indices m, n may be any positive integral number, and the number s_{mn} is defined, in accordance with some norm, for each pair of values of m and n, is said to form a *double sequence*.

If, for a given double sequence, a number s exists, such that, corresponding to each arbitrarily fixed positive number ϵ, the condition $|s - s_{mn}| < \epsilon$ be satisfied, for all values of m and n such that $m \geqq p$, $n \geqq p$, where p is some integer dependent on ϵ, then the double sequence is said to be *convergent*, and the number s is said to be the *limit of the double sequence*, or the *double limit of the sequence*. This is denoted by

$$s = \lim_{m\sim\infty,\, n\sim\infty} s_{mn}.$$

The theory of double sequences may be correlated with that given in I, §§ 302–306, of the double and repeated limits of a function of two variables. For, if we assume $x = 1/m$, $y = 1/n$, the number s_{mn} may be

taken to define the value of a function $f(x, y)$ at the point $x = 1/m$, $y = 1/n$. That the function is not defined for all positive values of x and y in the neighbourhood of the point $x = 0$, $y = 0$ makes no difference as regards the validity of the results obtained for a function of two variables. These results may now be interpreted as properties of the sequence $\{s_{mn}\}$.

The double limit $\lim\limits_{x\sim 0, y\sim 0} f(x, y)$, when it exists, is identical with $\lim\limits_{m\sim\infty, n\sim\infty} s_{mn}$, and the existence of either of these double limits implies that of the other*.

Corresponding to $\overline{\lim\limits_{y\sim 0}} f(x, y)$, $\underline{\lim\limits_{y\sim 0}} f(x, y)$, $\lim\limits_{y\sim 0} f(x, y)$, the notation $\overline{\lim\limits_{n\sim\infty}} s_{mn}$, $\underline{\lim\limits_{n\sim\infty}} s_{mn}$, $\lim\limits_{n\sim\infty} s_{mn}$ may be employed to denote respectively the upper limit, the lower limit, or the limit of s_{mn}, for a fixed value of m, as $n \sim \infty$. The limit exists when the upper and lower limits are identical. The notation $\overline{\underline{\lim\limits_{n\sim\infty}}} s_{mn}$ may be used to denote the upper and the lower limits, when either is to be taken indifferently. The corresponding notation

$$\overline{\lim_{m\sim\infty}} s_{mn}, \underline{\lim_{m\sim\infty}} s_{mn}, \overline{\underline{\lim_{m\sim\infty}}} s_{mn}, \lim_{m\sim\infty} s_{mn},$$

may be employed when n has a fixed value, and $m \sim \infty$. The repeated limits $\lim\limits_{m\sim\infty} \lim\limits_{n\sim\infty} s_{mn}$, $\lim\limits_{n\sim\infty} \lim\limits_{m\sim\infty} s_{mn}$ correspond precisely to the repeated limits $\lim\limits_{x\sim 0} \lim\limits_{y\sim 0} f(x, y)$, $\lim\limits_{y\sim 0} \lim\limits_{x\sim 0} f(x, y)$.

The following results are obtained by transposing those in I, § 303:

The existence of the double limit $s \equiv \lim\limits_{m\sim\infty, n\sim\infty} s_{mn}$ *implies the existence of the repeated limits* $\lim\limits_{m\sim\infty} \lim\limits_{n\sim\infty} s_{mn}$, $\lim\limits_{n\sim\infty} \lim\limits_{m\sim\infty} s_{mn}$, *and that these both have the value* s.

The existence of s *is not a necessary consequence of the existence and the equality of the two repeated limits.*

The existence of the repeated limit $\lim\limits_{m\sim\infty} \lim\limits_{n\sim\infty} s_{mn}$ *does not necessarily involve that of* $\lim\limits_{n\sim\infty} s_{mn}$, *as a definite number; but it implies that* $\lim\limits_{m\sim\infty} \overline{\lim\limits_{n\sim\infty}} s_{mn}$ *and* $\lim\limits_{m\sim\infty} \underline{\lim\limits_{n\sim\infty}} s_{mn}$ *have one and the same value. Thus* $\lim\limits_{m\sim\infty} \lim\limits_{n\sim\infty} s_{mn}$ *has a more general meaning than has* $\lim\limits_{m\sim\infty} \{\lim\limits_{n\sim\infty} s_{mn}\}$.

In case the sequence $\{s_{mn}\}$ be such that $s_{m'n'} \geqq s_{mn}$, for every set of values of m, n, m' and n', such that $m' \geqq m$, $n' \geqq n$, the sequence is said to be

* The theory of double sequences has been treated by Pringsheim, *Sitzungsber. Münch.* vol. XXVIII, and also in *Math. Annalen*, vol. LIII (1900), p. 289. See also a memoir by London in *Math. Annalen*, vol. LIII (1900), p. 322.

monotone. It is also said to be monotone when the condition is replaced by $s_{m'n'} \leqq s_{mn}$. This definition is equivalent to the corresponding definition in I, § 307.

The following theorem has already been established in I, § 307:

If the sequence $\{s_{mn}\}$ be monotone, the existence of any one of the three limits

$$\lim_{m\sim\infty, n\sim\infty} s_{mn}, \ \lim_{m\sim\infty}\lim_{n\sim\infty} s_{mn}, \ \lim_{n\sim\infty}\lim_{m\sim\infty} s_{mn}$$

involves the existence of the other two, and the equality of all three.

31. If the conditions are satisfied that $\overline{\lim}_{n\sim\infty} s_{mn}$, $\underline{\lim}_{n\sim\infty} s_{mn}$ are finite for each value of m, and that, corresponding to an arbitrarily chosen positive number ϵ, a value n_ϵ, of n, can be so chosen that s_{mn} lies between $\overline{\lim}_{n\sim\infty} s_{mn} + \epsilon$ and $\underline{\lim}_{n\sim\infty} s_{mn} - \epsilon$, provided $n \geqq n_\epsilon$, for every value of m, it may be said that the simple sequences $(s_{m1}, s_{m2}, s_{m3}, \ldots s_{mn}, \ldots)$ are *oscillatory, uniformly with respect to m.*

In case $\lim_{n\sim\infty} s_{mn}$ exists as a definite number, for each value of m, and in case, corresponding to an arbitrarily chosen positive number ϵ, an integer n_ϵ can be so chosen that $|s_{mn} - \lim_{n\sim\infty} s_{mn}| < \epsilon$, provided $n \geqq n_\epsilon$, for every value of m, the simple sequences $(s_{m1}, s_{m2}, \ldots s_{mn}, \ldots)$ are said to *converge uniformly with respect to m.*

In the present case the statement of the condition (2) of the theorem contained in I, § 304 may be simplified, it being observed (see I, p. 387) that the condition may be so strengthened that the theorem gives the necessary and sufficient conditions for the existence of the double limit. The condition there given is that corresponding to each arbitrarily fixed positive number ϵ, a number n_ϵ exists such that for each value n_1 of $n > n_\epsilon$, a positive number m_{n_1} exists such that s_{mn} lies between $\overline{\lim}_{n\sim\infty} s_{mn} + \epsilon$ and $\underline{\lim}_{n\sim\infty} s_{mn} - \epsilon$, for all values of n that are $\geqq n_1$ and for all values of m that are $> m_{n_1}$. It being assumed that $\overline{\lim}_{n\sim\infty} s_{mn}$, $\underline{\lim}_{n\sim\infty} s_{mn}$ are finite for all values of m, it is clear that, for $m = 1, 2, 3, \ldots m_{n_1}$, s_{mn} lies between $\overline{\lim}_{n\sim\infty} s_{mn} + \epsilon$ and $\underline{\lim}_{n\sim\infty} s_{mn} - \epsilon$, for all values of n that are $\geqq$ some fixed number n_1'. If now $\bar{n}_1$ be the greater of the two numbers n_1, n_1', we see that the condition is equivalent to the condition that s_{mn} lies between $\overline{\lim}_{n\sim\infty} s_{mn} + \epsilon$ and $\underline{\lim}_{n\sim\infty} s_{mn} - \epsilon$, for all values of $n \geqq \bar{n}_1$ and for all the values $1, 2, 3, \ldots$ of m. The condition may now be stated in the form that all the simple sequences

$$(s_{m1}, s_{m2}, \ldots s_{mn}, \ldots)$$

are oscillatory, uniformly with respect to m. It will clearly make no

difference if, for a finite number of values of m, $\overline{\lim}_{n\sim\infty} s_{mn}$ or $\underline{\lim}_{n\sim\infty} s_{mn}$ is not finite, or if both are infinite. The theorem so modified now takes the following form:

The necessary and sufficient conditions for the existence of the double limit $\lim_{m\sim\infty, n\sim\infty} s_{mn}$ *are* (1), *that* $\overline{\lim}_{n\sim\infty} s_{mn} - \underline{\lim}_{n\sim\infty} s_{mn}$ *should converge to zero, as* $m \sim \infty$, *and that* $\overline{\lim}_{m\sim\infty} s_{mn} - \underline{\lim}_{m\sim\infty} s_{mn}$ *should converge to zero, as* $n \sim \infty$, *and* (2), *that the sequences* $(s_{m1}, s_{m2}, \dots s_{mn}, \dots)$, *where* $m = 1, 2, 3, \dots$, *should be oscillatory, uniformly with respect to* m, *with the possible exception that this only holds when a finite number of them are disregarded.*

In case all the sequences $(s_{m1}, s_{m2}, \dots)$, with the possible exception of a finite number of them, are convergent, the condition (2) reduces to that of uniform convergence of the sequences, and the theorem may be stated as follows:

If $\lim_{n\sim\infty} s_{mn}$ *exists as a definite number, for all values of* m *with the possible exception of a finite set of such values, the necessary and sufficient conditions for the existence of the double limit, as a definite number, are* (1), *that*

$$\overline{\lim}_{m\sim\infty} s_{mn} - \underline{\lim}_{m\sim\infty} s_{mn}$$

should converge to zero, as $n \sim \infty$, *and* (2), *that the sequences*

$$(s_{m1}, s_{m2}, \dots s_{mn}, \dots)$$

are uniformly convergent with respect to m, *a finite set of these sequences being possibly omitted.*

32. Another form of the sufficient and necessary conditions for the existence of the double limit is contained in the following theorem:

The necessary and sufficient conditions for the existence of the double limit of s_{mn} *are* (1), *that* $\lim_{m\sim\infty} \overline{\lim}_{n\sim\infty} s_{mn}$ *should exist as a definite number, and* (2), *that, when possibly a finite set of values of* m *is omitted, the sequences*

$$(s_{m1}, s_{m2}, \dots)$$

are oscillatory, uniformly with respect to m.

In case the sequences $(s_{m1}, s_{m2}, \dots)$ *are convergent, at least when a finite set of these sequences is omitted, the necessary and sufficient conditions are* (1) *that* $\lim_{m\sim\infty} \lim_{n\sim\infty} s_{mn}$ *exists as a definite number, and* (2) *that the above sequences, when possibly a finite set is omitted, converge uniformly with respect to* m.

To prove that the conditions are sufficient, let $s \equiv \lim_{m\sim\infty} \overline{\lim}_{n\sim\infty} s_{mn}$; then, if $m > m_\epsilon$, we have $s + \epsilon > \overline{\lim}_{n\sim\infty} s_{mn} \geqq \underline{\lim}_{n\sim\infty} s_{mn} > s - \epsilon$, where m_ϵ is some integer dependent on ϵ.

Also, if n is greater than some fixed number n_ϵ, we have

$$\overline{\lim_{n\sim\infty}}\, s_{mn} + \epsilon > s_{mn} > \underline{\lim_{n\sim\infty}}\, s_{mn} - \epsilon,$$

for all values of m with the possible exception of those belonging to a finite set. When $m > m_\epsilon$, and $n > n_\epsilon$, we have $s + 2\epsilon > s_{mn} > s - 2\epsilon$, or

$$| s - s_{mn} | < 4\epsilon.$$

Since ϵ is arbitrary, it follows that s is the double limit of s_{mn}.

To prove that the conditions are necessary, we assume that

$$s \equiv \lim_{m\sim\infty,\, n\sim\infty} s_{mn}$$

exists. That the condition (1) is satisfied is at once clear. Also since

$$| s - s_{mn} | < \epsilon,$$

for $m > m_\epsilon$, $n > n_\epsilon$, where m_ϵ, n_ϵ are fixed integers dependent on ϵ, we have $s + \epsilon > s_{mn} > s - \epsilon$, for $m > m_\epsilon$, $n > n_\epsilon$. It follows that

$$s + \epsilon \geqq \overline{\lim_{n\sim\infty}}\, s_{mn} \geqq \underline{\lim_{n\sim\infty}}\, s_{mn} \geqq s - \epsilon,$$

provided $m > m_\epsilon$. From this we deduce that $s_{mn} < \overline{\lim\limits_{n\sim\infty}}\, s_{mn} + 2\epsilon$, and $s_{mn} > \underline{\lim\limits_{n\sim\infty}}\, s_{mn} - 2\epsilon$, provided $m > m_\epsilon$, $n > n_\epsilon$. Now consider the values $m = 1, 2, 3, \ldots m_\epsilon$, of m; for each of these values of m for which

$$(s_{m1}, s_{m2}, \ldots)$$

has finite upper and lower limits a value of n can be determined such that $s_{mn} < \overline{\lim\limits_{n\sim\infty}}\, s_{mn} + 2\epsilon$, and $s_{mn} > \underline{\lim\limits_{n\sim\infty}}\, s_{mn} - 2\epsilon$, for this and all greater values of n; it follows that a value n_ϵ', of n, can be determined so that these inequalities hold for all the values $1, 2, 3, \ldots m_\epsilon$, of m, provided the upper and lower limits exist. If $\bar{n}_\epsilon$ be greater than both n_ϵ and n_ϵ', we now see that s_{mn} lies between $\overline{\lim\limits_{n\sim\infty}}\, s_{mn} + 2\epsilon$ and $\underline{\lim\limits_{n\sim\infty}}\, s_{mn} - 2\epsilon$, provided $n > \bar{n}_\epsilon$, for all values of m, with the possible exception of a finite set. Since ϵ is arbitrary, the necessity of the condition has been established.

33. The preceding results may be applied to questions concerning the convergence of a double series $\sum\limits_{m=1,\, n=1}^{m,\, n} a_{mn}$, as m and n are indefinitely increased. Denoting this finite sum by s_{mn}, when the double limit

$$\lim_{m\sim\infty,\, n\sim\infty} s_{mn}$$

exists as a finite number s, this number is said to be the *sum of the double series*, and the double series is said to be convergent, and to converge to s.

The terms of the series may be arranged in rows and columns,

$$\begin{array}{l} a_{11} + a_{12} + a_{13} + \dots + a_{1n} + \dots \\ + a_{21} + a_{22} + a_{23} + \dots + a_{2n} + \dots \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ + a_{m1} + a_{m2} + a_{m3} + \dots + a_{mn} + \dots \\ \dots\dots\dots\dots\dots\dots \end{array}$$

which has been defined in § 29 as a series of type ω^2. The sum s_{mn} is that of the finite series

$$\begin{array}{l} a_{11} + a_{12} + \dots + a_{1n} \\ + a_{21} + a_{22} + \dots + a_{2n} \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ + a_{m1} + a_{m2} + \dots + a_{mn}. \end{array}$$

If $\lim\limits_{m\sim\infty,\, n\sim\infty} s_{mn}$ is $+\infty$ or $-\infty$, the double series is said to be divergent; if $\lim\limits_{m\sim\infty,\, n\sim\infty} s_{mn}$ does not exist either as a finite number, or as $+\infty$, or $-\infty$, the double series is said to oscillate.

If we denote by $\overline{\Sigma}_m$, $\underline{\Sigma}_m$ the upper and lower limits of indeterminancy of the series $a_{m1} + a_{m2} + \dots + a_{mn} + \dots$, which consists of the terms of the mth row of the given series when arranged in type ω^2, we have

$$\overline{\underline{\Sigma}}_m = \overline{\underline{\lim}}_{n\sim\infty} (s_{mn} - s_{m-1,n}), \text{ where } s_{0,n} = 0.$$

If now the double series be convergent, from which it follows that

$$\lim_{m\sim\infty} \overline{\lim_{n\sim\infty}}\, s_{mn} \text{ and } \lim_{m\sim\infty} \underline{\lim}_{n\sim\infty}\, s_{mn}$$

exist and have equal values, it follows that

$$\overline{\lim_{n\sim\infty}}\, s_{mn} - \underline{\lim}_{n\sim\infty}\, s_{mn}$$

converges to zero as $m \sim \infty$; it is then clear that $\overline{\Sigma}_m - \underline{\Sigma}_m$ must converge to zero as $m \sim \infty$. It follows that, although it is not necessary for the convergence of the double series that the single rows should converge, it is necessary that only a finite number of the rows should diverge, or have infinite limits of indeterminancy. Further it is necessary that the difference between the limits of indeterminancy of the sum of the series consisting of the mth row should converge to zero as $m \sim \infty$. Similar statements may be made as to the series $a_{1n} + a_{2n} + \dots + a_{mn} + \dots$ of which the terms are the constituents of the nth column, and of which the upper and lower limits of indeterminancy may be denoted by $\overline{\Sigma}_n'$ and $\underline{\Sigma}_n'$.

If all the rows are convergent, we may consider the series

$$\Sigma_1 + \Sigma_2 + \dots + \Sigma_m + \dots$$

obtained by summation first by rows and then by columns; and also, in case the columns are convergent, we may consider the series

$$\Sigma_1' + \Sigma_2' + \dots + \Sigma_n' + \dots$$

obtained by summation first by columns and then by rows.

The series

$$a_{11} + a_{12} + a_{21} + a_{13} + a_{22} + a_{31} + \dots + a_{1n} + a_{2(n-1)} + \dots + a_{n1} + \dots,$$

which is of type ω, is said to be the *diagonal* series corresponding to the double series. If this series converges, its sum is said to be the *diagonal sum* of the given double series.

The convergence of the two series

$$\Sigma_1 + \Sigma_2 + \dots + \Sigma_m + \dots, \qquad \Sigma_1' + \Sigma_2' + \dots + \Sigma_n' + \dots,$$

obtained respectively by summation first by rows and then by columns, and in the reverse order, does not necessarily involve the convergence of the double series; the double series may in fact be oscillatory.

If the double series be convergent, and all the rows be convergent, $\lim\limits_{m\sim\infty}\lim\limits_{n\sim\infty} s_{mn}$ must be equal to s, the double sum of the series, as has been observed in § 30; and since $\Sigma_m = \lim\limits_{n\sim\infty}(s_{mn} - s_{m-1,n})$ it follows that, if $\lim\limits_{n\sim\infty} s_{m-1,n}$ exists, so also does $\lim\limits_{n\sim\infty} s_{mn}$. But $\lim\limits_{n\sim\infty} s_{1n}$ exists, being equal to Σ_1; hence, by induction, we see that $\lim\limits_{n\sim\infty} s_{mn}$ exists for every value of m. We have clearly $\Sigma_1 + \Sigma_2 + \dots + \Sigma_m = \lim\limits_{n\sim\infty} s_{mn}$, and therefore the series $\Sigma_1 + \Sigma_2 + \dots + \Sigma_m + \dots$ converges to s. A similar result can be established for the series $\Sigma_1' + \Sigma_2' + \dots + \Sigma_n' + \dots$ in case all the series of columns are convergent. Thus it has been shewn that:

If the double series be convergent, and if every row be convergent, the series of the sums of the rows must converge to the double sum. A similar statement applies to the columns.

The double sum may exist when the rows, or columns, are not all convergent.

34. In accordance with a theorem in § 29, if all the terms of a double series be positive, the existence of the double sum ensures that, when all the terms are arranged in a single series of any type, that series converges to the double sum. We have thus the theorem that:

If all the terms of a double series are positive, and the double series be convergent, then the series obtained by summation first by rows and then by columns, or in the reverse order, and the diagonal series, are all convergent, and converge to the double sum.

It follows that all the rows and all the columns must be convergent.

A double series Σa_{mn} is said* to be *absolutely convergent* if the double series of which the terms are $| a_{mn} |$ is convergent. It is clear that the convergence of the double series $|a_{mn}|$ involves the convergence of Σa_{mn}, the given double series. For s_{mn} may be expressed as $s_{mn}^{(1)} - s_{mn}^{(2)}$, where $s_{mn}^{(1)}$ and $s_{mn}^{(2)}$ are both positive and monotone non-diminishing functions of m and n defined by

$$s_{mn}^{(1)} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \{\tfrac{1}{2} (| a_{\mu\nu} | + a_{\mu\nu})\},$$

and

$$s_{mn}^{(2)} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \{\tfrac{1}{2} (| a_{\mu\nu} | - a_{\mu\nu})\}.$$

The limits, as $m \sim \infty$, $n \sim \infty$, of $s_{mn}^{(1)}$ and $s_{mn}^{(2)}$ must be both finite and cannot oscillate, for $\lim (s_{mn}^{(1)} + s_{mn}^{(2)})$ is finite and $s_{mn}^{(1)}$, $s_{mn}^{(2)}$ are both monotone and non-diminishing; therefore $\lim\limits_{m\sim\infty,\, n\sim\infty} (s_{mn}^{(1)} - s_{mn}^{(2)})$ is a definite number.

In an absolutely convergent series, both $\sum\limits_{n=1}^{\infty} | a_{mn} |$, $\sum\limits_{m=1}^{\infty} | a_{mn} |$ must be convergent, the first for all values of m, and the second for all values of n, since each is less than the double sum of $\Sigma \,| a_{mn} |$. It follows that all the rows and all the columns of the given series Σa_{mn} are convergent. Therefore, in virtue of the theorem proved above, the series of the sums of the rows, and the series of the sums of the columns, must converge to the double sum. Moreover, in virtue of a theorem proved in § 29, it follows that, if the terms be arranged in a single series of any type, it is convergent and its sum is independent of the type. The following theorem has thus been established:

If a double series be absolutely convergent, it is absolutely convergent when summed by rows and by columns, in either order, or when arranged as a diagonal series, or as a single series of any type; moreover, in each case, the sum is equal to that of the double series.

So far as this theorem relates to summation by rows and columns it is due to Cauchy.

Moreover it can be shewn that:

A double series is absolutely convergent if the single series is convergent, of which the mth term is the sum of the absolute values of the terms in the mth row.

For, if the series

$$\sum_{n=1}^{\infty} | a_{1n} | + \sum_{n=1}^{\infty} | a_{2n} | + \sum_{n=1}^{\infty} | a_{3n} | + \dots$$

* It has been asserted by Jordan that there exist only absolutely convergent double series; see his *Cours d'Analyse*, vol. I, p. 302. This statement rests upon a narrow definition of the convergence of such series.

is convergent, it follows that $\sum_{m=1,n=1}^{m,n} |a_{mn}|$ is less than some fixed number, independent of m and n, and therefore that $\lim_{m\sim\infty, n\sim\infty} |s_{mn}|$ must be finite. Therefore the double series $\sum_{m,n} |a_{mn}|$ is convergent, and consequently $\sum_{m,n} a_{mn}$ is convergent. Moreover, since $\sum_{n=1}^{\infty} |a_{mn}|$ is absolutely convergent, $\sum_{n=1}^{\infty} a_{mn}$ is convergent, so that the rows, and similarly the columns, of the series Σa_{mn} are all convergent. It then follows from the previous theorem that the sum of the series is the same whether taken in either order by rows and columns, or in any other manner.

Thus, for a series in which any one of the sums

$$\sum_{m=1,n=1}^{\infty,\infty} |a_{mn}|, \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|, \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{mn}|,$$

$$\sum_{n=1}^{\infty} \{|a_{1n}| + |a_{2(n-1)}| + \ldots + |a_{n1}|\}$$

is known to exist, any one of the four equations

$$\sum_{m=1,n=1}^{\infty,\infty} a_{mn} = s, \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = s, \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} = s,$$

$$\sum_{n=1}^{\infty} \{a_{1,n} + a_{2,n-1} + \ldots + a_{n,1}\} = s$$

implies the other three.

35. The theorems of §§ 31, 32 can be applied to the consideration of double series which are not absolutely convergent.

Since $s_{mn} = (a_{11} + a_{21} + \ldots + a_{m1}) + (a_{12} + a_{22} + \ldots + a_{m2})$
$+ \ldots + (a_{1n} + a_{2n} + \ldots + a_{mn})$,

the sum of the infinite series

$$(a_{11} + a_{21} + \ldots + a_{m1}) + (a_{12} + a_{22} + \ldots + a_{m2}) + \ldots$$

when it exists is $\lim_{n\sim\infty} s_{mn}$. It is sufficient to consider the case in which these series are all convergent, except possibly a finite set of them. We have then the following theorem:

If all the series

$$(a_{11} + a_{21} + \ldots + a_{m1}) + (a_{12} + a_{22} + \ldots + a_{m2}) + \ldots$$

are convergent, except possibly for a finite set of values of m, the necessary and sufficient conditions for the convergence of the double series Σa_{mn} are (1), that the sum of the above series should converge to a definite number, as m is indefinitely increased, and (2), that all the above series except possibly a finite set should converge uniformly with respect to m.

The series when summed by rows and columns, and by columns and rows, may be convergent, and may have the same sum in both cases, and yet the double series is not necessarily convergent. A convergent double series which is not absolutely convergent can be replaced by a new series which diverges, or oscillates, and is such that each term a_{mn} occurs in a definite place in the new series, and that no terms occur in the new series which do not belong to the original one. This can be proved in a similar manner to the corresponding theorem for simple series. A general type of non-absolutely convergent double series has been given* by Hardy.

36. With a view to the investigation of the diagonal series corresponding to a double series, the following theorem† will be established:

If $|s_{mn}|$ *is less than a fixed positive number* g, *for all values of* m *and* n, *and if* $\lim\limits_{m\sim\infty, n\sim\infty} s_{mn}$ *has a definite value* s, *then*

$$\lim_{n\sim\infty} \frac{s_{1,n} + s_{2,n-1} + \dots + s_{n,1}}{n} = s.$$

We have

$$\sum_{r=1}^{r=n} s_{r,n-r+1} - ns = \sum_{r=1}^{r=p}\{s_{r,n-r+1} - s\} + \sum_{r=p+1}^{r=n-p}\{s_{r,n-r+1} - s\} + \sum_{r=n-p+1}^{r=n}\{s_{r,n-r+1} - s\},$$

where $1 < p$, and $n > 2p + 1$. If ϵ be an arbitrarily chosen positive number, p and n may be so chosen that $|s_{r,n-r+1} - s| < \epsilon$, for

$$r = p + 1, p + 2, \dots, n - p;$$

thus the second term on the right hand side of the equation is numerically less than $(n - 2p)\epsilon$. The sum of the first and third terms is numerically less than $2p\{g + |s|\}$. It follows that

$$\left|\frac{s_{1,n} + s_{2,n-1} + \dots + s_{n,1}}{n} - s\right| < \left(1 - \frac{2p}{n}\right)\epsilon + \frac{2p}{n}\{g + |s|\}.$$

Keeping p fixed, and letting n increase indefinitely, we see that the expression on the right hand side converges to ϵ; and since ϵ is arbitrary, the result stated in the theorem follows.

It is easily seen that

$$\frac{s_{1,n} + s_{2,n-1} + \dots + s_{n,1}}{n} = \frac{S_1 + S_2 + \dots + S_n}{n},$$

where S_n denotes the nth partial sum of the series

$$a_{11} + (a_{12} + a_{21}) + \dots + (a_{1n} + \dots + a_{n1}) + \dots;$$

therefore, if the limit of one of those expressions exists, that of the other also exists, and their values are identical. If now the diagonal series is convergent, in accordance with the theorem of § 27 the second of these limits exists and has the value of the sum of the diagonal series. Since the first of the above limits then exists, and has the same value,

* *Proc. Lond. Math. Soc.* (2), vol. I (1903), p. 124.

† See Pringsheim, *Sitzungsber. Münch.* vol. XXVII (1900), p. 123.

it follows that the sum of the diagonal series is the same as that of the double series, provided the conditions of the theorem are satisfied. If the diagonal series could diverge to $+\infty$, or to $-\infty$, then $(S_1 + S_2 + \ldots + S_n)/n$ would diverge to $+\infty$ or to $-\infty$, and this would be inconsistent with the convergence of the double sum. Thus the diagonal series cannot diverge, but may oscillate between finite or infinite limits. The following theorem has thus been established:

If the convergent double series $\sum\limits_{m=1, n=1} a_{mn}$, *of which s is the sum, is such that* $|s_{mn}|$ *is less than a fixed positive number, independent of m and n, then the diagonal series cannot diverge, but must be either convergent or oscillating. In case it converges, its sum is the same as that of the double series.*

EXAMPLES.

(1) Let $s_{mn} = (-1)^{m+n}(m^{-p} + n^{-q})$, where $p > 0$, $q > 0$. In this case $\lim\limits_{n\sim\infty} s_{mn}$, $\lim\limits_{m\sim\infty} s_{mn}$ do not exist as definite numbers, but the three limits

$$\lim_{m\sim\infty}\lim_{n\sim\infty} s_{mn}, \quad \lim_{n\sim\infty}\lim_{m\sim\infty} s_{mn}, \quad \lim_{m\sim\infty, n\sim\infty} s_{mn}$$

all exist, and are zero. In this case $\overline{\lim\limits_{n\sim\infty}}\, s_{mn} = \frac{1}{m^p}$, $\underline{\lim\limits_{n\sim\infty}}\, s_{mn} = -\frac{1}{m^p}$, and s_{mn} oscillates uniformly for all values of m.

(2) Let $s_{mn} = \frac{1}{1 + (m-n)^2}$. In this case the repeated limits both exist, and are zero; but the double limit does not exist. A similar case arises if $s_{mn} = \frac{mn}{m^2 + n^2}$.

(3) Let $a_{mn} = \frac{(-1)^{n+1}}{(2n-1)!}\left(\frac{2^m - 1}{2^m}\pi\right)^{2n-1}$; we have $\sum\limits_{n=1}^{\infty} a_{mn} = \sin\frac{2^m-1}{2^m}\pi$, and the series $\sum\limits_{m=1}^{\infty} \sin\frac{2^m-1}{2^m}\pi$ is also convergent, since the general term is less than $\frac{\pi}{2^m}$. But each of the series $\sum\limits_{m=1}^{\infty} a_{mn}$ is non-convergent, and consequently the double series is not convergent. Although the series $\sum\limits_{n=1}^{\infty} a_{mn}$ converge uniformly with respect to m, for $m > 1$, the series $\sum\limits_{n=1}^{\infty} a_{1n}$, $\sum\limits_{n=1}^{\infty} (a_{1n} + a_{2n})$, ... do not converge uniformly.

(4) Let* $a_{mn} = \frac{(-1)^{m+n}\, mn}{(m+n)^2}$. In this case the sum of the double series oscillates between the limits $\frac{1}{6}\log 2 - \frac{5}{48}$ and $\frac{1}{6}\log 2 + \frac{1}{48}$. The repeated sums are both equal to $\frac{1}{6}(\log 2 - \frac{1}{4})$; and the diagonal series oscillates between $+\infty$ and $-\infty$.

(5) Consider* the double series

$$\begin{array}{llll} +(a_0 + b_0) & +(a_1 - b_0) & + a_2 + a_3 + a_4 + \ldots \\ +(-a_0 + b_1) & +(-a_1 - b_1) & - a_2 - a_3 - a_4 - \ldots \\ + b_2 & - b_2 & + 0 + 0 + 0 + \ldots \\ + b_3 & - b_3 & + 0 + 0 + 0 + \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}$$

* See Bromwich and Hardy, *Proc. Lond. Math. Soc.* (2), vol. II (1904), p. 175.

The double sum is in any case zero; but the sums of the first two rows are not convergent if Σa_n does not converge. The diagonal sum is $\lim\limits_{n\sim\infty}(a_n+b_n)$ if this limit exists, and it will otherwise be oscillatory. The series $\sum\limits_{n=1}^{\infty}(a_{1n}+a_{2n}+\ldots+a_{mn})$ are uniformly convergent, when the series corresponding to $m=1$ is omitted, in case that series is non-convergent.

(6) Shew that the series $\sum\limits_{n=1,m=1}\frac{a_m a_n}{m+n}$ is convergent whenever $\sum\limits_{n=1} a_n^2$ is convergent. This theorem is due to Hilbert. Hilbert's proof was outlined by Weyl*; other proofs have been given by Wiener†, Schur‡, and Hardy§. The last deduces it from the theorem that, if $\sum\limits_{n=1} a_n^2$ is convergent, then $\Sigma\left(\frac{s_n}{n}\right)^2$ is convergent; where $s_n=a_1+a_2+\ldots+a_n$.

This last theorem is a particular case of the more general one that, if $k>1$, and $\sum\limits_{n=1} a_n^k$ is convergent, then $\sum\limits_{n=1}\left(\frac{s_n}{n}\right)^k$ is convergent.

THE CONVERGENCE OF THE CAUCHY-PRODUCT OF TWO SERIES.

37. Let us consider the two series

$$a_1+a_2+\ldots+a_n+\ldots,$$
$$b_1+b_2+\ldots+b_n+\ldots,$$

and let the series $c_1+c_2+\ldots+c_n+\ldots$ be formed, where

$$c_1=a_1b_1,\ c_2=a_1b_2+a_2b_1,\ \ldots,\ c_n=a_1b_n+a_2b_{n-1}+\ldots+a_nb_1,$$

then the last series may be termed the Cauchy-product series of the first two.

It is clear that the series $\sum\limits_{n=1} c_n$ is the diagonal series corresponding to the double series $\sum\limits_{m=1,n=1}\alpha_{mn}$, where $\alpha_{mn}=a_mb_n$. Also

$$\sum_{m=1,n=1}^{m,n}\alpha_{mn}=s_ms_n',$$

where s_m, s_n' denote the mth and nth partial sums of the series $\sum\limits_{n=1} a_n$, $\sum\limits_{n=1} b_n$ respectively.

In case both the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ are convergent, having s and s' for their respective sums, we see that the double series $\sum\limits_{m=1,n=1}\alpha_{mn}$ is convergent, its sum being ss', but it does not necessarily follow that the diagonal series is convergent. It can, however, be shewn that the diagonal series cannot diverge, but must either oscillate or converge. For it has

* Inaugural dissertation, *Singuläre Integralgleichungen*, Göttingen (1908), p. 83.

† *Math. Annalen*, vol. LXVIII (1910), p. 361.

‡ *Crelle's Journal*, vol. CXL (1912), p. 1.

§ *Math. Zeitschr.* vol. VI (1920), p. 314; also *Messenger of Math.* vol. XLVIII (1918), p. 107.

been shewn in § 36 that, S_n denoting the nth partial sum of the series Σc_n, we have $\lim_{n\sim\infty} \frac{S_1 + S_2 + \ldots + S_n}{n} = ss'$; and if the series Σc_n were divergent, the limit on the left hand side would be $+\infty$ or $-\infty$, which is not the case.

Since, when Σc_n is convergent, its sum is $\lim_{n\sim\infty} \frac{S_1 + S_2 + \ldots + S_n}{n}$, we obtain the following theorem due to Abel:

In case the three series Σa, Σb, Σc are all convergent, the sum S of the series Σc is equal to the product ss' of the sums of the series Σa, Σb.

Moreover it follows that:

If the series Σa, Σb are convergent, the series Σc has for its Cesàro sum the product ss'.

38. In case the series Σa, Σb are both absolutely convergent, the series Σc is also absolutely convergent, and its sum is the product of the sums of the two former series. This follows at once from the theorem of § 34, that an absolutely convergent double series $\sum_{m,n} a_m b_n$ is such that, when arranged as the diagonal series Σc_n, it is still absolutely convergent, the sum of the diagonal series being equal to the sum of the double series.

This result, which is due to Cauchy, is included in the following more general theorem due* to Mertens:

In case one at least of the two convergent series Σa, Σb be absolutely convergent, the Cauchy-product series Σc is convergent, and its sum is ss', the product of the sums of the two given series.

To prove this theorem, we have

$$\begin{aligned} S_n = (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) - b_2 a_n - b_3(a_n + a_{n-1}) - \ldots \\ - b_n(a_n + a_{n-1} + \ldots + a_2) \\ = s_n s_n' - b_2(s_n - s_{n-1}) - b_3(s_n - s_{n-2}) - \ldots - b_n(s_n - s_1); \end{aligned}$$

and therefore

$$\begin{aligned} | S_n - ss' | \leqq | s_n s_n' - ss' | + | b_2 | \, | s_n - s_{n-1} | + | b_3 | \, | s_n - s_{n-2} | \\ + \ldots + | b_n | \, | s_n - s_1 |. \end{aligned}$$

Let m be an integer $< n$, and so great that

$$| s_n - s_m |, \; | s_n - s_{m+1} |, \ldots | s_n - s_{n-1} |$$

are all less than a fixed positive number η; it is clear that n may be taken so large that the integer m exists, and that this holds for all greater values of n. We have then

$$\begin{aligned} | S_n - ss' | < | s_n s_n' - ss' | + \eta \{ | b_2 | + | b_3 | + \ldots + | b_{n-m+1} | \} \\ + | s_n - s_{m-1} | \, | b_{n-m+2} | + | s_n - s_{m-2} | \, | b_{n-m+3} | \\ + \ldots + | s_n - s_1 | \, | b_n |. \end{aligned}$$

* *Crelle's Journal*, vol. LXXIX (1875), p. 182.

Let it now be assumed that the series Σb is absolutely convergent, and let Σ_n be the nth partial sum of the series $\Sigma \mid b \mid$. The numbers

$$| s_n - s_{m-1} |, \ | s_n - s_{m-2} |, \ldots | s_n - s_1 |$$

are all less than some fixed number A, independent of n and m. We have accordingly

$$| S_n - ss' | < | s_n s_n' - ss' | + \eta (\Sigma_{n-m+1} - \Sigma_1) + A (\Sigma_n - \Sigma_{n-m+1}).$$

The numbers m, n can be so chosen that $| s_n s'_n - ss' | < \theta$, and that $| \Sigma_n - \Sigma_{n-m+1} | < \theta'$, where θ, θ' are arbitrarily chosen positive numbers; if this be done, we have

$$| S_n - ss' | < \theta + \eta B + \theta' A,$$

where B is some positive number independent of m and n. Since θ, η, θ' are arbitrarily small, provided n is sufficiently large, $| S_n - ss' |$ is less than an arbitrarily chosen positive number. Therefore the series Σc converges to ss'.

When both the series Σa, Σb are non-absolutely convergent, their Cauchy-product does not necessarily converge. Classes of cases in which the Cauchy-product of such series is convergent have been studied by Pringsheim*, Voss†, and Cajori‡.

THE CONVERGENCE OF INFINITE PRODUCTS.

39. Let $a_1, a_2, \ldots a_n, \ldots$ denote a sequence of positive numbers, none of which are zero, and let the product $a_1 a_2 \ldots a_n$ be denoted by p_n. If p_n have a definite limit P, as $n \sim \infty$, and $P \neq 0$, the infinite product $a_1 a_2 \ldots a_n \ldots$ is said to be convergent, and to converge to the value P. In case $\lim\limits_{n\sim\infty} p_n$ has the value zero, the infinite product is said to diverge to zero. The infinite product is also said to be divergent in case $\lim\limits_{n\sim\infty} p_n$, or P, is $+\infty$. In case $\lim\limits_{n\sim\infty} p_n$, or P, oscillates between limits of indeterminancy, one of which may be $+\infty$, the infinite product is said to oscillate.

Since $\log p_n = \log a_1 + \log a_2 + \ldots + \log a_n$, it is clear that in case p_n has a definite limit $P\ (\neq 0)$, $\log P$ is the limiting sum of the series $\Sigma \log a_n$. But if $\lim\limits_{n\sim\infty} p_n = 0$, the series $\Sigma \log a_n$ is divergent. It thus appears that the question of the convergence of an infinite product is reducible to that of the convergence of a series. The parallelism between the cases of an infinite sum and an infinite product is made complete by the convention that when $\lim\limits_{n\sim\infty} p_n = 0$ the infinite product is to be regarded as diverging to zero.

* *Math. Annalen*, vol. XXI (1883), p. 327; see also vol. XXVI (1886), where Pringsheim has considered the Cauchy-product of the trigonometrical series.

† *Math. Annalen*, vol. XXIV (1884), p. 42.

‡ *New York Bull.* (2), vol. I (1895) and *Amer. Journ. of Math.* vol. XV (1893), p. 339, and vol. XVIII, p. 196.

The theory of infinite products may however be developed independently of that of infinite series; and this has been carried out in detail by Pringsheim*. A short account of this theory will be given here.

If $\lim_{n\sim\infty} p_n$ have a value $P\ (\neq 0)$, the following two necessary and sufficient conditions are satisfied:

(*a*), $|p_n| > g$, where g is some fixed positive number, and $n = 1, 2, 3, \ldots$.

(*b*), $|p_{n+m} - p_n| < \epsilon$, where ϵ is a prescribed positive number, provided $n \geqq n_\epsilon$, an integer dependent on ϵ, and $m = 1, 2, 3, \ldots$.

These conditions (*a*) and (*b*) may both be included under one condition (*c*). Thus:

The necessary and sufficient condition that p_n should converge to a definite number $P\ (\neq 0)$ is that

(*c*), $\left|\dfrac{p_{n+m}}{p_n} - 1\right| < \eta$, *where η is an arbitrarily chosen positive number, for $n \geqq n_\eta$, a number dependent upon η, and $m = 1, 2, 3, \ldots$.*

When (*a*) and (*b*) are both satisfied, it is clear that (*c*) is satisfied; and thus the condition (*c*) is necessary.

In order to shew that it is sufficient, we see from (*c*), taking $\eta < 1$, that $|p_n|$ lies between $(1+\eta)\,|p_{n_\eta}|$ and $(1-\eta)\,|p_{n_\eta}|$, for all values of n that are $> n_\eta$. Thus, since $|p_n|$ is, for all values of n, with the possible exception of those of a finite set, greater than the fixed positive number $(1-\eta)\,|p_{n_\eta}|$, and less than the fixed positive number $(1+\eta)\,|p_{n_\eta}|$, it follows that $|p_n|$ is greater than some fixed positive number g, and less than a fixed positive number G, for all values of n. Thus the condition (*a*) is satisfied. Also, from (*c*) we have

$$|p_n - p_{n_\eta}| < \eta\,|p_{n_\eta}|,$$

for all values of $n > n_\eta$.

We have now, for $n \geqq n_\eta$, $|p_n - p_{n_\eta}| < G\eta < \frac{1}{2}\epsilon$, if η be chosen to be $\frac{1}{2}\epsilon/G$. From this it follows that $|p_{n+m} - p_n| < \epsilon$, for $n \geqq n_\eta$, and $m = 1, 2, 3, \ldots$; thus the condition (*b*) is satisfied. Since the conditions (*a*) and (*b*) are both deducible from (*c*), it follows that the condition (*c*) is sufficient for the convergence of the product.

Let $m = 1$, then the condition (*c*) shews that $|a_{n+1} - 1| < \eta$, and thus, since η is arbitrary, we have $\lim_{n\sim\infty}(a_n - 1) = 0$. Writing $a_n = 1 + c_n$, we have $\lim_{n\sim\infty} c_n = 0$; and it is consequently convenient to consider the product in the form $(1+c_1)(1+c_2)\ldots(1+c_n)\ldots$, where $\lim_{n\sim\infty} c_n = 0$ is a necessary condition for the convergence of the infinite product.

* *Math. Annalen*, vol. XXXIII (1889), p. 119.

Let it now be assumed that the series Σb is absolutely convergent, and let Σ_n be the nth partial sum of the series $\Sigma \mid b \mid$. The numbers

$$|s_n - s_{m-1}|, \; |s_n - s_{m-2}|, \ldots |s_n - s_1|$$

are all less than some fixed number A, independent of n and m. We have accordingly

$$|S_n - ss'| < |s_n s_n' - ss'| + \eta (\Sigma_{n-m+1} - \Sigma_1) + A (\Sigma_n - \Sigma_{n-m+1}).$$

The numbers m, n can be so chosen that $|s_n s'_n - ss'| < \theta$, and that $|\Sigma_n - \Sigma_{n-m+1}| < \theta'$, where θ, θ' are arbitrarily chosen positive numbers; if this be done, we have

$$|S_n - ss'| < \theta + \eta B + \theta' A,$$

where B is some positive number independent of m and n. Since θ, η, θ' are arbitrarily small, provided n is sufficiently large, $|S_n - ss'|$ is less than an arbitrarily chosen positive number. Therefore the series Σc converges to ss'.

When both the series Σa, Σb are non-absolutely convergent, their Cauchy-product does not necessarily converge. Classes of cases in which the Cauchy-product of such series is convergent have been studied by Pringsheim*, Voss†, and Cajori‡.

THE CONVERGENCE OF INFINITE PRODUCTS.

39. Let $a_1, a_2, \ldots a_n, \ldots$ denote a sequence of positive numbers, none of which are zero, and let the product $a_1 a_2 \ldots a_n$ be denoted by p_n. If p_n have a definite limit P, as $n \sim \infty$, and $P \neq 0$, the infinite product $a_1 a_2 \ldots a_n \ldots$ is said to be convergent, and to converge to the value P. In case $\lim\limits_{n\sim\infty} p_n$ has the value zero, the infinite product is said to diverge to zero. The infinite product is also said to be divergent in case $\lim\limits_{n\sim\infty} p_n$, or P, is $+\infty$. In case $\lim\limits_{n\sim\infty} p_n$, or P, oscillates between limits of indeterminancy, one of which may be $+\infty$, the infinite product is said to oscillate.

Since $\log p_n = \log a_1 + \log a_2 + \ldots + \log a_n$, it is clear that in case p_n has a definite limit P $(\neq 0)$, $\log P$ is the limiting sum of the series $\Sigma \log a_n$. But if $\lim\limits_{n\sim\infty} p_n = 0$, the series $\Sigma \log a_n$ is divergent. It thus appears that the question of the convergence of an infinite product is reducible to that of the convergence of a series. The parallelism between the cases of an infinite sum and an infinite product is made complete by the convention that when $\lim\limits_{n\sim\infty} p_n = 0$ the infinite product is to be regarded as diverging to zero.

* *Math. Annalen*, vol. XXI (1883), p. 327; see also vol. XXVI (1886), where Pringsheim has considered the Cauchy-product of the trigonometrical series.

† *Math. Annalen*, vol. XXIV (1884), p. 42.

‡ *New York Bull.* (2), vol. I (1895) and *Amer. Journ. of Math.* vol. XV (1893), p. 339, and vol. XVIII, p. 196.

The theory of infinite products may however be developed independently of that of infinite series; and this has been carried out in detail by Pringsheim*. A short account of this theory will be given here.

If $\lim_{n\sim\infty} p_n$ have a value P ($\neq 0$), the following two necessary and sufficient conditions are satisfied:

(*a*), $|p_n| > g$, where g is some fixed positive number, and $n = 1, 2, 3, \ldots$.

(*b*), $|p_{n+m} - p_n| < \epsilon$, where ϵ is a prescribed positive number, provided $n \geqq n_\epsilon$, an integer dependent on ϵ, and $m = 1, 2, 3, \ldots$.

These conditions (*a*) and (*b*) may both be included under one condition (*c*). Thus:

The necessary and sufficient condition that p_n should converge to a definite number P ($\neq 0$) is that

(*c*), $\left|\frac{p_{n+m}}{p_n} - 1\right| < \eta$, *where η is an arbitrarily chosen positive number, for $n \geqq n_\eta$, a number dependent upon η, and $m = 1, 2, 3, \ldots$.*

When (*a*) and (*b*) are both satisfied, it is clear that (*c*) is satisfied; and thus the condition (*c*) is necessary.

In order to shew that it is sufficient, we see from (*c*), taking $\eta < 1$, that $|p_n|$ lies between $(1 + \eta)|p_{n_\eta}|$ and $(1 - \eta)|p_{n_\eta}|$, for all values of n that are $> n_\eta$. Thus, since $|p_n|$ is, for all values of n, with the possible exception of those of a finite set, greater than the fixed positive number $(1 - \eta)|p_{n_\eta}|$, and less than the fixed positive number $(1 + \eta)|p_{n_\eta}|$, it follows that $|p_n|$ is greater than some fixed positive number g, and less than a fixed positive number G, for all values of n. Thus the condition (*a*) is satisfied. Also, from (*c*) we have

$$|p_n - p_{n_\eta}| < \eta\,|p_{n_\eta}|,$$

for all values of $n > n_\eta$.

We have now, for $n \geqq n_\eta$, $|p_n - p_{n_\eta}| < G\eta < \frac{1}{2}\epsilon$, if η be chosen to be $\frac{1}{2}\epsilon/G$. From this it follows that $|p_{n+m} - p_n| < \epsilon$, for $n \geqq n_\eta$, and $m = 1, 2, 3, \ldots$; thus the condition (*b*) is satisfied. Since the conditions (*a*) and (*b*) are both deducible from (*c*), it follows that the condition (*c*) is sufficient for the convergence of the product.

Let $m = 1$, then the condition (*c*) shews that $|a_{n+1} - 1| < \eta$, and thus, since η is arbitrary, we have $\lim_{n\sim\infty}(a_n - 1) = 0$. Writing $a_n = 1 + c_n$, we have $\lim_{n\sim\infty} c_n = 0$; and it is consequently convenient to consider the product in the form $(1 + c_1)(1 + c_2)\ldots(1 + c_n)\ldots$, where $\lim_{n\sim\infty} c_n = 0$ is a necessary condition for the convergence of the infinite product.

* *Math. Annalen*, vol. XXXIII (1889), p. 119.

It is clear that we may so far relax the condition that $1 + c_n$ be positive for all values of n, as to admit the case in which it is negative for a finite set of values of n. Those factors for which this is the case may be removed without affecting the convergence or divergence.

40. *If $a_n \geqq 0$, for every value of n, it is necessary and sufficient for the convergence of the infinite products*

$$(1 + a_1)(1 + a_2)(1 + a_3)\ldots(1 + a_n)\ldots,$$

$$(1 - a_1)(1 - a_2)(1 - a_3)\ldots(1 - a_n)\ldots,$$

that the series $\sum\limits_1^\infty a_n$ should be convergent.

Consider first the product $(1 + a_1)(1 + a_2)\ldots(1 + a_n)\ldots$. We have $p_n > 1 + a_1 + a_2 + \ldots + a_n$; and consequently Σa_n must converge to a value less than P, the limit of p_n; thus the condition is necessary.

$$\text{Again } \frac{p_{n+m}}{p_n} = (1 + a_{n+1})(1 + a_{n+2})\ldots(1 + a_{n+m})$$

$$< 1 + \sum_{r=n+1}^{n+m} a_r + \left(\sum_{r=n+1}^{n+m} a_r\right)^2 + \ldots + \left(\sum_{r=n+1}^{n+m} a_r\right)^m$$

If $\sum\limits_{r=1}^{\infty} a_r$ is convergent, n may be so chosen that $\sum\limits_{r=n+1}^{n+m} a_r < \epsilon$, for all values of n; thus

$$\frac{p_{n+m}}{p_n} - 1 < \epsilon + \epsilon^2 + \ldots + \epsilon^m < \frac{\epsilon}{1 - \epsilon} < 2\epsilon,$$

provided ϵ be chosen to be $< \frac{1}{2}$. Since this holds for all sufficiently large values of n, and since ϵ is arbitrarily small, it follows that p_n converges as $n \sim \infty$.

Consider next the product $(1 - a_1)(1 - a_2)\ldots(1 - a_n)\ldots$. It will be assumed that $a_n < 1$, for all values of n, for if the product be convergent, this condition will be satisfied if a finite set of terms is removed, and if $\sum\limits_1^\infty a_n$ is convergent this is also the case.

We have then

$$\frac{1}{(1 - a_1)(1 - a_2)\ldots(1 - a_n)} > (1 + a_1)(1 + a_2)\ldots(1 + a_n)$$

$$> 1 + a_1 + a_2 + \ldots + a_n.$$

If the series Σa_n is divergent, we therefore have

$$\lim_{n\sim\infty}(1 - a_1)(1 - a_2)\ldots(1 - a_n) = 0,$$

and the product diverges to zero. Therefore it is necessary for convergence of the product that the series Σa_n should be convergent.

Next, if Σa_n is convergent, n can be so chosen that

$$a_{n+1} + a_{n+2} + \ldots + a_{n+m} < \epsilon,$$

for $m = 1, 2, 3, \ldots$; and

$$(1 - a_{n+1})(1 - a_{n+2}) > 1 - (a_{n+1} + a_{n+2}),$$

$$(1 - a_{n+1})(1 - a_{n+2})(1 - a_{n+3}) > 1 - (a_{n+1} + a_{n+2} + a_{n+3}),$$

and generally

$$(1 - a_{n+1})(1 - a_{n+2}) \ldots (1 - a_{n+m}) > 1 - (a_{n+1} + \ldots + a_{n+m}) > 1 - \epsilon;$$

and therefore

$$|(1 - a_{n+1})(1 - a_{n+2}) \ldots (1 - a_{n+m}) - 1| < \epsilon.$$

The condition for the convergence of the product is therefore satisfied. The condition that Σa_n should be convergent has thus been shewn to be sufficient for the convergence of $(1 - a_1)(1 - a_2) \ldots$.

41. It can be shewn that if $a_n \geqq 0$, for all values of n, and the order of the factors of the product be rearranged, in accordance with any prescribed law, the type of the infinite product being unaltered, then the new infinite product converges to P, the limit of the infinite product in the original order. When this is the case for any infinite product which converges, the product is said to converge *unconditionally*.

Let p_n' denote the product of n factors when the new order is taken. Let the factors $1 + a_1, 1 + a_2, \ldots 1 + a_{n_1}$ all occur in p_n'; thus

$$p_n' = p_{n_1} q_{n_2}, \text{ where } p_{n_1} = (1 + a_1)(1 + a_2) \ldots (1 + a_{n_1}),$$

and q_{n_2} denotes

$$(1 + a_{\alpha_1})(1 + a_{\alpha_2}) \ldots (1 + a_{\alpha_{n_2}}), \text{ where } \alpha_1 < \alpha_2 < \ldots < \alpha_{n_2};$$

thus $\alpha_1 > n_1$. It is clear that, as $n \sim \infty$, also $n_1 \sim \infty$.

Now $$q_{n_2} \leqq (1 + a_{\alpha_1})(1 + a_{\alpha_1+1})(1 + a_{\alpha_1+2}) \ldots (1 + a_{\alpha_{n_2}})$$
$$\leqq \frac{p_{\alpha_{n_2}}}{p_{\alpha_1 - 1}}.$$

The integer n can be chosen so large that n_1 is sufficiently large for the inequality $\frac{p_{n'+m}}{p_{n'}} - 1 < \epsilon$ to hold, provided $n' \geqq n_1$. It then follows that $q_{n_2} \leqq 1 + \epsilon$; and therefore p_n' lies between p_{n_1} and $p_{n_1}(1 + \epsilon)$, provided n is sufficiently large. Since ϵ is arbitrary, it follows that $\lim_{n \sim \infty} p_n' = \lim_{n_1 \sim \infty} p_{n_1}$. It has thus been established that:

If $a_1, a_2, \ldots a_n, \ldots$ are all $\geqq 0$, and the infinite product

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \ldots$$

is convergent, which is the case when Σa_n is convergent, then the convergence of the product is unconditional.

42. *When $a_1, a_2, \ldots a_n, \ldots$ are not all positive, the infinite product $(1 + a_1) \ldots (1 + a_n) \ldots$ is convergent if the product*

$$(1 + |a_1|)(1 + |a_2|) \ldots (1 + |a_n|) \ldots$$

is convergent, that is if $\sum_{n=1} |a_n|$ is convergent.

The converse of this theorem does not hold good; thus

$$(1 + a_1) \dots (1 + a_n) \dots$$

may be convergent whilst $(1 + |a_1|) \dots (1 + |a_n|) \dots$ is divergent.

To prove the theorem, we have

$$(1 + a_{n+1})(1 + a_{n+2}) \dots (1 + a_{n+m}) - 1 = a_{n+1} + a_{n+2} + a_{n+1}a_{n+2} + \dots;$$

hence

$$\begin{aligned} |(1 + a_{n+1})(1 + a_{n+2}) \dots (1 + a_{n+m}) - 1| & \\ &\leqq |a_{n+1}| + |a_{n+2}| + |a_{n+1}a_{n+2}| + \dots \\ &\leqq (1 + |a_{n+1}|)(1 + |a_{n+2}|) \dots (1 + |a_{n+m}|) - 1 \\ &< \epsilon, \end{aligned}$$

for all values of n that are sufficiently large, and for $m = 1, 2, 3, \dots$. Thus the condition of convergence of the product $(1 + a_1) \dots (1 + a_n) \dots$ is satisfied. In particular, if $a_1, a_2, \dots a_n, \dots$ are all positive and

$$(1 + a_1)(1 + a_2) \dots (1 + a_n) \dots$$

is convergent, so also is

$$(1 - a_1)(1 - a_2) \dots (1 - a_n) \dots .$$

The convergence of the product $(1 + a_1)(1 + a_2) \dots (1 + a_n) \dots$ is said to be *absolute*, in case the product $(1 + |a_1|)(1 + |a_2|) \dots (1 + |a_n|) \dots$ is convergent, which is the case if the series $\Sigma |a_n|$ is convergent.

It can be shewn that:

An infinite product which converges absolutely is also unconditionally convergent, and conversely, if the convergence is unconditional it is absolute.

Let p_n, p_n' denote the product of the first n factors in the original and the new orders. Choosing any value of n, a greater value n_1 can be so chosen that all the factors in p_n' are contained in p_{n_1}; we can regard n_1 as a function of n. If $\bar{p}_n, \bar{p}_n'$ are the corresponding products when $|a_n|$ is taken in each factor instead of a_n, we see that all the factors in $\bar{p}_n'$ are contained in $\bar{p}_{n_1}$. Since $\dfrac{\bar{p}_{n_1}}{\bar{p}_n'} - 1$ consists of the same terms as $\dfrac{p_{n_1}}{p_n'} - 1$, but with all the terms positive, we have

$$\left| \frac{p_{n_1}}{p_n'} - 1 \right| \leqq \frac{\bar{p}_{n_1}}{\bar{p}_n'} - 1.$$

Since $p_n' \leqq \bar{p}_n'$, we have

$$|p_{n_1} - p_n'| \leqq \bar{p}_{n_1} - \bar{p}_n'.$$

On account of the absolute and unconditional convergence of

$$(1 + |a_1|)(1 + |a_2|) \dots,$$

we see that $|p_{n_1} - p_n'| < \epsilon$, for all sufficiently great values of n. Now p_{n_1} differs from p by less than ϵ, if n be taken sufficiently large, hence p_n'

differs from p by less than 2ϵ, for all sufficiently large values of n, and thus $\lim_{n\sim\infty} p_n' = p = \lim_{n\sim\infty} p_n$. The convergence of the absolutely convergent product has thus been shewn to be unconditional.

Next, let it be assumed that the product converges unconditionally. Let those factors for which $a_n \geqq 0$ be denoted by

$$1 + b_1,\ 1 + b_2,\ \ldots,\ 1 + b_r,\ \ldots,$$

and those for which $a_n < 0$, by

$$1 - c_1,\ 1 - c_2,\ \ldots,\ 1 - c_s,\ \ldots.$$

By hypothesis the product

$$(1 + b_1)(1 + b_2)\ldots(1 + b_r)(1 - c_1)(1 - c_2)\ldots(1 - c_s)$$

converges to a definite number p, when to r and s are assigned the corresponding values r_n, s_n in any two divergent increasing sequences $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$; the number p being independent of the particular sequences chosen. If $(1 + b_1)(1 + b_2)\ldots(1 + b_r)$ diverges as $r \sim \infty$, it is clear that $(1 - c_1)(1 - c_2)\ldots(1 - c_s)$ diverges to zero as $s \sim \infty$, for otherwise the product $(1 + b_1)\ldots(1 + b_r)(1 - c_1)\ldots(1 - c_s)$ could not be convergent; and thus both the series Σb_n, Σc_n are divergent. When this is the case, it can be shewn that sequences of values of r and s can be so chosen that the product converges to any assigned value A which may differ from p, and thus the given product is not unconditionally convergent. Let r_1 be the smallest value of r such that $(1 + b_1)(1 + b_2)\ldots(1 + b_{r_1}) > A$, and s_1 the smallest value of s such that

$$(1 + b_1)(1 + b_2)\ldots(1 + b_{r_1})(1 - c_1)(1 - c_2)\ldots(1 - c_{s_1})$$

is $< A$; then let r_2 be the smallest value of r such that

$$(1 + b_1)\ldots(1 + b_{r_2})(1 - c_1)(1 - c_2)\ldots(1 - c_{s_1}) > A,$$

and s_2 the smallest value of s such that

$$(1 + b_1)\ldots(1 + b_{r_2})(1 - c_1)\ldots(1 - c_{s_2})$$

is $< A$. Proceeding in this manner we obtain a sequence of the numbers

$$(1 + b_1)(1 + b_2)\ldots(1 + b_{r_n})(1 - c_1)(1 - c_2)\ldots(1 - c_{s_n}),$$

where $n = 1, 2, 3, \ldots$.

The ratio of

$$(1 + b_1)(1 + b_2)\ldots(1 + b_{r_n})(1 - c_1)(1 - c_2)\ldots(1 - c_{s_n})$$

to A is less than 1, and greater than $(1 - c_{s_n})$. Since c_{s_n} converges to zero as $n \sim \infty$, on account of the convergence of the product, it follows that the ratio converges to 1; hence a sequence of products has been obtained which converges to the arbitrarily chosen number A. This is inconsistent with the assumption that the product converges to p, whatever sequences of values are assigned to r and s. It follows that the product

$$(1 + b_1)(1 + b_2)\ldots(1 + b_r)\ldots$$

is convergent, and thus that the series Σb_r is convergent. Also

$$(1 - c_1)(1 - c_2) \ldots (1 - c_s) \ldots$$

must accordingly be convergent, and consequently the series Σc_s is convergent. It now follows that the product $(1 + a_1)(1 + a_2) \ldots$ is absolutely convergent, since $\Sigma \mid a_n \mid = \Sigma b_n + \Sigma c_n$. It has thus been shewn that an unconditionally convergent product is also absolutely convergent*. This is the analogue of the theorem of § 25.

It follows from the above theorem that, *if a_n is $\geqq 0$ for all values of n, and the product $(1 - a_1)(1 - a_2) \ldots (1 - a_n) \ldots$ is convergent, then the convergence is unconditional.*

By a modification of the process in the above proof it can be shewn that the terms of a conditionally convergent product can be so arranged that the new product either diverges, or so that it oscillates with assigned limits of indeterminancy, as in the case of conditionally convergent series (see § 26).

43. The following theorem will be established, which is due to Cauchy†:

If the series Σa_n is conditionally convergent, then the product

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \ldots$$

is convergent, or diverges towards zero, according as the series Σa_n^2 is convergent or divergent.

Employing Maclaurin's theorem (§ 142) we have

$$\log(1 + a_n) = a_n - \frac{a_n^2}{2(1 + \theta_n a_n)^2}, \text{ where } 0 < \theta_n < 1.$$

Hence $$\sum_{n=1} \log(1 + a_n) = \sum_{n=1} a_n - \frac{1}{2}\sum_{n=1} \frac{a_n^2}{(1 + \theta_n a_n)^2}.$$

If a_n be positive $\frac{a_n^2}{(1 + \theta_n a_n)^2} < a_n^2$; and if a_n be negative, it must, except possibly for a finite set of values of n, be > -1, so that

$$1 + \theta_n a_n > 1 + a_n > g,$$

and thus $\frac{a_n^2}{(1 + \theta_n a_n)^2} < \frac{a_n^2}{g^2}$. In case Σa_n^2 is convergent, it is now clear that $\Sigma \frac{a_n^2}{(1 + \theta_n a_n)^2}$ is convergent, and it then follows that $\Sigma \log(1 + a_n)$ is convergent, and thus the product $(1 + a_1)(1 + a_2) \ldots$ is convergent.

* The proof of this theorem given by Pringsheim in *Math. Annalen*, vol. XXXIII (1889), p. 140, contains a hiatus. He argues that, if an unconditionally convergent product is split in any manner into two V_{n_1}, W_{n_2} such that $\lim V_{n_1} W_{n_2} = U$, when n_1, n_2 diverge to infinity independently of one another, then $\mid V_{n_1} W_{n_2} - U \mid < \delta$, if $n_1 \geqq N_1$, $n_2 \geqq N_2$. This amounts to the assumption that $V_{n_1} W_{n_2}$ converges to U, not only for all pairs of sequences of values of n_1 and n_2, but also *uniformly* for all such pairs of sequences; that this is the case does not appear to be immediately obvious.

† *Cours d'Analyse algébrique* (1821), p. 563.

We have also, when Σa_n^2 is divergent if a_n is positive,

$$\frac{a_n^2}{(1+\theta_n a_n)^2} > \frac{a_n^2}{(1+a_n)^2} > \frac{a_n^2}{G^2},$$

where $1 + a_n < G$, for all values of n. Since a_n must be positive for an infinite set of values of n, it follows that $\Sigma \frac{a_n^2}{(1+\theta_n a_n)^2}$ is divergent, and therefore the series $\Sigma \log(1 + a_n)$ diverges to $-\infty$, and the product $(1 + a_1)(1 + a_2) \ldots$ diverges to zero.

It is clear that, if Σa_n^2 is convergent, the series $\Sigma \log(1 + a_n)$, and therefore also the product $(1 + a_1)(1 + a_2) \ldots$, oscillates or diverges in case the series Σa_n oscillates or diverges.

A proof has been given of the above theorem by Pringsheim, in which the series of logarithms are not employed (*loc. cit.* p. 150).

EXAMPLE.

If $a > 0$, the product $\left(1 + \frac{a}{1}\right)\left(1 + \frac{a}{2}\right) \ldots \left(1 + \frac{a}{n}\right) \ldots$ and the product

$$\left(1 - \frac{a}{1}\right)\left(1 - \frac{a}{2}\right) \ldots \left(1 - \frac{a}{n}\right) \ldots$$

are both divergent, since $\sum\limits_{n=1} \frac{1}{n}$ is divergent.

The series $\sum\limits_{n=1} \left\{\log\left(1 + \frac{a}{n}\right) - \frac{a}{n}\right\}$ is convergent, as has been shewn above; thus

$$\left(1 + \frac{a}{1}\right)e^{-a}\left(1 + \frac{a}{2}\right)e^{-\frac{a}{2}} \ldots \left(1 + \frac{a}{n}\right)e^{-\frac{a}{n}}$$

converges as $n \sim \infty$. Since $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \log n$ converges to a fixed number C, it follows that $n^{-a}\left(1 + \frac{a}{1}\right)\left(1 + \frac{a}{2}\right) \ldots \left(1 + \frac{a}{n}\right)$ is convergent.

The product $\frac{n^a\, n!}{(a+1)(a+2)\ldots(a+n)}$ is accordingly convergent as $n \sim \infty$; it is defined to be the Gaussian function $\Pi(a)$.

THE SUMMABILITY OF SERIES.

44. Various definitions have been introduced in recent years of what may be termed the conventional sum of an arithmetic series. Such a conventional sum should satisfy the *consistency condition*, that its value for a convergent series coincides with the ordinary sum of the series, whilst the conventional sum should have a definite value for oscillating series belonging to some class wider than, but including, the class of convergent series. The value of the introduction of such conventional sums will be clearly exhibited when we consider the case of series of which the terms are functions of a variable, as for example, in the case of Fourier's series, or in that of power series.

Assuming that the consistency condition is satisfied by a particular conventional sum of a series, the utility of such conventional sum will be affected by the fact of its possessing or not possessing various simple properties which appertain to the ordinary sums of convergent series. Examples of such properties are the following:

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$

$$k \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} k a_n$$

$$0 + a_1 + a_2 + \ldots = a_1 + a_2 + a_3 + \ldots,$$

$$a_1 + a_2 + \ldots = a_1 + (a_2 + a_3 + \ldots).$$

It can be seen that all these properties hold good for the conventional sum as defined by Cesàro, which will be considered here, but the last of them is not necessarily satisfied* in the case of Borel's definition (§ 46); for it can be seen that the series of which a_1 is the first term may be summable, but not that of which a_2 is the first term. None of the conventional sums have the property denoted by

$$a_1 + a_2 + a_3 + \ldots = (a_1 + a_2) + (a_3 + a_4) + \ldots;$$

which holds good for the ordinary sums of convergent series.

The simplest definition of a conventional sum of an arithmetic series is that involved in the summation by arithmetic means, which has already been treated in some detail in §§ 27, 28. This definition has been extended in two different manners by Hölder† and by Cesàro‡. Taking Hölder's method first, let $\frac{s_1 + s_2 + \ldots + s_n}{n}$ be denoted by $h_n^{(1)}$, and let

$$\frac{h_1^{(1)} + h_2^{(1)} + \ldots + h_n^{(1)}}{n}$$

be denoted by $h_n^{(2)}$; and generally let $h_n^{(k)}$ denote

$$\frac{h_1^{(k-1)} + h_2^{(k-1)} + \ldots + h_n^{(k-1)}}{n};$$

where k is any positive integer. In case $\lim_{n\sim\infty} h_n^{(k)}$ has a definite value, for some integral value of k, the series $\sum_{n=1} a_n$ is said to be summable in accordance with Hölder's definition, of order k, or to be summable (H, k), and the value of $\lim_{n\sim\infty} h_n^{(k)}$ is said to be the sum (H, k) of the series.

Cesàro's own extension of the method of arithmetic means is as follows: Let s_n be denoted by $s_n^{(0)}$; let $s_1^{(0)} + s_2^{(0)} + \ldots + s_n^{(0)}$ be denoted by $s_n^{(1)}$; and generally let $s_1^{(k-1)} + s_2^{(k-1)} + \ldots + s_n^{(k-1)}$ be denoted by $s_n^{(k)}$, for each positive integral value of k.

* See Hardy, *Camb. Phil. Trans.* vol. XIX (1904), p. 300.

† *Math. Annalen*, vol. XX (1882), p. 535. ‡ *Bulletin d. Sc. Mat.* vol. XIV (1890).

Let $C_n^{(k)} = \dfrac{k!\,(n-1)!}{(n+k-1)!}\, s_n^{(k)}$; then if, for some positive integral value of k, $\lim\limits_{n\sim\infty} C_n^{(k)}$ has a definite value, the series is said to be summable (C, k), that is to be summable by Cesàro's method of order k; the sum (C, k) of the series is then taken to be the limit of $C_n^{(k)}$, as $n \sim \infty$.

It can easily be shewn by means of the theorem given in § 6 that, when the series Σa_n is convergent, so that $\lim s_n = s$, both $\lim\limits_{n\sim\infty} C_n^{(k)}$ and $\lim\limits_{n\sim\infty} h_n^{(k)}$ exist, and have the value s.

For, if $\alpha_n = s_n^{(k)}$, $\beta_n = \dfrac{(n+k-1)!}{k!\,(n-1)!}$, we have $\alpha_n - \alpha_{n-1} = s_n^{(k-1)}$, $\beta_n - \beta_{n-1} = \dfrac{(n+k-2)!}{(k-1)!\,(n-1)!}$. Thus, if $C_n^{(k-1)}$ converges, as $n \sim \infty$, so also does $C_n^{(k)}$, and the two limits are the same; letting k have the values $r, r-1, \ldots 1$, successively, we see that if $C_n^{(0)}$ converges, so also does $C_n^{(r)}$, and both have the same limit.

Again, if $\alpha_n = nh_n^{(k)}$, $\beta_n = n$, we have $\alpha_n - \alpha_{n-1} = h_n^{(k-1)}$, $\beta_n - \beta_{n-1} = 1$; and thus, if $h_n^{(k-1)}$ converges, as $n \sim \infty$, so also does $h_n^{(k)}$, and the two limits are the same; letting k have the values $r, r-1, \ldots 1$, successively, we see that if $h_n^{(0)}$, or s_n, converges, so also does $h_n^{(k)}$, and the two limits are the same.

It was first shewn by Knopp* that a series which is summable (H, k) is necessarily summable (C, k). The converse was established by Schnee† and by Ford‡. The simplest proof of the complete equivalence of the two definitions of summability of order k is that given by Schur§; this will be given in a modified form in § 55. A proof has also been given|| by Hahn. In view of this complete equivalence it is unnecessary to consider any further Hölder's method. The method of Cesàro, in an extended form, in which k is not necessarily a positive integer, will be dealt with below.

45. Another method of summation was introduced by M. Riesz. Let $[n]$ denote the greatest integer less than a positive variable n, and let $\lambda(n)$ denote a positive monotone function of n, which diverges to ∞ with n.

Let $$\sum_{m=0}^{[n]} u_m \left\{1 - \frac{\lambda(m)}{\lambda(n)}\right\}^r = \Sigma_n^{(r)};$$

then, if $\Sigma_n^{(r)}$ has a definite limit as the continuous variable n is increased

* *Grenzwerte von Reihen bei der Annäherung an die Convergenzgrenze*, Inaugural Dissertation, Berlin, 1907.

† *Math. Annalen*, vol. LXVII (1909), p. 110.

‡ *Amer. Journ. of Math.* vol. XXXII (1910), p. 315.

§ *Math. Annalen*, vol. LXXIV (1913), p. 447; see also a memoir by Knopp in the same volume, p. 459.

|| *Monatshefte f. Math. u. Physik*, vol. XXXIII (1923), p. 135.

indefinitely, the series $u_0 + u_1 + u_2 + \dots$ is said to be summable (R, λ, r), or summable by Riesz's method of order r. The order r is not necessarily a positive integer; but may have any value > 0.

The simplest and most important case of Riesz's method is that which $\lambda(n) = n$; in this case

$$\Sigma_n^{(r)} = \sum_{m=0}^{m=[n]} u_m \left(1 - \frac{m}{n}\right)^r;$$

if $\Sigma_n^{(r)}$ has a definite limit as the continuous variable n is indefinitely increased, we shall say that the series $\sum\limits_{n=0} u_n$ is summable (R, r); r denoting any positive number. It will hereafter be shewn (§ 58) that this method of Riesz is completely equivalent, for positive values of r, to that of Cesàro, when the latter is extended to include non-integral values of r. Another important case of Riesz's general method of defining a conventional sum is obtained by taking $\lambda(n) = \log_e n$. Riesz's method has received an important application* to the case of Dirichlet's series $\sum\limits_{n=1} a_n n^{-s}$, or more generally to the series $\sum\limits_{n=1} a_n e^{-\lambda_n s}$, where $\lambda_1, \lambda_2, \dots$ is a diverging sequence of real increasing numbers.

46. A general mode† of defining conventional sums of a series depends upon a principle which may be stated as follows:

If a repeated limit of a function of two variables has a definite value X, but the repeated limit taken in the reverse order has no definite value, it may be agreed to consider X as the conventional value of the second repeated limit. A method which was employed by Poisson in connection with certain series may be considered as an example of this method. Let us consider the repeated limits of

$$u_0 + u_1 h + u_2 h^2 + \dots + u_n h^n,$$

where $0 \leq h < 1$, as a function of the two variables n and h. It may happen that the series $u_0 + u_1 h + \dots + u_n h^n + \dots$ is convergent for $0 \leq h < 1$, and has $S(h)$ for its sum; thus

$$S(h) = \lim_{n \sim \infty} (u_0 + u_1 h + u_2 h^2 + \dots + u_n h^n).$$

If $S(h)$ converges to a definite number s, as $h \sim 1$, we have

$$s = \lim_{h \sim 1} \lim_{n \sim \infty} (u_0 + u_1 h + u_2 h^2 + \dots + u_n h^n).$$

It may happen that the series $u_0 + u_1 + u_2 + \dots$ is not convergent, so that

$$\lim_{n \sim \infty} \lim_{h \sim 1} (u_0 + u_1 h + \dots + u_n h^n)$$

has no definite value. In that case s may be regarded as the conventional Poisson sum of the series $u_0 + u_1 + u_2 + \dots$.

* See the Cambridge tract by G. H. Hardy and M. Riesz on *The general theory of Dirichlet's series*, p. 21.

† See Hardy, *Quarterly Journal*, vol. xxxv (1904), p. 22.

As another example of the principle, the conventional value of

$$\lim_{n\sim\infty}\lim_{h\sim\infty}\int_0^h f(x, n)\,dx, \text{ or } \lim_{n\sim\infty}\int_0^\infty f(x, n)\,dx,$$

may be taken to be the value of $\lim\limits_{h\sim\infty}\lim\limits_{n\sim\infty}\int_0^h f(x, n)\,dx$, assuming that this latter repeated limit exists.

If $$f(x, n) = e^{-x}\left(a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \dots + a_n\frac{x^n}{n!}\right);$$

we have $$\int_0^\infty e^{-x}\left(a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \dots + a_n\frac{x^n}{n!}\right)dx$$
$$= a_0 + a_1 + a_2 + \dots + a_n.$$

Thus the conventional sum of the series $\sum\limits_{n=0} a_n$ is taken to be

$$\lim_{h\sim\infty}\lim_{n\sim\infty}\int_0^h e^{-x}\left(a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!}\right)dx.$$

If now the series $a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!} + \dots$ converges for all values of x in the interval $(0, h)$, we have (see § 214 (1)), since the convergence is uniform,

$$\lim_{n\sim\infty}\int_0^h e^{-x}\left(a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!}\right)dx$$
$$= \int_0^h e^{-x}\left(a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!} + \dots\right)dx.$$

Thus the conventional sum of the series $a_0 + a_1 + a_2 + \dots$ will be taken to be

$$\int_0^\infty e^{-x}\left(a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!} + \dots\right)dx,$$

where it is assumed that this integral exists.

We have thus the method of Borel, in accordance with which the conventional sum of the series $a_0 + a_1 + a_2 + \dots$ is $\int_0^\infty e^{-x}u(x)\,dx$, provided this integral exists; where $u(x)$ denotes the sum of the series

$$a_0 + a_1\frac{x}{1!} + \dots + a_n\frac{x^n}{n!} + \dots,$$

which is assumed to be convergent for all values of x.

It is necessary to shew that this holds good in case $a_0 + a_1 + a_2 + \dots$ is convergent, and that the conventional sum then agrees with the ordinary sum. For a general theory of this method of summation, and for its relations with other methods, reference may be made to a treatise by Borel*, where, however, the question of consistency is not considered.

* *Leçons sur les séries divergentes*, Paris (1901), p. 98. See also a memoir by G. H. Hardy and S. Chapman, *Quarterly Journal*, vol. XLII (1911), p. 181, and further, a memoir by G. H. Hardy and Littlewood, *Proc. Lond. Math. Soc.* (2), vol. XI (1911), p. 1. See also Bromwich's *Theory of Infinite Series*, pp. 267–273.

EXTENSION OF CESÀRO'S THEORY OF SUMMABILITY.

47. The theory of summability introduced by Cesàro (see § 44) has been extended by S. Chapman*, and Knopp† to the case in which the order of summability may be any number, not necessarily integral.

It is convenient to denote the series by $a_0 + a_1 + a_2 + \dots + a_n + \dots$, and $a_0, a_0 + a_1, \dots, a_0 + a_1 + \dots + a_n$, by $s_0, s_1, \dots s_n$.

Let $S_n^{(r)}$ denote

$$s_n + \binom{r}{1} s_{n-1} + \binom{r+1}{2} s_{n-2} + \binom{r+2}{3} s_{n-3} + \dots + \binom{r+n-1}{n} s_0, \quad \dots(1)$$

which is equivalent to

$$a_n + \binom{r+1}{1} a_{n-1} + \binom{r+2}{2} a_{n-2} + \dots + \binom{r+n}{n} a_0; \quad \dots\dots(2)$$

where $\binom{r+s}{s}$ denotes

$$\frac{(r+1)(r+2)\dots(r+s)}{s!}, \text{ or } \frac{\Gamma(r+s+1)}{\Gamma(s+1)\,\Gamma(r+1)}.$$

We then define $C_n^{(r)}$ to be equal to $S_n^{(r)} \Big/ \binom{r+n}{n}$; where r may have any real (or also complex) value, except that negative integral values are excluded. In the following investigations r will in general be confined to have real values > -1. If, for any such value of r, $\lim_{n \sim \infty} C_n^{(r)}$ has a definite value, that value is said to be the Cesàro sum, of order r, of the given series, or to be the sum (C, r) of that series.

It will be seen that when r is a positive integer, this definition is in agreement with that of § 44, allowance being made for the difference of notation.

A series which is summable $(C, 0)$ is by definition convergent; and such a series may be summable (C, r) for values of r which are negative. The consideration of such cases may be expected to throw light upon the nature of the convergence of such a convergent series.

In case $|C_n^{(r)}|$ is bounded for all values of n, the series is said to be bounded (C, r).

Taking Stirling's‡ asymptotic value of $\Gamma(1 + x)$, which is

$$e^{-x} x^x (2\pi x)^{\frac{1}{2}} (1 + \epsilon_x),$$

* *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 369.

† Inaugural Dissertation, Berlin, 1907. Also *Archiv d. Math. u. Physik* (3), vol. XII (1907). A very complete treatment of the Cesàro method for unrestricted orders is contained in A. F. Andersen's work, *Studier over Cesàro's Summabilitetsmethode*, Copenhagen, 1921. For the case of double series see C. N. Moore, *Trans. Amer. Math. Soc.* vol. XVI (1913), p. 73.

‡ A proof of Stirling's theorem is given in Bromwich's *Theory of Infinite Series*, p. 461.

where $\epsilon_x \sim 0$, as $x \sim \infty$, we have

$$\binom{r+n}{n} = \frac{1}{\Gamma(r+1)} \frac{\Gamma(r+n+1)}{\Gamma(n+1)} = \frac{n^r}{\Gamma(r+1)}(1+\zeta_n),$$

where $\zeta_n \sim 0$, as $n \sim \infty$. It thus appears that

$$C_n^{(r)} = \Gamma(r+1)\frac{S_n^{(r)}}{n^r}(1+\zeta_n'),$$

where $\zeta_n' \sim 0$, as $n \sim \infty$. Accordingly, $\frac{S_n^{(r)}}{n^r}$ has a definite limit, when the series is summable (C, r); and it is bounded when the series is bounded (C, r).

48. Since the series

$$1 + \binom{r}{1}x + \binom{r+1}{2}x^2 + \ldots + \binom{r+n-1}{n}x^n + \ldots$$

is absolutely convergent when $|x| < 1$, and has the sum $(1-x)^{-r}$, if the series is multiplied by the finite series $s_0 + s_1x + s_2x^2 + \ldots + s_nx^n$, the Cauchy-product series is absolutely convergent, and has for its sum the product of the sums of the two series that are multiplied together. The coefficient of x^n in the product series is $S_n^{(r)}$. Hence

$$(s_0 + s_1x + \ldots + s_nx^n)(1-x)^{-r}$$

is the sum of an absolutely convergent series, of which the first $n+1$ terms are

$$S_0^{(r)} + S_1^{(r)}x + S_2^{(r)}x^2 + \ldots + S_n^{(r)}x^n.$$

No assumption is here made as regards the convergence of the infinite series $\sum\limits_{n=0} S_n^{(r)}x^n$.

It follows that $s_0 + s_1x + \ldots + s_nx^n$ is the product of $(1-x)^r$ into the sum of the series of which the first $n+1$ terms have been stated. Therefore

$$s_n = S_n^{(r)} - \binom{r}{1}S_{n-1}^{(r)} + \binom{r}{2}S_{n-2}^{(r)} - \ldots + (-1)^n\binom{r}{n}S_0^{(r)}; \quad \ldots\ldots(3)$$

and this holds for every value of n. In case r is a positive integer, the series stops after $r+1$ terms, if $n > r$. In a precisely similar manner, by the employment of the series for $(1-x)^{-(r+1)}$, it is found that

$$a_n = S_n^{(r)} - \binom{r+1}{1}S_{n-1}^{(r)} + \binom{r+1}{2}S_{n-2}^{(r)} - \ldots + (-1)^n\binom{r+1}{n}S_0^{(r)}, \ldots(4)'$$

where the series stops after $r+2$ terms, in case r is a positive integer $< n-1$.

When $|x| < 1$, it has been shewn that the sum of a certain absolutely convergent series of which the sum of the first $n+1$ terms is

$$S_0^{(r)} + S_1^{(r)}x + \ldots + S_n^{(r)}x^n \text{ is } (1-x)^{-r}\{s_0 + s_1x + \ldots + s_nx^n\},$$

which is equal to

$$(1-x)^{-(r-r')}\{(1-x)^{-r'}(s_0 + s_1x + \ldots + s_nx^n)\}.$$

This last expression is the product of $(1-x)^{-(r-r')}$ into the sum of an absolutely convergent of which the sum of the first $n+1$ terms is

$$S_0^{(r')} + S_1^{(r')}x + \dots + S_n^{(r')}x^n.$$

The Cauchy-product of this last series and the series for $(1-x)^{-(r-r')}$ is a series of which the sum of the first $n+1$ terms is

$$S_0^{(r)} + S_1^{(r)}x + \dots + S_n^{(r)}x^n.$$

There cannot be two different series in powers of x which converge, for $|x| < 1$ to the same sum (see § 134); it follows that

$$S_n^{(r)} = S_n^{(r')} + \binom{r-r'}{1} S_{n-1}^{(r')} + \binom{r-r'+1}{2} S_{n-2}^{(r')} + \dots \\ + \binom{r-r'+n-1}{n} S_0^{(r')}. \quad \dots\dots(5)$$

We shall assume that $r' < r$, in this expression. In case $r' > r$, we have $\binom{r-r'+p-1}{p} = (-1)^p \binom{r'-r}{p}$, and the formula becomes, on interchange of r and r',

$$S_n^{(r')} = S_n^{(r)} - \binom{r-r'}{1} S_{n-1}^{(r)} + \binom{r-r'}{2} S_{n-2}^{(r)} - \dots + (-1)^n \binom{r-r'}{n} S_0^{(r)}, \\ \dots\dots(6)$$

where we assume that $r' < r$. In case $r-r'$ is an integer less than n, the expression breaks off after $r-r'+1$ terms.

If, in (5), we have $r > 0$, we may take $r' = 0$, and the formula reduces to (1) in § 47. If $r > 1$, we may take $r' = -1$; and since $S_n^{(-1)} = a_n$, the formula reduces to (2).

If, in (6), we take $r > 0$, $r' = 0$, the formula reduces to (3); and if $r > -1$, $r' = -1$, it reduces to (4).

49. If, in (6), we write $r+1$ for r, and r for r', we have

$$S_n^{(r)} = S_n^{(r+1)} - S_{n-1}^{(r+1)}.$$

Taking also the relation

$$\binom{r+n}{n} = \binom{r+n+1}{n} - \binom{r+n}{n-1},$$

it is seen, by employing Stolz's theorem given in § 6, that:

If a series is summable (C, r), *where* $r > -1$, *it is also summable* $(C, r+1)$.

For we have only to write $a_n = S_{n-1}^{(r+1)}$, $\beta_n = \binom{r+n}{n-1}$ in the theorem of § 6.

This theorem is only a particular case of the more general theorem* that:

If a series is summable (C, r), it is also summable (C, r'), where r, r' are any real numbers such that $r' > r > -1$; and the sums (C, r), (C, r') have one and the same value.

In order to prove this theorem, the following theorem, due to Cesàro, will be established:

If $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of numbers such that $\frac{\alpha_n}{n^r}$, $\frac{\beta_n}{n^s}$ converge respectively to definite limits α, β, as $n \sim \infty$, then

$$\lim_{n\sim\infty} \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \ldots + \alpha_n\beta_1}{n^{r+s+1}} = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\alpha\beta,$$

provided r and s are numbers both > -1.

Let $\alpha_n = \alpha n^r(1+\eta_n)$, $\beta_n = \beta n^s(1+\zeta_n)$; then, since η_n, ζ_n both converge to zero as $n \sim \infty$, they are both bounded; thus $|\eta_n| < A$, $|\zeta_n| < B$, where A and B are fixed numbers.

Since $\binom{r+n}{n} = \frac{n^r}{\Gamma(r+1)}(1+\epsilon_n)$, $\binom{s+n}{n} = \frac{n^s}{\Gamma(s+1)}(1+\delta_n)$, where ϵ_n, δ_n converge to zero, and are therefore bounded for all values of n, we have

$$\alpha_n = \Gamma(r+1)\,\alpha\,(1+\epsilon_n')\binom{r+n}{n}, \quad \beta_n = \Gamma(s+1)\,\beta\,(1+\delta_n')\binom{s+n}{n},$$

where ϵ_n', δ_n' are bounded; and thus $|\epsilon_n'| < A'$, $|\delta_n'| < B'$. We now have

$$\alpha_1\beta_n + \alpha_2\beta_{n-1} + \ldots + \alpha_n\beta_1 = \Gamma(r+1)\,\Gamma(s+1)\,\alpha\beta \\ \times \sum_{t=1}^{t=n}(1+\epsilon'_t)(1+\delta'_{n-t+1})\binom{r+t}{t}\binom{s+n-t+1}{n-t+1}.$$

Since $\sum_{t=1}^{t=n}\binom{r+t}{t}\binom{s+n-t+1}{n-t+1}$, being the coefficient of x^n in the product $(1-x)^{-(r+1)}(1-x)^{-(s+1)}$, or $(1-x)^{-(r+s+2)}$, is equal to $\binom{r+s+n+1}{n}$, we see that the part of $\frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \ldots + \alpha_n\beta_1}{n^{r+s+1}}$ obtained by omitting ϵ_p', δ'_{n-t+1} is

$$\Gamma(r+1)\,\Gamma(s+1)\,\alpha\beta\binom{r+s+n+1}{n}\frac{1}{n^{r+s+1}},$$

which converges, as $n \sim \infty$, to $\frac{\alpha\beta\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}$.

Consider next the sum $\sum_{t=1}^{t=n}\epsilon_t'\binom{r+t}{t}\binom{s+n-t+1}{n-t+1}$; we may choose

* See Knopp, Inaugural Dissertation, Berlin (1907), p. 46; also *Archiv d. Math. u. Physik* (3), vol. XII (1907); the theorem is also proved by Chapman, *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 377. See also Ottolenghi, *Giorn. Battaligni* (3), vol. XLIX (1911), p. 239.

the integer m so that $|\epsilon_t'| < \eta$, for $t > m$; and we may take n to be greater than m. The part of the sum taken from $t = 1$ to $t = m$ is less than $A' \sum_{t=1}^{t=m} \binom{r+t}{t} \binom{s+n-t+1}{n-t+1}$, or than

$$A' \sum_{t=1}^{t=m} \binom{r+t}{t} \frac{(n-t+1)^s}{\Gamma(s+1)} (1 + \delta_{n-t+1});$$

when this is multiplied by $\frac{1}{n^{r+s+1}}$, it converges to zero, as $n \sim \infty$, the integer m being kept fixed. The part of the sum from $t = m+1$ to $t = n$ is numerically less than $\eta \sum_{t=1}^{t=n} \binom{r+t}{t} \binom{s+n-t+1}{n-t+1}$; and when this is multiplied by $\frac{1}{n^{r+s+1}}$, it converges, as $n \sim \infty$, to a fixed multiple of η. The sum $\sum_{t=1}^{t=n} \delta'_{n-t+1} \binom{r+t}{t} \binom{s+n-t+1}{n-t+1}$ can be divided into two parts by taking m so that $|\delta'_{n-t+1}| < \eta$, for $t < n - m$, and considering separately the sums for $t = 1$ to $n - m - 1$, and from $t = n - m$ to $t = n$; as before, when $n \sim \infty$, the sum when multiplied by $\frac{1}{n^{s+t+1}}$, converges to a fixed multiple of η.

We have lastly to consider

$$\frac{1}{n^{r+s+1}} \sum_{t=1}^{t=n} \epsilon_t' \delta'_{t-n+1} \binom{r+t}{t} \binom{s+n-t+1}{n-t+1};$$

this is numerically less than

$$B' \cdot \frac{1}{n^{r+s+1}} \sum_{t=1}^{t=n} |\epsilon_t'| \binom{r+t}{t} \binom{s+n-t+1}{n-t+1};$$

and this has been shewn to converge to a fixed multiple of η. Since $\frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \dots + \alpha_n\beta_1}{n^{r+s+1}}$ has been shewn to have upper and lower limits which differ from $\frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \alpha\beta$ by a fixed multiple of the arbitrarily chosen number η, Cesàro's theorem has been established.

In this theorem, let α_n denote $S_n^{(r)}$, where $\frac{S_n^{(r)}}{n^r}$ is assumed to converge to the definite limit $\frac{s^{(r)}}{\Gamma(r+1)}$; the series being taken to be summable (C, r); and let $\beta_n = \binom{r'-r+n-1}{n}$, and is thus such that $\frac{\beta_n}{n^{r'-r-1}}$ converges to $\frac{1}{\Gamma(r'-r)}$; when $r' - r - 1 = s$, and $r' > r$.

By employing the expression (5), of § 48, for $S_n^{(r')}$, we see that $\frac{S_n^{(r')}}{n^{r'}}$

converges to $\dfrac{s^{(r)}}{\Gamma(r'+1)}$; and therefore $S_n^{(r')}\Big/\dbinom{r'+n}{n}$ converges to $s^{(r)}$, the sum (C, r) of the given series.

The particular case of Cesàro's theorem which arises when $r = 0$, $s = 0$ is that:

If $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequences of numbers which converge to* α *and* β, *respectively, then* $\dfrac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \dots + \alpha_n\beta_1}{n}$ *converges to* $\alpha\beta$.

50. *If a series is divergent* (C, r), *where* $r > -1$, *it is also divergent* $(C, r+1)$.

By divergent (C, r) is meant that the Cesàro sum (C, r) is $+\infty$, or $-\infty$.

The condition of the theorem is that $\lim\limits_{n\sim\infty} \dfrac{S_n^{(r)}}{\dbinom{n+r}{n}} = +\infty$, or $-\infty$, or

$\lim\limits_{n\sim\infty} \dfrac{S_n^{(r+1)} - S_{n-1}^{(r+1)}}{\dbinom{n+r+1}{n} - \dbinom{n+r}{n-1}} = \infty$, or $-\infty$. Applying the theorem of § 6, it follows that $\lim\limits_{n\sim\infty} \dfrac{S_n^{(r+1)}}{\dbinom{n+r+1}{n}} = +\infty$, or $-\infty$; therefore the given series is divergent $(C, r+1)$.

Now, if $r = 0$, it follows that, if the given series is divergent, the sum $(C, 1)$ is infinite. This has already been shewn to be the case in § 27. It now follows, by letting r successively have the values 1, 2, 3, ..., that the Cesàro sums of all integral orders are infinite. By employing the theorem of § 49, it now follows that the series cannot be summable for any positive value of r, for if it were so it would also be summable of order the integer next greater than r. Hence we have the theorem that:

If a series is summable (C, r), *for any positive value of* r, *the given series either converges or oscillates between finite or infinite limits, but it cannot be divergent.*

In case the series Σa_n is summable (C, k) for $k > r$, but not summable (C, k) for $k < r$, the number r may be called the *index of summability* of Σa_n. If the series is summable (C, r), the index of summability is said to be attained. Clearly the index of summability of a series may be zero, and yet the series may not be convergent, the index zero being in that case unattained.

51. The following multiplication theorem due to Knopp, and of which Chapman has given a simpler proof, is applicable to the Cauchy-product of two series which are summable (C, r) and (C, s) respectively.

If $\sum\limits_{n=0} a_n$ *is summable* (C, r), *and* $\sum\limits_{n=0} b_n$ *is summable* (C, s), *where* $r > -1$, $s > -1$, *the series* $\sum\limits_{n=0} c_n$, *where* $c_n = a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n$, *is summable* $(C, r + s + 1)$. *In particular, if both the given series are convergent,* Σc_n *is summable* $(C, 1)$. *The Cesàro sum* $(C, r + s + 1)$ *of* Σc_n *is the product of the sums* (C, r) *and* (C, s) *of the series* Σa_n, Σb_n.

Cesàro established this theorem for the case in which r and s are both positive integers, or zero. To prove the theorem generally, we employ Cesàro's theorem given in § 49. Let $\alpha_n = S_n^{(r)}$, for the series Σa_n, and let $\beta_n = S'^{(s)}_n$, for the series Σb_n; then since $S_n^{(r)}/n^r$, $S'^{(s)}_n/n^s$ converge to

$$\frac{1}{\Gamma(r+1)} \lim_{n\sim\infty} C_n^{(r)},\ \frac{1}{\Gamma(s+1)} \lim_{n\sim\infty} C'^{(s)}_n \text{ respectively,}$$

$$\frac{S_n^{(r)} S'^{(s)}_0 + S_{n-1}^{(r)} S'^{(s)}_1 + \ldots + S_0^{(r)} S'^{(s)}_n}{n^{r+s+1}}$$

converges to $\lim\limits_{n\sim\infty} C_n^{(r)} . \lim\limits_{n\sim\infty} C'^{(s)}_n / \Gamma(r + s + 2)$.

The product $(a_0 + a_1 x + \ldots + a_n x^n)(1 - x)^{-(r+1)}$ is, when arranged in powers of x, an absolutely convergent series, for $|x| < 1$, of which the first $n + 1$ terms are $S_0^{(r)} + S_1^{(r)} x + \ldots + S_n^{(r)} x^n$; and a similar statement applies to the product $(b_0 + b_1 x + \ldots + b_n x^n)(1 - x)^{-(s+1)}$.

The first $n + 1$ terms of the Cauchy-product of the two series are $\sum\limits_{m=0}^{n} (S_m^{(r)} S_0^{(s)} + S_{m-1}^{(r)} S_1^{(s)} + \ldots + S_0^{(r)} S_m^{(s)}) x^m$; the Cauchy-product series being absolutely convergent. This series has the same sum as the Cauchy-product series of the two series for

$$(a_0 + a_1 x + \ldots + a_n x^n)(1 - x)^{-(r+1)},\ (b_0 + b_1 x + \ldots + b_n x^n)(1 - x)^{-(s+1)},$$

and the first $n + 1$ terms of this series are the same as the first $n + 1$ terms of the series for $(c_0 + c_1 x + \ldots + c_n x^n)(1 - x)^{-(r+s+2)}$, and are thus

$$\bar{S}_0^{(r+s+1)} + \bar{S}_1^{(r+s+1)} x + \ldots + \bar{S}_n^{(r+s+1)} x^n$$

where $\bar{S}_n^{(r+s+1)}$ refers to the series $\sum\limits_{n=0} c_n$. Consequently we have

$$\bar{S}_n^{(r+s+1)} = S_n^{(r)} S'^{(s)}_0 + S_{n-d}^{(r)} S_1^{(s)} + \ldots + S_0^{(r)} S'^{(s)}_n.$$

It now follows that $\lim\limits_{n\sim\infty} \dfrac{S_n^{(r+s)+1}}{\binom{n+r+s+1}{n}} = \lim\limits_{n\sim\infty} C_n^{(r)} . \lim\limits_{n\sim\infty} C'^{(s)}_n$; which establishes the theorem.

52. *If* $\sum\limits_{n=0} a_n$ *is summable* (C, r), *where* r *has a value* > -1, *then* $S_n^{(r')} = o(n^r)$, *where* r' *has any value* $< r$. *Also, if* $\sum\limits_{n=0} a_n$ *is bounded* (C, r) *then* $S_n^{(r')} = O(n^r)$, *where* $r' < r$.

This general theorem was given by Andersen*, but particular cases

* *Loc. cit.* p. 11.

were known previously. In case $r > -1$, we can take $r' = -1$; and in case $r > 0$, we can take $r' = 0$. We thus have:

If $r > -1$, *then* $a_n = o(n^r)$, *or* $a_n = O(n^r)$, *according as the series* $\sum_{n=0} a_n$ *is summable* (C, r), *or bounded* (C, r). *If* $r > 0$, *then* $s_n = o(n^r)$, *or* $s_n = O(n^r)$, *according as the series is summable, or is bounded* (C, r).

In order to prove the general theorem, we employ the relation (6) of § 48. We have

$$S_n^{(r')} = S_n^{(r)} - \binom{r-r'}{1} S_{n-1}^{(r)} + \ldots + (-1)^n \binom{r-r'}{n} S_0^{(r)},$$

where the series stops after $r - r' + 1$ terms in case $r - r'$ is an integer $< n$. When the series is bounded (C, r), with U and L as the upper and lower limits of indeterminancy, we may write

$$S_n^{(r)} \Big/ \binom{n+r}{n} = \tfrac{1}{2}(U + L) + \tfrac{1}{2}(U - L)\,\theta_n + \epsilon_n;$$

where $|\theta_n| \leqq 1$, and ϵ_n converges to zero, as $n \sim \infty$, and is accordingly bounded.

The part of $S_n^{(r')}$ which involves $\frac{1}{2}(U + L)$ is

$$\tfrac{1}{2}(U + L)\left\{\binom{n+r}{n} - \binom{r-r'}{n}\binom{n+r-1}{n-1} + \ldots + (-1)^n \binom{r-r'}{n}\right\};$$

the series in the bracket is the coefficient of x^n in the product of the series for $(1 - x)^{r-r'}(1 - x)^{-r}$, where $|x| < 1$, or in the series for $(1 - x)^{-r'}$, and it is thus equal to $\binom{r'+n}{n}$. When multiplied by $\frac{1}{n^r}$, we see that this part of $S_n^{(r')}/n^r$ converges to zero, since $\frac{1}{n^r}\binom{r'+n}{n}$ is bounded.

The remaining part of $S_n^{(r')}/n^r$ is

$$\frac{1}{\Gamma(r+1)} \sum_{p=0}^{p=n-1} \left[(-1)^p \binom{r-r'}{p}\left(1 - \frac{p}{n}\right)^r (1 + \zeta_{n-p})\{\tfrac{1}{2}(U - L)\,\theta_{n-p} + \epsilon_{n-p}\}\right] + (-1)^n \frac{a_0}{n^r}\binom{r-r'}{n}.$$

If $r - r'$ is an integer less than n the series stops after $r - r' + 1$ terms and the last term does not occur. When this is the case the expression is less for all values of n $(> r - r' + 1)$ than a fixed number, since θ_{n-p}, ϵ_{n-p}, ζ_{n-p} are bounded; and thus $S_n^{(r')}/n^r$ is bounded.

If $U = L$, since $\lim\limits_{n\sim\infty} \epsilon_{n-p} = 0$, for each of the $r - r' + 1$ values of p, it is clear that $S_n^{(r')}/n^r$ converges to zero, as $n \sim \infty$.

Next let it be assumed that $r - r'$ is not integral; we then have to consider the whole of the above expression, the number of terms being now no longer independent of n.

We have $\left|\binom{r-r'}{p}\right| = \left|\binom{p-(r-r'+1)}{p}\right| < \dfrac{K}{p^{r-r'+1}}$, where K is a fixed number; hence $\dfrac{1}{n^r}\left|\binom{r-r'}{n}\right| < \dfrac{K}{n^{r-r'}}\cdot\dfrac{1}{n^{r+1}}$, which converges to zero, when $r+1>0$, $r-r'>0$, as $n \sim \infty$; it thus appears that the last term in the above expression converges to zero; it may therefore be omitted.

If $r>0$, the part of $S_n^{(r)}/n^r$, is, since $\left(1-\dfrac{p}{n}\right)^r < 1$, numerically less than

$$\frac{1}{\Gamma(r+1)}(1+\delta)\{\tfrac{1}{2}(U-L)+\delta\}\sum_{p=0}^{p=m}\left|\binom{r-r'}{p}\right| + K'\sum_{p=m+1}^{p=n}\left|\binom{r-r'}{p}\right|;$$

where $m\ (<n)$ and n are so chosen that $|\zeta_{n-p}|$, $|\epsilon_{n-p}|$ are less than an arbitrarily chosen positive number δ, for $p=0, 1, 2, \dots m$; and K' is a fixed number.

The series $\sum\limits_{p=0}\left|\binom{r-r'}{p}\right|$ is convergent, when $r-r'>0$; hence $\sum\limits_{p=0}^{p=m}\left|\binom{r-r'}{p}\right|$ is less, for all values of m, than a fixed number, independent of n. Also m can be chosen so large that $\sum\limits_{p=m+1}^{\infty}\left|\binom{r-r'}{p}\right| < \delta$. It thus appears that $|S_n^{(r)}/n^r|$ is bounded; and, if $U=L$, it is less, for all sufficiently large values of n, than a fixed multiple of δ; consequently it converges to zero, as $n \sim \infty$.

If $o > r > -1$, we divide the expression for $S_n^{(r')}/n^r$ into three parts; the first of these is

$$\frac{1}{\Gamma(r+1)}\sum_{p=0}^{p=m}\left[(-1)^p\binom{r-r'}{p}\left(1-\frac{p}{n}\right)^r(1+\zeta_{n-p})\{\tfrac{1}{2}(U-L)\theta_{n-p}+\epsilon_{n-p}\}\right],$$

where m is a fixed number less than $\frac{1}{2}n$, and n is so large that $|\zeta_{n-p}|$, $|\epsilon_{n-p}|$ are $<\delta$, for $p=0, 1, 2, \dots m$. We have then $\left(1-\dfrac{p}{n}\right)^r < \dfrac{1}{2^r}$; and as before, this expression is seen to be bounded; and when $U=L$, it is less than a fixed multiple of δ; the number m being kept fixed.

We next take the summation from $p=m+1$ to $p=[\frac{1}{2}n]$, which denotes $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$. The number m may be so chosen that it satisfies the condition $\sum\limits_{m+1}^{\infty}\left|\binom{r-r'}{p}\right| < \delta$; the part of the expression under consideration is then numerically less than a fixed multiple of $\sum\limits_{m+1}^{[\frac{1}{2}n]}\left|\binom{r-r'}{p}\right|$ which is $<\delta$. The last part of the expression is less than a fixed multiple of $\sum\limits_{p=[\frac{1}{2}n]+1}^{n}\left(1-\dfrac{p}{n}\right)^r\dfrac{1}{p^{r-r'+1}}$, which is less than a fixed multiple of

$$\frac{1}{n^{r-r'}}\int_{\frac{1}{2}}^{1}\frac{(1-x)^r}{x^{r-r'+1}};$$

and this converges to 0, as $n \sim \infty$. It has now been shewn that, for $r > -1$, $r - r' > 0$, $S_n^{(r')}/n^r$ is bounded, or converges to zero, when $S_n^{(r)}/n^r$ is bounded, or is convergent.

53. *If Σa_n is either summable (C, r), or bounded (C, r), where $r > 0$, and $\{v_n\}$ is a sequence of numbers, then $\sum\limits_{n=0} a_n v_n$ is convergent provided* (1), $v_n = o\left(\frac{1}{n^r}\right)$, *and* (2), $\Sigma n^r \nabla^{r+1} v_n$ *is absolutely convergent; and the sum of $\sum\limits_{n=0} a_n v_n$ is that of the series $\Sigma S_n^{(r)} \nabla^{r+1} v_n$, which is absolutely convergent. If the condition* (1) *be replaced by* (1)′, $v_n = O\left(\frac{1}{n^r}\right)$, *and* (2) *remains unchanged, the series $\sum\limits_{n=0} a_n v_n$ is bounded.*

The sum of the series $v_n - \binom{r+1}{1} v_{n+1} + \binom{r+1}{2} v_{n+2} - \dots$ is here denoted by $\nabla^{r+1} v_r$. It is infinite when r is not integral, and it may be regarded as a generalization of the definition of differences of integral order. This series is convergent because

$$1 + \left|\binom{r+1}{1}\right| + \left|\binom{r+1}{2}\right| + \dots$$

is convergent and $|v_{n+m}|$ is bounded. Since $a_n = \sum\limits_{t=0}^{t=n} (-1)^t \binom{r+1}{t} S_{n-t}^{(r)}$, the finite sum $\sum\limits_{n=0}^{n=m} a_n v_n$ is equal to

$$\sum_{n=0}^{n=m} S_n^{(r)} \left\{v_n - \binom{r+1}{1} v_{n+1} + \binom{r+1}{2} v_{n+2} - \dots + (-1)^{m-n} \binom{r+1}{m-n} v_m\right\}.$$

The coefficient of $S_n^{(r)}$ differs from $\nabla^{r+1} v_n$ by

$$(-1)^{m-n+1} \left[\binom{r+1}{m-n+1} v_{m+1} - \binom{r+1}{m-n+2} v_{m+2} + \dots\right],$$

which is, in absolute value, less than

$$\frac{K\eta_m}{m^r} \left\{\frac{1}{(m-n+1)^{r+2}} + \frac{1}{(m-n+2)^{r+2}} + \dots\right\},$$

or than $\dfrac{K'\eta_m}{m^r} \dfrac{1}{(m-n)^{r+1}}$; where K and K' are fixed positive numbers, and $|v_p p^r| < \eta_m$, for all values of $p \geqq m$. When $n = m$, the absolute difference is $< \dfrac{K\eta_m}{m^r}\left(\dfrac{1}{1^{r+2}} + \dfrac{1}{2^{r+2}} + \dots\right)$, which is less than $\dfrac{K''\eta_m}{m^r}$. We now see that $\sum\limits_{n=0}^{n=m} a_n v_n$ differs from $\sum\limits_{n=0}^{n=m} S_n^{(r)} \nabla^{r+1} v_n$ by less than

$$\frac{K'\eta_m}{m^r}\left(\frac{|S_{m-1}^{(r)}|}{1^{r+1}} + \frac{|S_{m-2}^{(r)}|}{2^{r+1}} + \dots\right) + \frac{K''\eta_m |S_m^{(r)}|}{m^r}$$

or by less than $L\eta_m$, where L is a fixed positive number, independent of m, provided $\left|\frac{S_p^{(r)}}{p^r}\right|$ is bounded, in which case $|S_p^{(r)}|$ is less than a fixed multiple of m^r, for $p \leqq m$.

If $m_2 > m_1 \geqq m$, $\sum_{n=m_1}^{n=m_2} a_n v_n$ differs from $\sum_{n=m_1}^{n=m_2} S_n^{(r)} \nabla^{r+1} v_n$ by less than $2L\eta_m$. If $\Sigma n^r \nabla^{r+1} v_n$ is absolutely convergent, so also is $\Sigma S_n^{(r)} \nabla^{r+1} v_n$; and if $v_n = o\left(\frac{1}{n^r}\right)$, η_m is arbitrarily small, for a sufficiently large value of m. It then follows that, provided $m_2 > m_1 \geqq m$, where m is sufficiently large, $\sum_{n=m_1}^{n=m_2} a_n v_n$ is less, in absolute magnitude, than an arbitrarily chosen positive number ϵ; and therefore $\Sigma a_n v_n$ is convergent, although not necessarily absolutely convergent, and it converges to the sum of the series $\Sigma S_n^{(r)} \nabla^{r+1} v_n$. If the condition which v_n satisfies is $v_n = O\left(\frac{1}{n^r}\right)$, η_m is bounded, and $\sum_{n=0}^{n=m} a_n v_n$ differs from $\sum_{n=0}^{n=m} S_n^{(r)} \nabla^{r+1} v_n$ by less than a fixed number; consequently the series $\Sigma a_n v_n$ is bounded.

As a particular case, let $v_n = \frac{1}{(n+1)^s}$, where $s \geqq r > 0$, then the condition (1) is satisfied when $s > r$, and the condition (1)′ is satisfied when $s = r$. It will be shewn that, in either case, the condition (2) is satisfied.

We have

$$\begin{aligned}\nabla^{r+1} v_n &= \frac{1}{(n+1)^s} - \binom{r+1}{1} \frac{1}{(n+2)^s} + \dots \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-(n+1)x} x^{s-1} \sum_{m=0}^{\infty} (-1)^m \binom{r+1}{m} e^{-mx} dx;\end{aligned}$$

since, when the factors $(-1)^m$ are omitted in the integrand, the new integrand converges to a function that is summable in $(0, \infty)$, in accordance with a criterion given at the end of § 214, term by term integration is permissible. It has thus been shewn that

$$\nabla^{r+1} v_n = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(n+1)x} x^{s-1} (1 - e^{-x})^{(r+1)} dx,$$

which shews that $\nabla^{r+1} v_n$ has a positive value, decreasing as n increases. We have

$$\sum_{n=0}^{n=m} n^r \nabla^{r+1} v_n = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (1 - e^{-x})^{r+1} \sum_{n=0}^{n=m} n^r e^{-(n+1)x} dx,$$

which is less than

$$M \int_0^\infty x^{s-1} (1 - e^{-x})^{r+1} \sum_0^\infty \binom{r+n}{n} e^{-(n+1)x} dx,$$

where M is a fixed positive number; and this expression reduces to $M\int_0^\infty x^{s-1}e^{-x}dx$, or $M\Gamma(s)$. The absolute convergence of the series $\sum_{n=0} n^r \nabla^{r+1} v_n$ has now been established. From the general theorem we now obtain the following special theorem:

If Σa_n is bounded (C, r), or summable (C, r) where $r > 0$, the series $\Sigma \dfrac{a_n}{(n+1)^s}$, where $s > r$, is convergent, and the series $\Sigma \dfrac{a_n}{(n+1)^r}$ is bounded.

The first part of this theorem and the corresponding part of the general theorem were established by Chapman*.

54. It has been shewn in § 52 that, if $\sum_{n=0} a_n$ is summable (C, r), then a_n/n^r converges to zero, as n increases indefinitely. This may be expressed by the statement that the summation (C, r) is inapplicable, for all values of r, if $|a_n|$ increases too rapidly with n; for example, if $a_n = (-1)^n k^n$, where $k > 1$, the series $\sum_{n=0} (-1)^n k^n$ is not summable (C, r) for any value of r. It can, however, also be shewn that the method of summation may also be inapplicable in case $|a_n|$ diminishes too rapidly as n is increased. This appears from the following important theorem, which is due to Hardy†:

If na_n is bounded, the series $\sum_{n=0}^{\infty} a_n$ cannot be summable (C, r), for any positive values of r, unless the series is convergent.

In particular, if $\lim_{n\sim\infty} na_n = 0$, the series cannot be summable (C, r) unless the series $\sum_{n=0} a_n$ is convergent. We may take r to be an integer.

Let it be assumed that $|(n+1)a_n|$ is less than a positive number k, for all values of n, and let $(n+1)a_n = b_n$. We have

$$S_n^{(r)} = a_n + \binom{r+1}{1} a_{n-1} + \binom{r+2}{2} a_{n-2} + \dots + \binom{r+n}{n} a_0$$

$$S_n^{(r-1)} = a_n + \binom{r}{1} a_{n-1} + \binom{r+1}{2} a_{n-2} + \dots + \binom{r+n-1}{n} a_0$$

where we take r to be a positive integer; this involves no loss of generality since r can always be replaced by the next greater integer. We may define $T_n^{(r-1)}$ by

$$T_n^{(r-1)} = b_n + \binom{r}{1} b_{n-1} + \binom{r+1}{2} b_{n-2} + \dots + \binom{r+n-1}{n} b_0;$$

and it can easily be verified that $rS_n^{(r)} + T_n^{(r-1)} = (n+r+1)\, S_n^{(r-1)}$, and that $rS_n^{(r)} + T_{n+1}^{(r-1)} = (n+2)\, S_{n+1}^{(r-1)}$.

* *Proc. Lond. Math. Soc.* (2), vol. IX (1911), pp. 382–387.

† *Proc. Lond. Math. Soc.* (2), vol. VIII (1909), p. 301.

The first of these equations may be written in the form

$$C_n^{(r-1)} = \frac{r+n}{r+n+1} C_n^{(r)} + \frac{T_n^{(r-1)}}{(n+r+1)\binom{r+n-1}{n}},$$

from which it follows that, if the series $\sum\limits_{n=0} a_n$ is summable (C, r), the necessary and sufficient condition that it should be also summable $(C, r-1)$ is that $\lim\limits_{n\sim\infty} T_n^{(r-1)}/n^r = 0$.

We have $$s_n = b_0 + \frac{b_1}{2} + \frac{b_2}{3} + \ldots + \frac{b_n}{n+1};$$

and remembering that $T_n^{(0)} = b_0 + b_1 + \ldots + b_n$, we have

$$s_n = \sum_{\nu=0}^{\nu=n-1} T_\nu^{(0)} \Delta\left(\frac{1}{\nu+1}\right) + \frac{T_n^{(0)}}{n+1},$$

where $\Delta f(x)$ denotes $f(x) - f(x+1)$. Proceeding in the same manner, we have

$$s_n = \sum_{\nu=0}^{\nu=n-2} T_\nu^{(1)} \Delta^2\left(\frac{1}{\nu+1}\right) + T_{n-1}^{(1)} \Delta\left(\frac{1}{n}\right) + \frac{T_n^{(0)}}{n+1},$$

and generally

$$s_n = \sum_{\nu=0}^{\nu=n-r} T_\nu^{(r-1)} \Delta^r\left(\frac{1}{\nu+1}\right) + \sum_{\nu=0}^{\nu=r-1} T_{n-\nu}^{(\nu)} \Delta^\nu\left(\frac{1}{n+1-\nu}\right) \quad \ldots(A).$$

It can be easily verified that

$$s_n - \frac{T_n^{(0)}}{n+1} = \frac{1}{n+1} S_{n-1}^{(1)},$$

and $$s_n - \frac{T_n^{(0)}}{n+1} - T_{n-1}^{(1)} \Delta\left(\frac{1}{n}\right) = \frac{2}{(n+1)\,n} S_{n-2}^{(2)}.$$

These are particular cases of the identity

$$s_n - \sum_{\nu=0}^{\nu=r-1} T_{n-\nu}^{(\nu)} \Delta^\nu\left(\frac{1}{n+1-\nu}\right) = \frac{r!\,(n+1-r)!}{(n+1)!} S_{n-r}^{(r)} \quad \ldots(B),$$

which may be proved by induction. Assuming it to be true for $r = p$, it will also hold for $r = p+1$ if

$$\frac{p!\,(n+1-p)!}{(n+1)!} S_{n-p}^{(p)} - \frac{(p+1)!\,(n-p)!}{(n+1)!} S_{n-p-1}^{(p+1)} = T_{n-p}^{(p)} \Delta^p \frac{1}{n+1-p};$$

since $\Delta^p \dfrac{1}{n+1-p} = \dfrac{p!\,(n-p)!}{(n+1)!}$, it has to be shewn that

$$(n+1-p)\, S_{n-p}^{(p)} - (p+1)\, S_{n-p-1}^{(p+1)} = T_{n-p}^{(p)},$$

which is obtained by writing $n-p-1$ for n, and $p+1$ for r, in the identity $rS_n^{(r)} + T_{n+1}^{(r-1)} = (n+2)\, S_{n+1}^{(r-1)}$. Since the relation (B) holds for $r = 1$, it holds generally.

From (A) and (B) we have

$$\frac{r!\,(n+1-r)!}{(n+1)!}S_{n-r}^{(r)} = \sum_{\nu=0}^{\nu=n-r} T_\nu^{(r-1)}\Delta^r\left(\frac{1}{\nu+1}\right);$$

and from this it follows that the necessary and sufficient condition that the given series is summable (C, r) is that the series $\sum_{\nu=0}^{\infty} T_\nu^{(r-1)}\Delta^r\left(\frac{1}{\nu+1}\right)$ should be convergent. It will be shewn that, if the condition

$$\lim_{n\sim\infty} T_n^{(r-1)}/n^r = 0$$

is not satisfied, this series cannot be convergent, it being assumed that $|b_n| < k$.

Let it be assumed that $T_n^{(r-1)}/n^r$ does not converge to zero; then, for arbitrarily large values N, of n, we have $T_N^{(r-1)} > k_1 N^r$, or else $T_N^{(r-1)} < -k_1 N^r$, where k_1 is a positive number which we may take to be $< k$. The second inequality is reduced to be first by changing the sign of a_n, for every value of n; consequently we may assume that $T_N^{(r-1)} > k_1 N^r$, for an infinite set of values of N. Let N_1 be the least integer such that $N_1 \geqq \left(1 - \frac{k_1}{2k}\right)N$; thus $N_1 < N$, and N_1 increases indefinitely as N does so. We may take $N_1 = k_2 N$ and $N_1 < k_3 N$, where $k_2 < k_3 < 1$.

Let $N_1 \leqq n \leqq N$, then the value of $T_N^{(r-1)} - T_n^{(r-1)}$ is

$$\sum_{s=0}^{s=N}(s+1)a_s\binom{r+N-s-1}{N-s} - \sum_{s=0}^{s=n}(s+1)a_s\binom{r+n-s-1}{n-s},$$

and since
$$\binom{r+N-s-1}{N-s} \geqq \binom{r+n-s-1}{n-s},$$

we have

$$|T_N^{(r-1)} - T_n^{(r-1)}| < k\left\{\sum_{s=0}^{s=N}\binom{r+N-s-1}{N-s} - \sum_{s=0}^{s=n}\binom{r+n-s-1}{n-s}\right\}.$$

The expression on the right hand side is equal to

$$k\sum_{s=0}^{s=N-n-1}\binom{r+N-s-1}{N-s},$$

and this is less than $k(N-n)\binom{r+N-1}{N}$, or than

$$k(N-n)(N+1)(N+2)\ldots(N+r-1);$$

and this is, for every value of n, less than

$$k(1-k_2)N^r\left(1+\frac{1}{N}\right)\left(1+\frac{2}{N}\right)\ldots\left(1+\frac{r-1}{N}\right).$$

Since $k_2 \geqq 1 - \frac{k_1}{2k}$, we have $(1-k_2)k \leqq \frac{1}{2}k_1$; it follows that, for all sufficiently large values of n, $|T_N^{(r-1)} - T_n^{(r-1)}| < \frac{1}{2}\bar{k}N^r$ where $\bar{k}$ is a constant

which may be taken to be greater than k_1 by as small a difference as we please.

Since $T_N^{(r-1)} > k_1 N^r$, it follows that $T_n^{(r-1)} > (k_1 - \frac{1}{2}\bar{k}) N^r$; thus $T_n^{(r-1)} > \frac{1}{2}\bar{k}_1 N^r$, for $N_1 \leqq n \leqq N$, where $\bar{k}_1$ is a number which we may suppose less than k_1 by as little as we please.

We now have

$$\sum_{\nu=N_1}^{\nu=N} T_\nu^{(r-1)} \Delta^r \left(\frac{1}{\nu+1}\right) > \tfrac{1}{2}\bar{k}_1 N^r \sum_{\nu=N_1}^{\nu=N} \Delta^r \frac{1}{\nu+1};$$

and the expression on the right hand side is

$$\tfrac{1}{2}\bar{k}_1 N^r (r-1)! \left\{\frac{1}{(N_1+1)(N_1+2)\dots(N_1+r)} - \frac{1}{(N+2)(N+3)\dots(N+r+1)}\right\}.$$

We have $\dfrac{1}{(N_1+1)(N_1+2)\dots(N_1+r)} > \dfrac{k_4}{N_1{}^r}$, where k_4 is less than 1 by as small a difference as we please, provided N_1 is sufficiently large; also $\dfrac{1}{(N+2)(N+3)\dots(N+r+1)} < \dfrac{1}{N^r}$. Thus $\sum\limits_{\nu=N_1}^{\nu=N} T_\nu^{(r-1)} \Delta^r \dfrac{1}{\nu+1}$ is greater than $\frac{1}{2}\bar{k}_1 N^r (r-1)! \left(\dfrac{k_4}{N_1{}^r} - \dfrac{1}{N^r}\right)$, or than $\frac{1}{2}\bar{k}_1 (r-1)! \left(\dfrac{k_4}{k_2{}^r} - 1\right)$. The number k_2 having been fixed, for sufficiently large values of N, we may choose k_4 so that $k_2{}^r < k_4 < 1$; thus, for all sufficiently large values of N, such that $T_N^{(r-1)} > kN^r$ the value of $\sum\limits_{\nu=N_1}^{\nu=N} T_\nu^{(r-1)} \Delta^r \dfrac{1}{\nu+1}$ exceeds a fixed positive number. It is therefore impossible that the series $\sum\limits_{\nu=0}^{\infty} T_\nu^{(r-1)} \Delta^r \dfrac{1}{\nu+1}$ can converge; and therefore the given series cannot be summable (C, r).

If $\lim\limits_{n\sim\infty} T_n^{(r-1)}/n^r = 0$, and the series is summable (C, r) it has been shewn also to be summable $(C, r-1)$. Thus, if na_n is bounded, and the series $\sum\limits_{n=0} a_n$ is summable (C, r), for any integral value of r, it is also summable $(C, r-1)$, and therefore also it is summable $(C, r-2), \dots$; it is consequently summable $(C, 0)$, that is, it is convergent.

The theorem of Hardy which has now been established has been extended by Landau* to the case in which na_n is bounded on one side only; thus:

If $na_n < k$, or $na_n > -k$, for all values of n, where k is some positive number, then the series $\sum\limits_{n=0}^{\infty} a_n$ cannot be summable (C, r) unless it is convergent.

* *Prac. matematyczno-fizycznych*, vol. XXI (1910), p. 97.

EXAMPLES.

(1) If $\sum\limits_{n=0} a_n$ is bounded, or summable, (C, r), where r is any real number, shew that $\sum\limits_{n=0} \frac{a_n}{n+1}$ is summable $(C, r-1)$. This theorem was given by Chapman*; the case when r is a positive integer, and the series is summable (C, r) had been given by H. Bohr† and M. Riesz‡.

(2) If r is a positive integer, and $\sum\limits_{n=0} a_n$ is bounded (C, r), shew that $\sum\limits_{n=1} \frac{a_n}{n^s}$ is summable (C, r), provided $s > 0$, whether s be integral or not. This theorem is due to H. Bohr†.

(3) If $\sum\limits_{n=0} a_n$ is summable (C, r), where r is a positive integer, prove that $\sum\limits_{n=1} \frac{a_n}{n^s}$ is summable $(C, r-s)$, where $s > 0$. This result was given by M. Riesz‡.

(4) If§ the series $\sum\limits_{n=0} a_n$ is bounded (C, r), and summable $(C, r+1)$, where $r \geqq -1$, show that the series is summable $(C, r+\delta)$, provided $\delta > 0$.

(5) Prove that the series $1^s - 2^s + 3^s - \ldots$, is summable (C, r), provided $r > s$. The number s may have either sign. This theorem was given by Chapman‖. It had been shewn by Bromwich¶ that, when s is integral, the series is summable $(C, s+1)$; and by the same method it could be proved that it is bounded (C, s).

(6) If the series $\sum\limits_{n=0} a_n$ is bounded (C, r), where $r > -1$, and the series $\sum\limits_{n=0} b_n$ is bounded (C, s), where $s > -1$, then the Cauchy-product $\sum\limits_{n=0} c_n$ is bounded $(C, r+s+1)$.

THE EQUIVALENCE OF CESÀRO'S AND HÖLDER'S METHODS OF SUMMATION.

55. The notation of § 44, in which the given series is taken to be $a_1 + a_2 + \ldots + a_n + \ldots$, will be employed here. If $x_1, x_2, x_3, \ldots$, be an infinite set of variables, and $\{a_{nm}\}$ a set of numbers such that $a_{nm} = 0$, when $m > n$, and $a_{nm} \neq 0$, for $m \leqq n$, let us consider the set of equations

$$y_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_n, \quad n = 1, 2, 3, \ldots.$$

Denoting the matrix

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 & \ldots \\ a_{21} & a_{22} & 0 & 0 & \ldots \\ a_{31} & a_{32} & a_{33} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

by A, we may regard the set of variables $\{y_n\}$ as obtained from the set $\{x_n\}$ by means of the operation A, and this fact may be expressed by $(y) = A\,(x)$.

If, whenever $\lim\limits_{n\sim\infty} x_n$ has a definite value, $\lim\limits_{n\sim\infty} y_n$ has also the same definite value, the operation A is said to be regular.

The set of equations by which $x_1, x_2, \ldots x_n, \ldots$ may be expressed in terms of $y_1, y_2, \ldots y_n, \ldots$ define an operation which may be denoted by

* *Loc. cit.* p. 388.

† *Comptes Rendus*, vol. CXLIII (1909), p. 75.

‡ *Ibid.* p. 1658. More general theorems are there given.

§ See Andersen, *loc. cit.* p. 56.

‖ *Loc. cit.* p. 397.

¶ *Theory of Infinite Series*, p. 317.

A^{-1}; thus $(x) = A^{(-1)}(y)$. If A and A^{-1} are both regular, A may be termed a *reversible* operation.

If a third set of variables z_n are obtained from the variables y_n by means of an operation B, or

$$\begin{vmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

so that $z = B(y)$, the operation by which the variables z_n are obtained from the variables x_n may be denoted by BA; thus $(z) = BA(x)$. In case $AB = BA$, the operations A and B are said to be *interchangeable*.

If A and B are two regular operations, AB is also a regular operation. Moreover the operation $\alpha A + (1 - \alpha) B$, for which the constants are $\alpha a_{nm} + (1 - \alpha) b_{nm}$, is also regular. Also AB is reversible if A and B are reversible operations.

If two operations A, B are such that, when $y = A(x)$, $z = B(x)$, for every sequence $\{x_n\}$ the sequences $\{y_n\}$, $\{z_n\}$ are either both convergent, with one and the same limit, or both non-convergent, the operations A and B are said to be *equivalent*. Since $(y) = AB^{-1}(z)$, $(z) = BA^{-1}(y)$, the operations A and B are equivalent only if AB^{-1}, BA^{-1} are regular operations.

The identical operation E represents the set of equations $x_n = x_n$, for which $a_{nn} = 1$, $a_{nm} = 0$, when $m \neq n$.

56. If we take $a_{nm} = \dfrac{1}{n}$, for $m \leqq n$, $a_{nm} = 0$, for $m > n$, we have $h^{(1)} = M(s)$, $h^{(2)} = M(h^{(1)})$, ... $h^{(r)} = M(h^{(r-1)})$, and thus $h^{(r)} = M^r(s)$, in accordance with the definition of Hölder's means, given in § 44; where M denotes the operation with the matrix

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

In accordance with the definition of Cesàro's means, given in § 44, we have

$$s_n^{(r)} = \binom{n+r-1}{r} C_n^{(r)} = s_1^{(r-1)} + s_2^{(r-1)} + \dots + s_n^{(r-1)} = s_{n-1}^{(r)} + s_n^{(r-1)}$$

and therefore $\displaystyle\binom{n+r-1}{r} C_n^{(r)} = \sum_{m=1}^{m=n} \binom{m+r-2}{r-1} C_m^{(r-1)}$;

and this can be written in the form

$$\binom{n+r-1}{r} C_n^{(r)} = \binom{n+r-2}{r} C_{n-1}^{(r)} + \binom{n+r-2}{r-1} C_n^{(r-1)},$$

from which we have $rC_n^{(r-1)} = (r-1) C_n^{(r)} + nC_n^{(r)} - (n-1) C_{n-1}^{(r)}$.

It then follows that

$$r\,\frac{C_1^{(r-1)} + C_2^{(r-1)} + \dots + C_n^{(r-1)}}{n} = (r-1)\,\frac{C_1^{(r)} + C_2^{(r)} + \dots + C_n^{(r)}}{n} + C_n^{(r)},$$

which may be written in the form

$$rM\,(C^{(r-1)}) = (r-1)\,M\,(C^{(r)}) + (C^{(r)}),$$

or $M\,(C^{(r-1)}) = S_r\,(C^{(r)})$, where S_r denotes the operation $\frac{1}{r}E + \left(1 - \frac{1}{r}\right) M$.

Since $h_n^{(1)} = C_n^{(1)}$, we have $(h^{(2)}) = M\,(h^{(1)}) = M\,(C^{(1)}) = S_2\,(C^{(2)})$. Also, since the operations $S_2, S_3, \dots$ are clearly interchangeable with M, and with one another, we obtain

$$(h^{(3)}) = M\,(h^{(2)}) = MS_2\,(C^{(2)}) = S_2M\,(C^{(2)}) = S_2S_3\,(C^{(3)});$$

and proceeding in this manner we find that

$$(h^{(r)}) = S_2S_3 \dots S_r\,(C^{(r)}) = P_r\,(C^{(r)}),$$

where P_r denotes the operation $S_2S_3 \dots S_r$.

Now M is a regular operation (see § 55), and consequently $S_2, S_3, \dots S_r$ are regular operations, and therefore P_r is a regular operation. Consequently, if $\lim\limits_{n\sim\infty} C_n^{(r)}$ exists, so also does $\lim\limits_{n\sim\infty} h_n^{(r)}$, and the values are the same; thus one part of the theorem of equivalence has been established.

In order to prove the second part of the theorem it is necessary and sufficient to shew that the operation P_r is reversible, for every value of r; and in order to prove this it is sufficient to show that the operation $S_r \equiv \frac{1}{r}E + \left(1 - \frac{1}{r}\right) M$ is reversible. Writing α for $\frac{1}{r}$, it will be shewn* that, if $0 < \alpha \leqq 1$, and the limit

$$\lim_{n\sim\infty} \left\{\alpha x_n + (1-\alpha)\,\frac{x_1 + x_2 + \dots + x_n}{n}\right\}$$

exists, as a definite number, then $\lim\limits_{n\sim\infty} x_n$ exists, and the two limits have the same value.

Denoting $\frac{x_1 + x_2 + \dots + x_n}{n}$ by X_n, we see that, if $x_n < X_n$, then $X_{n-1} > X_n$; that, if $x_n = X_n$, then $X_{n-1} = X_n$; and that, if $x_n > X_n$, then $X_{n-1} < X_n$. These results follow at once from the identity

$$x_n = X_n + (n-1)\,(X_n - X_{n-1}).$$

It has been shewn in § 27 that, if $\lim\limits_{n\sim\infty} x_n = +\infty$, or $\lim\limits_{n\sim\infty} x_n = -\infty$, then $\lim\limits_{n\sim\infty} X_n = +\infty$, in the first case, and $\lim\limits_{n\sim\infty} X_n = -\infty$, in the second

* The proof of this part of the theorem given here is a modification of that given by Knopp, *Math. Annalen*, vol. LXXIV (1913), p. 459, in a remark upon Schur's proof of the complete theorem given in the same volume. The limit was first investigated by Mercer, *Proc Lond. Math. Soc.* (2), vol. v (1906), p. 206.

case. It then follows that, since $1 - \alpha \geqq 0$, $\lim_{n \sim \infty} \{\alpha x_n + (1 - \alpha) X_n\}$ is $+\infty$ in the first case, and is $-\infty$ in the second case. It has thus been shewn that the sequence $\{X_n\}$ cannot be divergent, and must therefore, unless it converges, oscillate between limits, either of which may be finite or infinite. Let it be assumed, if possible, to oscillate. Let P and Q be two numbers such that

$$\overline{\lim_{n \sim \infty}} X_n > P > Q > \underline{\lim_{n \sim \infty}} X_n,$$

whether the upper and lower limits be finite or infinite. If N be an arbitrarily chosen integer, there is an infinite set of values of n ($> N$) such that $X_n > P$, and there is also an infinite set of values of n ($> N$) such that $X_n < Q$. If, for every value of n ($> N$), $x_n \leqq X_n$, we have $X_{n-1} \geqq X_n$, and thus the sequence $\{X_n\}$ is monotone non-increasing, for $n > N$, and it therefore converges, since it cannot diverge. Since X_n, $\alpha x_n + (1 - \alpha) X_n$ both converge, as $n \sim \infty$, it follows that x_n converges. Similarly, if $x_n > X_n$, for every value of n ($> N$), it can be shewn that x_n converges. The case in which there is only a finite set of values of n ($> N$), for which $x_n \leqq X_n$, or that in which there is only a finite set of values of n ($> N$) for which $x_n > X_n$, can be reduced to the above, by proper choice of N. We have therefore only to consider the case in which there is an infinite set of values of n ($> N$) for which $x_n \leqq X_n$, and also an infinite set for which $x_n > X_n$. The integer m ($> N$) can be chosen so large that, amongst the integers $N + 1, N + 2, \ldots m$, there are values of n for which $x_n \leqq X_n$, and also values for which $x_n > X_n$. Let m be also so chosen that $X_m > P$; if $x_m > X_m$, we have $x_m > P$ and therefore $\alpha x_m + (1 - \alpha) X_m > P$. If, on the other hand, $x_m \leqq X_m$, we have $X_{m-1} \geqq X_m$; let m_1 be the greatest integer $< m$, and necessarily $> N$, for which $x_{m_1} > X_{m_1}$. We have then

$$X_{m_1} \geqq X_{m_1+1} \geqq \ldots \geqq X_m,$$

and $x_{m_1} > X_{m_1} > X_m$; therefore $x_{m_1} > P$. It has thus been shewn that a value of n, greater than the arbitrarily chosen integer N, exists, such that $x_n > P$, and also $X_n > P$.

For this value of n, $\alpha x_n + (1 - \alpha) X_n > P$. In a similar manner, taking $x_m < Q$, it can be shewn that a value of n ($> N$) exists, for which $\alpha x_n + (1 - \alpha) x_n < Q$. Since N is arbitrary, the results are incompatible with the convergence of $\alpha x_n + (1 - \alpha) X_n$. It follows that X_n must be convergent, as $n \sim \infty$, and therefore x_n converges, as $n \sim \infty$.

The reversibility of the operation P_r having been established, if the sequence $\{h_n^{(r)}\}$ converges, so also does the sequence $\{C_n^{(r)}\}$, and the two limits are the same. The equivalence of the two modes of summation has now been completely established.

57. It may be remarked that, in the theorem that has been proved above that, if $\alpha x_n + (1 - \alpha) X_n$ converges to a definite limit, x_n also does

so, the condition $0 < \alpha \leqq 1$ may be replaced by the wider condition $0 < \alpha$. In fact the theorem can be very easily established in the case $\alpha > 1$, and this, taken together with the case $0 \leqq \alpha \leqq 1$, shews that it holds for $0 < \alpha$.

Let $u_n = \alpha x_n + (1 - \alpha) X_n$, or $\alpha x_n = u_n + (\alpha - 1) X_n$, we then have, if $\alpha > 1$, $\alpha \overline{\lim\limits_{n\sim\infty}} x_n \leqq \overline{\lim\limits_{n\sim\infty}} u_n + (\alpha - 1) \overline{\lim\limits_{n\sim\infty}} X_n$, and also

$$\alpha \underline{\lim\limits_{n\sim\infty}} x_n \geqq \underline{\lim\limits_{n\sim\infty}} u_n + (\alpha - 1) \underline{\lim\limits_{n\sim\infty}} X_n.$$

It follows that

$$\alpha (\overline{\lim\limits_{n\sim\infty}} x_n - \underline{\lim\limits_{n\sim\infty}} x_n) \leqq (\overline{\lim\limits_{n\sim\infty}} u_n - \underline{\lim\limits_{n\sim\infty}} u_n) + (\alpha - 1)(\overline{\lim\limits_{n\sim\infty}} X_n - \underline{\lim\limits_{n\sim\infty}} X_n);$$

and we have (see § 27)

$$\overline{\lim\limits_{n\sim\infty}} x_n \geqq \overline{\lim\limits_{n\sim\infty}} X_n \geqq \underline{\lim\limits_{n\sim\infty}} X_n \geqq \underline{\lim\limits_{n\sim\infty}} x_n,$$

and thus

$$\overline{\lim\limits_{n\sim\infty}} x_n - \underline{\lim\limits_{n\sim\infty}} x_n \leqq (\overline{\lim\limits_{n\sim\infty}} u_n - \underline{\lim\limits_{n\sim\infty}} u_n);$$

it then follows that, if $\lim\limits_{n\sim\infty} u_n$ exists, so also does $\lim\limits_{n\sim\infty} x_n$. The theorem is a particular case of the following general theorem, due to Knopp*:

If $b_n \geqq 0$, and $\sum\limits_{n=0}^{\infty} b_n$ is divergent, and $\alpha > 0$; then if

$$\alpha x_n + (1 - \alpha) \frac{b_0 x_0 + b_1 x_1 + \dots + b_n x_n}{b_0 + b_1 + \dots + b_n}$$

is convergent, so also is x_n, and the two converge to the same number.

The theorem can be proved in exactly the same manner as in the case $b_n = 1$, established above.

In the memoir by Knopp (*loc. cit.*), the following theorem is given:

If, for a given sequence $\{s_n\}$ the relation

$$\lim_{n\sim\infty} \left\{ \sum_{\nu=0}^{\nu=n} \binom{n+k-\nu}{k} \binom{p+\nu}{p} s_\nu \right\} \Big/ \binom{n+k+p+1}{k+p+1} = s$$

holds for a particular integer k ($\geqq 0$) and a particular integer p ($\geqq 0$), then it also holds when k is replaced by a greater integer, or when p is replaced by any integer ($\geqq 0$), or when both changes are made.

This theorem and the foregoing are employed by Knopp to obtain a new proof of the equivalence of Cesàro's and Hölder's methods of summation, and to obtain a new proof of Hardy's theorem (see § 54), that if a series Σa_n is summable (C, r), and if na_n is bounded, then the series is convergent.

* *Math. Zeitschr.* vol. XIX (1924), p. 99. In the proof there given, the possibility that x_n and X_n may both diverge to $+\infty$, or to $-\infty$, is left out of account.

THE EQUIVALENCE OF CESÀRO'S AND RIESZ'S METHODS OF SUMMATION.

58. If we denote by $\sigma_k(\omega)$, where $k \geqq 0$, the sum $\sum\limits_{r<\omega} a_r(\omega - r)^k$; in accordance with Riesz's method of summation (§ 45) the series $\sum\limits_{r=0}^{\infty} a_r$ is summable if $\lim\limits_{\omega\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k}$ has a definite value; where ω is a continuous variable, and not merely a sequence of integers. If $\lim\limits_{n\sim\infty} \dfrac{S_n^{(k)}}{\binom{n+k}{n}}$ has a definite value (§ 47), the series $\sum\limits_{r=0}^{\infty} a_r$ is summable (C, k). It will be shewn that, if either of these limits exists, then the other exists, and both have the same value; and thus that the two methods of summation are equivalent.

Throughout the following proof, when a relation $\phi(n) = o\{\psi(n)\}$ is employed, where $\phi(n)$, $\psi(n)$ involve one or more parameters besides n, it will be understood to mean that, if ϵ be arbitrarily chosen, then for each fixed set of values of the parameters, n_ϵ can be so chosen that

$$\left|\frac{\phi(n)}{\psi(n)}\right| < \epsilon, \text{ for } n > n_\epsilon;$$

but that n_ϵ cannot necessarily be so chosen as to be independent of the values of the parameters. A similar remark applies to a relation

$$\phi(n) = O\{\psi(n)\}.$$

If $\lim\limits_{\omega\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k} = s$, by changing a_0 into $a_0 - s$, we see that $\lim\limits_{\omega\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k} = 0$; similarly if $\lim\limits_{n\sim\infty} \dfrac{S_n^{(k)}}{C_n^{(k)}}$ exists, it is seen that, by changing the value of a_0, the limit becomes zero. It is therefore sufficient, in order to prove the theorem of equivalence, to shew* that, if either of the two limits exists and is zero, then the other exists and is also zero.

Some preliminary propositions, required in the proof, will be first established.

(*a*) *If* Δx^k, $\Delta^r x^k$ *denote the differences*

$$x^k - (x-1)^k,\ x^k - r(x-1)^k + \frac{r(r-1)}{2!}(x-2)^k - \ldots + (-1)^r(x-r)^k,$$

for $r = 1, 2, 3, \ldots$, *where* $x - r > 0$, *then*

$$\Delta^r x^k = k(k-1)\ldots(k-r+1)\,x^{k-r} + O(x^{k-r-1}).$$

* The proof here given is founded upon that indicated by M. Riesz, *Comptes Rendus*, vol. CLII (1911), p. 1651, and it has been supplemented by reference to a letter written by Riesz, containing further details of the proof.

It is easily seen that $\Delta^r x^k$ may be expressed in the form

$$k(k-1)\dots(k-r+1)\int_{x-1}^{x} du_1 \int_{u_1-1}^{u_1} du_2 \dots \int_{u_{r-1}-1}^{u_{r-1}} u_r^{k-r} du_r;$$

where $x - r > 0$. Thus $|\Delta^r x^k - k(k-1)\dots(k-r+1)x^{k-r}|$ is expressed by

$$k(k-1)\dots(k-r+1)\int_{x-1}^{x} du_1 \int_{u_1-1}^{u_1} du_2 \dots \int_{u_{r-1}-1}^{u_{r-1}} (x^{k-r} - u_r^{k-r})\, du_r.$$

The integrand $x^{k-r} - u_r^{k-r}$ is positive and less than $x^{k-r} - (x-r)^{k-r}$, since u_r is in the interval $(x - r, x)$; and this is less than

$$x^{k-r}\left[1 - \left(1 - \frac{r}{x}\right)^{k-r}\right],$$

which is equal to $O(x^{k-r-1})$. Therefore

$$\Delta^r x^k - k(k-1)\dots(k-r+1)x^{k-r} = O(x^{k-r-1}).$$

(*b*) $\dfrac{\Gamma(n+k+1)}{n!} - n^k = O(n^{k-1})$.

From Stirling's theorem we find that

$$\lim_{n\sim\infty} \frac{(k+1)(k+2)\dots(k+n)}{n!}\frac{1}{n^k} = \frac{1}{\Gamma(k+1)};$$

thus
$$\frac{(k+1)(k+2)\dots(k+n)}{n!}\frac{1}{n^k} = O(1).$$

Denoting the expression on the left hand side by $f(n)$, we have

$$\begin{aligned} f(n) - f(n+1) &= \frac{(k+1)\dots(k+n)}{n^k n!}\left\{1 - \frac{k+n+1}{n+1}\left(1+\frac{1}{n}\right)^{-k}\right\} \\ &= O(1)\left\{1 - \frac{k+n+1}{n+1}\left[1 - \frac{k}{n} + O\left(\frac{1}{n^2}\right)\right]\right\} \\ &= O(1)\left\{1 - \left(1 + \frac{k}{n+1}\right)\left[1 - \frac{k}{n+1} + O\left(\frac{1}{n^2}\right)\right]\right\} \\ &= O\left(\frac{1}{n^2}\right). \end{aligned}$$

It follows that

$$f(n) - \frac{1}{\Gamma(k+1)} = \sum_{m=n}^{\infty}\{f(m) - f(m+1)\};$$

and this is less than a fixed multiple of $\sum_{m=n}^{\infty} \frac{1}{m^2}$, or of $\frac{1}{n-1}$; hence

$$f(n) - \frac{1}{\Gamma(k+1)} = O\left(\frac{1}{n}\right),$$

or
$$\frac{\Gamma(n+k+1)}{n!} - n^k = O(n^{k-1}).$$

(*c*) *If i is an integer, $S_n^{(i)}$ can be expressed as a linear function of*

$$\sigma_i(n), \sigma_{i-1}(n), \sigma_{i-2}(n), \dots \sigma_0(n).$$

We have $S_n^{(i)} = \sum_{r=0}^{r=n} \frac{(n-r+1)(n-r+2)\dots(n-r+i)}{i!} a_r$; and the numerator of the coefficient is of the form $(n-r)^i + \lambda_1 (n-r)^{i-1} + \dots + i!$; where $\lambda_1, \lambda_2, \dots$ are integers dependent only on i. Thus we have

$$S_n^{(i)} = \frac{1}{i!}\{\sigma_i(n) + \lambda_1 \sigma_{i-1}(n) + \dots + i!\sigma_0(n)\}.$$

(*d*) *$\sigma_k(\omega)$ can be expressed as a linear function of $S_0^{(K)}, S_1^{(K)}, \dots S_n^{(K)}$, where K is the integer next less than k, and $n < \omega \leq n+1$.*

We have $a_r = S_r^{(K)} - \binom{K+1}{1} S_{r-1}^{(K)} + \dots + (-1)^r \binom{K+1}{r} S_0^{(K)}$; where the series stops after $K+2$ terms, or after $r+1$ terms, whichever is the smaller of these numbers. Substituting this value of a_r in the expression $\Sigma a_r (\omega - r)^k$, we have

$$\sigma_k(\omega) = \sum_{r=0}^{n-K-1} S_r^{(K)} \Delta^{K+1} (\omega - r)^k + \sum_{r=n-K}^{n} S_r^{(K)} \times \{(\omega - r)^k - (K+1)(\omega - r - 1)^k + \dots\},$$

where the number of terms in the bracket is, for every value of r, $n - r + 1$.

59. In order to prove the equivalence theorem, three lemmas will be required. Lemma I:

If $\lim\limits_{n\sim\infty} \frac{S_n^{(k)}}{n^k} = 0$, *then* $\lim\limits_{n\sim\infty} \frac{S_n^{(k')}}{n^k} = 0$, *where* $k' < k$.

This has already been proved in § 52.

We proceed to the proof of Lemma II:

If $\lim\limits_{\omega\sim\infty} \frac{\sigma_k(\omega)}{\omega^k} = 0$, *and i be any integer less than k, then* $\lim\limits_{n\sim\infty} \frac{S_n^{(i)}}{n^k} = 0$.

The lemma will be first proved in the case in which k is an integer, so that i has any one of the values $k-1, k-2, \dots 0$. Let n be the integer next less than ω, so that $n < \omega \leq n+1$.

Let us consider

$$\sigma_k(n) - (k-p)\sigma_k\left(n + \frac{1}{k}\right) + \frac{(k-p)(k-p-1)}{2!}\sigma_k\left(n + \frac{2}{k}\right) - \dots + (-1)^{k-p}\sigma_k\left(n + \frac{k-p}{k}\right),$$

where p may have the values $0, 1, 2, \dots k-1$.

The coefficient of a_r in this expression is

$$(n-r)^k - (k-p)\left(n + \frac{1}{k} - r\right)^k + \frac{(k-p)(k-p-1)}{2!}\left(n + \frac{2}{k} - r\right)^k - \dots;$$

and this is the coefficient of $\frac{x^k}{k!}$ in $e^{(n-r)x} - (k-p)e^{\left(n+\frac{1}{k}-r\right)x} + \dots$ or in $(-1)^{k-p}e^{(n-r)x}(e^{\frac{x}{k}} - 1)^{k-p}$. It follows that the coefficient of a_r is of the

form $A_0(n-r)^p + A_1(n-r)^{p-1} + \ldots + A_p$, where $A_0, A_1, \ldots A_p$ depend only on k and p. It then follows that

$$\sigma_k(n) - (k-p)\,\sigma_k\left(n+\frac{1}{k}\right) + \ldots + (-1)^{k-p}\,\sigma_k\left(n+\frac{k-p}{k}\right)$$

has the form $A_0\sigma_p(n) + A_1\sigma_{p-1}(n) + \ldots + A_p\sigma_0(n)$.

Let $p = 0$, this expression then becomes $A_0\sigma_0(n)$; thus $\sigma_0(n)$ is expressed as a linear function of $\sigma_k(n)$, $\sigma_k\left(n+\frac{1}{k}\right), \ldots \sigma_k(n+1)$. Next let $p=1$, we have then the form $A_0\sigma_1(n) + A_1\sigma_0(n)$, hence $\sigma_1(n)$ can be expressed as a linear function of $\sigma_k(n)$, $\sigma_k\left(n+\frac{1}{k}\right), \ldots$. Letting p have the values $2, 3, \ldots k-1$, we see that $\sigma_0(n), \sigma_1(n), \ldots \sigma_{k-1}(n)$ are all expressible as linear functions of $\sigma_k(n)$, $\sigma_k\left(n+\frac{1}{k}\right), \ldots \sigma_k(n+1)$.

Now $S_n^{(k)}$ has been shewn in theorem (c), of § 58, to be a linear function of $\sigma_0(n), \sigma_1(n), \ldots \sigma_k(n)$; therefore $S_n^{(k)}$ is a linear function of

$$\sigma_k(n),\ \sigma_k\left(n+\frac{1}{k}\right), \ldots \sigma_k(n+1).$$

If $\dfrac{\sigma_k(\omega)}{\omega^k}$ converges to zero, $\dfrac{\sigma_k(n)}{n^k}, \dfrac{\sigma_k\left(n+\frac{1}{k}\right)}{n^k}, \ldots \dfrac{\sigma_k(n+1)}{n^k}$ all converge to zero, since $n < \omega \leq n+1$; it then follows that $\dfrac{S_n^{(k)}}{n^k}$ converges to zero; hence also, using Lemma I, $\dfrac{S_n^{(i)}}{n^k}$ converges to zero, where i is an integer less than k.

Next let k not be an integer, and let K be the integer next less than k.

It will be proved that

$$\sigma_k(\omega) = \frac{k(k-1)\ldots(k-K)}{K!}\int_0^\omega \sigma_K(t)\,(\omega-t)^{k-K-1}\,dt.$$

Integrating by parts, we have

$$\begin{aligned}\int_0^\omega \sigma_K(t)\,(\omega-t)^{k-K-1}\,dt &= \left[-\,\sigma_K(t)\,\frac{(\omega-t)^{k-K}}{k-K}\right]_0^\omega + \int_0^\omega \sigma_K'(t)\,\frac{(\omega-t)^{k-K}}{k-K}\,dt\\ &= \frac{1}{k-K}\int_0^\omega \sigma_K'(t)\,(\omega-t)^{k-K}\,dt.\end{aligned}$$

Proceeding in this manner, we have, since $\sigma_K^{(K)}(t) = K!\sum_{r<K} a_r$,

$$\begin{aligned}&\int_0^\omega \sigma_K(t)\,(\omega-t)^{k-K-1}\\ &\quad= \frac{1}{(k-K)(k-K+1)\ldots(k-1)}\int_0^\omega K!\sum_{r<K} a_r\,(\omega-t)^{k-1}\,dt\\ &\quad= \frac{K!}{(k-K)\ldots(k-1)\,k}\sum_{r<\omega} a_r\,(\omega-r)^k.\end{aligned}$$

Thus $\sigma_k(\omega)$ is a fixed multiple of $\int_0^\omega \sigma_K(t)(\omega - t)^{k-K-1}\,dt$. Dividing the integral in two parts, in which $(0, n)$, (n, ω) are the intervals of integration, respectively, we shall consider first $\int_0^n \sigma_K(t)(\omega - t)^{k-K-1}\,dt$.

It can be shewn that $\sigma_K(t) = M\int_0^t \sigma_k'(u)(t-u)^{-(k-K)}\,du$; for

$$\sigma_k'(u) = k\sum_{r<u} a_r (u-r)^{k-1},$$

and $$\int_0^t \sigma_k'(u)(t-u)^{-(k-K)}\,du = k\sum_{r<t} a_r \int_r^t (u-r)^{k-1}(t-u)^{-(k-K)}\,du.$$

Changing the variable in the integral on the right hand side to w, where $t - u = (t-r)w$, the expression becomes

$$k\sum_{r<t} a_r \int_0^1 (t-r)^K (1-w)^{k-1} w^{-(k-K)}\,dw$$

which is a fixed multiple $\frac{1}{M}$ of $\sigma_K(t)$.

We now have

$$\int_0^n \sigma_K(t)(\omega-t)^{k-K-1}\,dt = M\int_0^n (\omega-t)^{k-K-1}\,dt\int_0^t \sigma_k'(u)(t-u)^{-(k-K)}\,du$$
$$= M\int_0^n \sigma_k'(u)\,du\int_u^n (t-u)^{-(k-K)}(\omega-t)^{k-K-1}\,dt;$$

the validity of the inversion of the order of integration following from the fact that the repeated integral exists when the integrand is changed into its absolute value (see I, § 429).

Let $\int_u^n (t-u)^{-(k-K)}(\omega-t)^{k-K-1}\,dt$ be denoted by $\psi(u)$; thus

$$\int_0^n \sigma_K(t)(\omega-t)^{k-K-1}\,dt = M\int_0^n \sigma_k'(u)\,\psi(u)\,du.$$

The function $\psi(u)$ diminishes as u increases, for it is the difference of

$$\int_u^\omega (t-u)^{-(k-K)}(\omega-t)^{k-K-1}\,dt \text{ and } \int_n^\omega (t-u)^{-(k-K)}(\omega-t)^{k-K-1}\,dt;$$

and the latter integral increases with u, whereas the former reduces, by substituting the new variable w, given by $(t-u) = (\omega-u)w$, to a constant.

Employing the second mean value theorem, we have

$$\int_0^n \sigma_K(t)(\omega-t)^{k-K-1}\,dt = M\,\psi(0)\,\sigma_k(\tau)$$

where τ is in the interval $(0, n)$. To express $\psi(0)$, we have

$$\psi(0) = \int_0^n t^{-(k-K)}(\omega-t)^{k-K-1}\,dt.$$

Let $t = nt'$, we have then

$$\psi(0) = \int_0^{t'} n^{1-(k-K)} t'^{-(k-K)} (\omega - nt')^{k-K-1}\, dt' < \int_0^{t'} t'^{-(k-K)} (1 - t')^{k-K-1}\, dt.$$

Thus $\psi(0)$ is bounded for all values of n; or

$$\left| \int_0^n \sigma_K(t)\, (\omega - t)^{k-K-1}\, dt \right| < A\sigma_k(\tau),$$

where τ is in the interval $(0, n)$, and A is independent of n and ω.

Let it now be assumed that $\lim\limits_{\omega\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k} = 0$; if we suppose that $\overline{\lim\limits_{\omega\sim\infty}} \left| \dfrac{\sigma_k(\tau)}{\omega^k} \right| > 0$, it is possible to choose a positive number η, and a sequence $\{\omega_m\}$, of values of ω, so that $\left| \dfrac{\sigma_k(\tau_m)}{\omega_m^k} \right| > \eta$, for all values of m, where τ_m is the value of τ that corresponds to ω_m. Now $\left| \dfrac{\sigma_k(\tau)}{\tau^k} \right| < \eta$, if τ is greater than some fixed number β; and it thus follows that $\tau_m \leq \beta$, for all values of m; and then $\lim\limits_{m\sim\infty} \dfrac{\tau_m}{\omega_m} = 0$, from which it follows that $\lim\limits_{m\sim\infty} \dfrac{\sigma_k(\tau_m)}{\omega_m^k} = 0$; contrary to the hypothesis. Hence $\overline{\lim\limits_{\omega\sim\infty}} \left| \dfrac{\sigma_k(\tau)}{\omega^k} \right|$ must be zero, and therefore

$$\lim_{\omega\sim\infty} \frac{1}{\omega^k} \int_0^n \sigma_K(t)\, (\omega - t)^{k-K-1}\, dt = 0.$$

Next, let us consider $\int_n^\omega \sigma_K(t)\, (\omega - t)^{k-K-1}\, dt$; on successive integration by parts, this is equal to

$$\Big[\sigma_K(t) \frac{(\omega - t)^{k-K}}{k - K} + \sigma_K'(t) \frac{(\omega - t)^{k-K+1}}{(k - K)(k - K + 1)} + \dots + \sigma_K^{(K)} \frac{(\omega - t)^k}{(k - K)\dots k} \Big]_\omega^n,$$

or to

$$\sigma_K(n) \frac{(\omega - n)^{k-K}}{k - K} + \sigma_K'(n) \frac{(\omega - n)^{k-K+1}}{(k - K)(k - K + 1)} + \dots + \sigma_K^{(K)} \frac{(\omega - n)^k}{(k - K)\dots k};$$

and this is a linear function of $\sigma_K(n), \sigma_{K-1}(n), \dots \sigma_0(n)$.

If we assign to ω, $K + 1$ different values $\omega_1, \omega_2, \dots \omega_{K+1}$, all within the interval $(n, n + 1)$, we obtain $K + 1$ such linear functions. As the determinant of these linear functions is a multiple of

$$\begin{vmatrix} 1, & \omega_1 - n, & (\omega_1 - n)^2, & \dots & (\omega_1 - n)^K \\ 1, & \omega_2 - n, & (\omega_2 - n)^2, & \dots & (\omega_2 - n)^K \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \omega_{K+1} - n, & \dots & \dots & (\omega_{K+1} - n)^K \end{vmatrix}$$

which is a multiple of the product of the differences of pairs of

$$\omega_1, \omega_2, \dots \omega_{K+1},$$

the determinant does not vanish, and therefore the linear functions are independent. Therefore $\sigma_K(n)$, $\sigma_{K-1}(n)$, ... $\sigma_0(n)$ are all expressible as linear functions of the $K+1$ expressions $\int_n^{\omega_r} \sigma_K(t)(\omega_r - t)^{k-K-1}\,dt$.

From theorem (*c*), $S_n^{(i)}$ is expressible as a linear function of

$$\sigma_0(n),\ \sigma_1(n),\ \ldots\ \sigma_i(n);$$

and it follows that $S_n^{(i)}$, for all integers $i \leqq K < k$ is expressible as a linear function of the $K+1$ expression $\int_n^{\omega_r} \sigma_K(t)(\omega_r - t)^{k-K-1}\,dt$. If now we assume that $\lim\limits_{\omega\sim 0} \dfrac{\sigma_k(\omega)}{\omega^k} = 0$, since $\lim\limits_{\omega\sim\infty} \dfrac{1}{\omega^k}\int_0^n \sigma_K(t)(\omega - t)^{k-K-1}\,dt = 0$, we have $\lim\limits_{\omega\sim 0} \dfrac{1}{\omega^k}\int_n^{\omega} \sigma_K(t)(\omega - t)^{k-K-1}\,dt = 0$.

If, as n increases, $\omega_1, \omega_2, \ldots \omega_{K+1}$ so increase that the differences $\omega_1 - n$, $\omega_2 - n$, ... $\omega_{K+1} - n$ remain constant, we see that $\lim \dfrac{S_n^{(i)}}{n^k} = 0$, for all integers $i < k$; thus the Lemma has been proved.

60. It remains to investigate Lemma III:

If $\lim\limits_{n\sim\infty} \dfrac{S_n^{(i)}}{n^k} = 0$, *for every integer i, less than k, then*

$$\lim_{n\sim\infty}\left(\frac{\sigma_k(\omega)}{\omega^k} - \frac{S_n^{(k)}}{c_n^{(k)}}\right) = 0, \text{ where } c_n^{(k)} \equiv \frac{\Gamma(k+n+1)}{\Gamma(k+1)\,n!}.$$

On account of Lemma I it follows that $\lim\limits_{n\sim\infty} \dfrac{S_n^{(K)}}{n^k} = 0$, where K is the integer next less than k.

Let $\overline{S}_n^{(K)}$ denote the maximum of $|S_r^{(K)}|$ for $r \leqq n$; it can be shewn that, if $\lim\limits_{n\sim\infty} \dfrac{S_n^{(K)}}{n^k} = 0$, then $\lim\limits_{n\sim\infty} \dfrac{\overline{S}_n^{(K)}}{n^k} = 0$. A number ν can be so fixed that $\dfrac{|S_r^{(K)}|}{r^k} < \epsilon$, for $r > \nu$; there exists a number M such that $\dfrac{|S_r^{(K)}|}{r^k} < M$, for $r = 1, 2, 3, \ldots \nu$. We have then, taking $n > \nu$,

$$\frac{\overline{S}_n^{(K)}}{n^k} \leqq \frac{r^k}{n^k} \cdot \frac{\textit{maximum of } |S_r^{(K)}|, \textit{ for } r \leqq n}{r^k}$$

$$\leqq \text{greater of the numbers } \epsilon,\ M\left(\frac{\nu}{n}\right)^k;$$

n can be so chosen that $M\left(\dfrac{\nu}{n}\right)^k < \epsilon$, and therefore

$$\frac{\overline{S}_n^{(K)}}{n^k} < \epsilon; \text{ and thus } \lim_{n\sim\infty}\frac{\overline{S}_n^{(K)}}{n^k} = 0.$$

We have now

$$\left|\frac{\sigma_k(\omega)}{\omega^k} - \frac{S_n^{(k)}}{c_n^{(k)}}\right| \leqq \frac{|\sigma_k(\omega) - S_n^{(k)}\Gamma(k+1)|}{\omega^k} + \left|S_n^{(k)}\left(\frac{\Gamma(k+1)}{\omega^k} - \frac{1}{c_n^{(k)}}\right)\right|;$$

and $\left|S_n^{(k)}\left(\frac{\Gamma(k+1)}{\omega^k} - \frac{1}{c_n^{(k)}}\right)\right|$

$$\leqq \left|S_n^{(k)}\Gamma(k+1)\left(\frac{1}{\omega^k} - \frac{1}{n^k}\right)\right| + \left|S_n^{(k)}\left(\frac{\Gamma(k+1)}{n^k} - \frac{1}{c_n^{(k)}}\right)\right|$$

$$\leqq \left|S_n^{(k)}\right| O\left(\frac{1}{n^{k+1}}\right)$$

as is seen by employing theorem (*b*), and remembering that

$$c_n^{(k)} = \frac{\Gamma(k+n+1)}{\Gamma(k+1)\,n!}.$$

Employing (5), of § 48, we have

$$\frac{S_n^{(k)}}{n^{k+1}} = \frac{1}{n^{k+1}}\left\{S_n^{(K)} + \sum_{p=1}^{p=n}\binom{k-K+p-1}{p} S_{n-p}^{(K)}\right\};$$

and therefore $$\frac{|S_n^{(k)}|}{n^{k+1}} < \frac{1}{n^{k+1}}\binom{k-K+n}{n}\bar{S}_n^{(K)}$$

$$< \frac{1}{n}\frac{\bar{S}_n^{(K)}}{n^k} O(n^{k-K}) = o(1).$$

It has now been shown that

$$\left|S_n^{(k)}\left(\frac{\Gamma(k+1)}{\omega^k} - \frac{1}{c_n^{(k)}}\right)\right| = o(1).$$

Next, we consider $\frac{|\sigma_k(\omega) - S_n^{(k)}\Gamma(k+1)|}{\omega^k}$. Employing theorem (*d*) and the formula (5) of § 48, we see that this is not greater than

$$\frac{1}{\omega^k}\left|\sum_{r=0}^{n-K-1} S_r^{(K)}\{\Delta^{K+1}(\omega-r)^k - \Gamma(k+1)\,c_{n-r}^{(k-K-1)}\}\right|$$

$$+ [S_n^{(K)}O(1) + S_{n-1}^{(K)}O(1) + \dots + S_{n-K}^{(K)}O(1)]\frac{1}{\omega^k}.$$

Since $\lim_{n\sim\infty}\frac{S_n^{(i)}}{\omega^k} = 0$, each of the $K+1$ expressions $\frac{S_n^{(K)}}{\omega^k}, \frac{S_{n-1}^{(K)}}{\omega^k}, \dots \frac{S_{n-K}^{(K)}}{\omega^k}$ is $o(1)$. In accordance with theorem (*a*), we have

$$\Delta^{K+1}(\omega-r)^k = k(k-1)\dots(k-K)(\omega-r)^{k-K-1} + O\{(\omega-r)^{k-K-2}\}$$

$$= k(k-1)\dots(k-K)(n-r)^{k-K-1} + O\{n-r)^{k-K-2}\}.$$

Also

$$\Gamma(k+1)\, c_{n-r}^{(k-K-1)} = k(k-1)\ldots(k-K)\,\frac{\Gamma(k-K+n-r)}{(n-r)!};$$

and $$\left| \Delta^{K+1}(\omega-r)^k - \Gamma(k+1)\, c_{n-r}^{k-K-1} \right| = O\{(n-r)^{k-K-2}\}$$

for $r \leq n-K-1$, from theorem (*b*). Thus we have

$$\begin{aligned}\frac{\left| \sigma_k(\omega) - S_n^{(k)}\Gamma(k+1) \right|}{\omega^k} &\leq \frac{1}{\omega^k} S_n^{(K)} \lambda \sum_{r=1}^{\infty} r^{k-K-2} + o(1) \\ &\leq o(1) + \omega^{-k} S_n^{(K)} O(1) \\ &= o(1);\end{aligned}$$

where λ denotes some fixed number.

The Lemma has now been established.

The equivalence theorem can be at once deduced from the three Lemmas. For, if it be assumed that $\lim\limits_{n\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k} = 0$, we have, from Lemma II, $\lim\limits_{n\sim\infty} \dfrac{S_n^{(i)}}{n^k} = 0$, $(i < k)$, and then, by Lemma III, $\lim\limits_{n\sim\infty} \dfrac{S_n^{(k)}}{n^k} = 0$. If it be assumed that $\lim\limits_{n\sim\infty} \dfrac{S_n^{(k)}}{n^k} = 0$, by Lemma I we have $\lim\limits_{n\sim\infty} \dfrac{S_n^{(i)}}{n^k} = 0$, $i < k$, and then, by Lemma III, $\lim\limits_{n\sim\infty} \dfrac{\sigma_k(\omega)}{\omega^k} = 0$.

CHAPTER II

FUNCTIONS DEFINED BY SEQUENCES OR SERIES

61. Let $s_1(x)$, $s_2(x)$, $s_3(x)$, ... $s_n(x)$, ... be a sequence of functions defined for the values of x in some given set of points E. All the functions $s_n(x)$ will, in the first instance, be taken to be single-valued functions of x, in the sense that, at each point of E, $s_n(x)$ has a single finite value, either finite, or $+\infty$, or $-\infty$. The function $s_n(x)$, when everywhere finite, is not necessarily bounded in the set E. The set E may be a linear set, or a p-dimensional set, in which case x symbolizes a point $(x_1, x_2, \ldots x_p)$, of the set. The set E is said to be the field, or domain, of the variable x, for which the functions are defined. It need not be assumed in the first instance to be restricted in any special manner; thus it is not necessarily bounded or closed.

At any point ξ, of the domain of the functions, the sequence of numbers $s_1(\xi)$, $s_2(\xi)$, ... $s_n(\xi)$, ... may either (1), have a single finite limiting point, in which case the sequence $\{s_n(x)\}$ is said to be convergent at the point ξ; or (2), it may have a single improper limiting point ∞, or $-\infty$, but no further limiting point, in which case the sequence $\{s_n(x)\}$ is said to be divergent at the point ξ; or (3), it may have a set of limiting points, either finite but containing more than one point, or infinite, which may include either, or both, of the improper points $+\infty$, $-\infty$; in this last case the sequence is said to oscillate at the point ξ.

Let the function $s(x)$ be defined in the field E, for which the functions of the sequence are defined, by the rules that, at any point ξ at which the sequence $\{s_n(x)\}$ is convergent, $s(\xi)$ is the number to which the sequence converges; at any point at which the sequence diverges, $s(\xi)$ has the value $+\infty$, or $-\infty$, as the case may be; and at any point ξ, at which the sequence oscillates, $s(\xi)$ is multiple-valued, having for its stock of values those defined by the limiting points, finite or infinite, of the sequence $\{s_n(\xi)\}$.

If $U(\xi)$, $L(\xi)$ are respectively the upper and the lower boundaries of all the numbers $s_n(\xi)$, the two functions $U(x)$, $L(x)$ are single-valued functions which may be termed the *upper boundary function* and the *lower boundary function* of the sequence $\{s_n(x)\}$. Either or both of the numbers $U(\xi)$, $L(\xi)$ may be infinite.

The set of values of $s(x)$ at any point ξ, when it does not consist of a single point, necessarily consists of a closed linear set of points, the term closed set being extended, when necessary, to include cases in which one or both of the points $+\infty$, $-\infty$ belong to the set. It should be re-

membered that if, for an infinite set of values of n, the values of $s_n(\xi)$ are all identical, their common value must be reckoned as belonging to the closed linear set of values of $s(\xi)$.

The upper and lower boundaries of this closed set, of values of $s(\xi)$, may be denoted by $\bar{s}(\xi)$, $\underline{s}(\xi)$, where either, or both, of these may be either finite or infinite. The single-valued functions $\bar{s}(x)$, $\underline{s}(x)$ defined in the field E, as having at each point ξ, the values respectively of $\bar{s}(\xi)$, $\underline{s}(\xi)$ are termed the *upper limiting function* and the *lower limiting function*, respectively, or simply the *upper and lower functions*, defined by the sequence $\{s_n(x)\}$.

At a point ξ, of convergence, or divergence of $\{s_n(x)\}$, we have

$$\bar{s}(\xi) = \underline{s}(\xi).$$

If ϵ be an arbitrarily chosen positive number, and if $\bar{s}(\xi)$, $\underline{s}(\xi)$ are both finite, $s_n(\xi)$ must lie in the interval $(\underline{s}(\xi) - \epsilon, \bar{s}(\xi) + \epsilon)$, for every value of n, with the possible exception of a finite number of such values. If $\bar{s}(\xi) = \infty$, and $\underline{s}(\xi)$ is finite, only a finite set of the numbers $s_n(\xi)$ can be less than $\underline{s}(\xi) - \epsilon$.

It is clear that, at every point x, the relations $U(x) \geqq \bar{s}(x) \geqq \underline{s}(x) \geqq L(x)$, are satisfied.

In case, at each point ξ, of E, the sequence $\{s_n(x)\}$ is convergent, $\bar{s}(x) = \underline{s}(x)$, and the limiting function $s(x)$ is single-valued and finite. If, at each point of E, the sequence $\{s_n(x)\}$ is either convergent or divergent, $s(x)$ is also single-valued, but at each point of divergence of the sequence it has for its value either ∞, or $-\infty$, as the case may be.

If $(p_1, p_2, \dots p_n, \dots)$ be a sequence of increasing positive integers, the sequence $\{s_{p_n}(x)\}$ may be said to be a *sub-sequence of the sequence* $\{s_n(x)\}$. Such a sub-sequence will have an upper function that is $\leqq \bar{s}(x)$, and a lower function that is $\geqq \underline{s}(x)$. If a sub-sequence be convergent, it may have for its limiting function either $\bar{s}(x)$ or $\underline{s}(x)$ or some function whose value at each point is a limiting point of $\{s_n(x)\}$ in the interval bounded by $\bar{s}(x)$ and $\underline{s}(x)$. When all possible sub-sequences of $\{s_n(x)\}$ are taken into account, the totality of their upper functions may be spoken of as the *set of upper functions* of the sequence $\{s_n(x)\}$. Similarly the set of lower functions of the sequence is defined as the totality of the lower functions of all sub-sequences.

62. If the method of transformation given in I, § 219 be applied to the functions of the sequence $\{s_n(x)\}$, defining the function $\sigma_n(x)$ by

$$\sigma_n(x) = \frac{s_n(x)}{1 + |s_n(x)|},$$

and $\sigma(x)$ by $\sigma(x) = \dfrac{s(x)}{1 + |s(x)|}$, we observe the fact that, for any point

ξ, corresponding to the set of values of $s(\xi)$ in $(-\infty, \infty)$, which is closed either in the ordinary sense, or in the extended sense in which $+\infty$ and $-\infty$ are admissible points of the set, there exists a closed set of values of $\sigma(\xi)$ in the interval $(-1, 1)$; moreover the converse of this holds good. Divergence of $\{s_n(\xi)\}$ to $+\infty$ implies convergence of $\{\sigma_n(\xi)\}$ to the point 1, and divergence of $\{s_n(\xi)\}$ to $-\infty$ implies convergence of $\{\sigma_n(\xi)\}$ to the value -1.

From this point of view, the distinction between convergence and divergence of a sequence, at a point, is unessential, whereas oscillation is essentially distinct from either. Thus, for example, if $\{s_n(x)\}$ be at all points of E, either convergent or divergent, the sequence $\{\sigma_n(x)\}$ is, at all points of E, convergent.

In case $+\infty$, or $-\infty$, is the value of $s_n(x)$ at a particular point ξ, the corresponding value of $\sigma_n(x)$ is 1, or -1, as the case may be.

It is sometimes convenient to modify the transformation just employed. If $\{\sigma_n(x)\}$ be a sequence of functions of which the values all lie in the interval $(-1, 1)$, we may take $s_n(x) = \dfrac{k_n \sigma_n(x)}{1 - k_n |\sigma_n(x)|}$, $s(x) = \dfrac{\sigma(x)}{1 - |\sigma(x)|}$, where $\{k_n\}$ is a monotone increasing sequence of positive numbers converging to 1, as limit. The advantage which this transformation has, over the one above which corresponds to the case $k_n = 1$, is that $s_n(x)$ is necessarily bounded, for each value of n, being numerically $\leqq \dfrac{k_n}{1 - k_n}$.

63. If $u_1(x), u_2(x), \ldots u_n(x), \ldots$ be a sequence of functions defined in a given linear, or p-dimensional, set of points E, let

$$s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x);$$

then the sequence $\{s_n(x)\}$ defines, as explained above, the limiting function $s(x)$. This function may be termed the sum-function of the infinite series

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots.$$

It thus appears that the theory relating to the sum-function defined by an infinite series, each term of which is a function of one or more variables, is identical with the theory of the limiting functions of a sequence of functions defined in the given domain E. Thus any theorem relating to the theory of infinite series of functions of one or more variables can be stated as a theorem relating to sequences of functions. The functions $\bar{s}(x)$, $\underline{s}(x)$ may be termed the *upper sum-function*, and the *lower sum-function* of the given series. At a point ξ, of convergence or of divergence of the series, we have $s(\xi) = \bar{s}(\xi) = \underline{s}(\xi)$, the number $s(\xi)$ being finite at a point of convergence, and either ∞ or $-\infty$, as the case may be, at a point of divergence of the series.

FUNCTIONS RELATED WITH A GIVEN FUNCTION.

64. In I, § 220, the maximal and minimal functions at a point x, of the domain of a single-valued function of a single variable, have been defined. These definitions can be extended to the case of a function $s(x)$, of any number of variables, when the function is not necessarily single-valued, but may have, at each point x, upper and lower boundaries $U(x)$, $L(x)$, and upper and lower limits $u(x)$, $l(x)$, each of which may be finite or infinite. The values of $s(x)$, for each x, form a closed set, when $s(x)$ is defined, as in § 61, by means of a sequence $\{s_n(x)\}$; but in case $s(x)$ is not defined in that manner it need not be assumed that the set of values of $s(x)$, at x, is closed.

Let E denote the domain of the function $s(x)$, and let ξ be a limiting point of E which belongs to the set. The upper boundary of the numbers $U(x)$, for all points x, of E, in a neighbourhood Δ, of ξ, converges as the span of Δ converges to zero, to a number $M(\xi)$ which is the value, at ξ, of the *maximal function* $M(x)$, *associated with* $s(x)$. Similarly, the lower boundary of the numbers $L(x)$, for all points x, of E, in the neighbourhood Δ, of ξ, converges, as the span of Δ converges to zero, to a number $m(\xi)$, which is the value, at ξ, of the *minimal function* $m(x)$ associated with $s(x)$.

In these definitions, the values of x include ξ itself; but if the value ξ be excluded from the permissible values of x, so that the values of $s(\xi)$ are irrelevant, we obtain, instead of $M(\xi)$, and $m(\xi)$, numbers $A(\xi)$, $a(\xi)$, the values of which are termed respectively those of *the upper associated function* $A(x)$, and *the lower associated function* $a(x)$. The number $M(\xi)$ is clearly the greater of the numbers $U(\xi)$, $A(\xi)$; and the number $m(\xi)$ is the lesser of the two numbers $L(\xi)$, $a(\xi)$. The definitions are applicable also to a point ξ, of E', which does not belong to E, and at such a point $M(\xi) = A(\xi)$, and $m(\xi) = a(\xi)$. At an isolated point ξ, of E, the associated functions do not exist, but $M(\xi) = U(\xi)$, and $m(\xi) = L(\xi)$.

The above definitions may be stated more explicitly in the following form:

If ξ be a limiting point of the set E in which the single or multiple valued function $s(x)$ is defined, and if $\{\Delta_m\}$ be a sequence of neighbourhoods of ξ, each of which contains the next, and which converge to the point ξ, then if $\bar{U}(\Delta_m)$ denote the upper boundary of $U(x)$, for all points x, of E, in Δ_m, the non-increasing sequence $\{\bar{U}(\Delta_m)\}$ has a lower limit $M(\xi)$, as $m \sim \infty$, which is taken to be the value, at ξ, of the maximal function $M(x)$. Similarly, if $\bar{L}(\Delta_m)$ denote the lower boundary of $L(x)$, for all points x, of E, in Δ_m, the non-diminishing sequence $\{\bar{L}(\Delta_m)\}$ has an upper limit $m(\xi)$, which is taken to be the value, at ξ, of the minimal function $m(x)$.

If $\overline{\overline{U}}(\Delta_m)$ *denote the upper boundary of* $U(x)$, *for all points* x, *of* E, *except* ξ, *in* Δ_m; *and* $\overline{\overline{L}}(\Delta_m)$ *denote the lower boundary of* $L(x)$, *for all points* x, *of* E, *except* ξ, *in* Δ_m, *the limits* $A(\xi)$, $a(\xi)$ *of the two monotone sequences* $\{\overline{\overline{U}}(\Delta_m)\}$, $\{\overline{\overline{L}}(\Delta_m)\}$, *define the values, at* ξ, *of the upper and lower associated functions* $A(x)$, $a(x)$.

It is easily seen that the four numbers defined are independent of the particular sequence, $\{\Delta_m\}$, of neighbourhoods employed. For, if $\{\Delta'_m\}$ be any other such sequence, and m be sufficiently large, Δ'_m is contained in a neighbourhood $\Delta_{m'}$; and also Δ'_m contains $\Delta_{m''}$, if m'' be sufficiently large; so that $U(\Delta'_m)$ lies between $U(\Delta_{m'})$ and $U(\Delta_{m''})$. A similar argument applies to all four numbers.

It is seen from the definitions of the maximal, minimal, and associated functions that, at every point, they satisfy the conditions

$$M(x) \geqq A(x) \geqq a(x) \geqq m(x).$$

65. In accordance with the definition of $M(\xi)$, having given an arbitrarily chosen positive number ϵ, a neighbourhood Δ, of ξ, can be so determined that the upper boundary $U(\Delta)$, of $s(x)$, for all points of E, in Δ, is $< M(\xi) + \epsilon$, and that there exists one point x of E, at least, in Δ, at which $U(x) > M(\xi) - \epsilon$.

Similarly Δ can be so determined that the lower boundary $L(\Delta)$, of $s(x)$, in Δ, is $> m(\xi) - \epsilon$, and which contains at least one point at which

$$L(x) < m(\xi) + \epsilon.$$

It is clear that $M(\xi) = m(\xi)$ is the necessary and sufficient condition that $s(x)$ should be continuous at the point ξ. This condition may also be expressed by $s(\xi) = A(\xi) = a(\xi)$. It is also clear that the necessary and sufficient condition that $s(x)$ should be upper semi-continuous at ξ is that $M(\xi) = s(\xi)$, and that $m(\xi) = s(\xi)$ is the necessary and sufficient condition that $s(x)$ should be lower semi-continuous at ξ. It is here assumed that $s(\xi)$ has a single value.

For the case of a single-valued function it has been shewn by W. H. Young* that the relation $A(x) \geqq s(x) \geqq a(x)$ holds, except possibly at points of an enumerable set.

It can be shewn that:

The functions $M(x)$, $A(x)$ *are both upper semi-continuous, and the functions* $m(x)$, $a(x)$ *are both lower semi-continuous.*

For if, in every neighbourhood of ξ, there are points at which

$$M(x) > M(\xi) + \epsilon,$$

for some fixed value of ϵ, there must be points at which $U(x) > M(\xi) + \epsilon$,

* *Quart. Journ.* vol. XXXIX (1908), p. 82, and *Proc. Lond. Math. Soc.* (2), vol. VIII (1910), p. 119.

and this is inconsistent with the fact that $M(\xi)$ is the value of the maximal function at ξ. A similar argument applies to the function $A(x)$. The property of $m(x)$ and $a(x)$ is established in a similar manner.

UNIFORM CONVERGENCE OF SEQUENCES AND SERIES.

66. If, in any domain E, of one or more dimensions, for which the sequence $\{s_n(x)\}$, of single valued functions $s_n(x)$, is defined, the limiting function $s(x)$ has, at each point of the domain, a single finite value, the sequence $\{s_n(x)\}$ is said to be convergent in the given domain.

At each point ξ, of the given domain, the condition is satisfied that, if ϵ be an arbitrarily prescribed positive number, $|s(\xi) - s_n(\xi)| < \epsilon$, for all values of n which are not less than some definite integer, dependent upon ϵ, and in general also upon the particular point ξ. We may denote the smallest integer which satisfies this condition by $n(\epsilon, \xi)$. A very important case of convergence arises when the numbers $n(\epsilon, \xi)$ have, for each fixed value of ϵ, a finite upper boundary, when all points ξ, of E, are taken into account. In this case, $n(\epsilon)$, or n_ϵ, can be so chosen that, $|s(x) - s_n(x)| < \epsilon$, for $n \geqq n_\epsilon$, everywhere in E. The convergence of the sequence $\{s_n(x)\}$ is then said to be uniform in the given domain. We have thus the following definition of uniform convergence:

If, in a given domain, of one or more dimensions, the sequence $\{s_n(x)\}$ of single-valued functions, everywhere converges to the value of a function $s(x)$, finite at each point of the domain, and if, corresponding to each arbitrarily prescribed positive number ϵ, an integer n_ϵ can be so determined that

$$|s(x) - s_n(x)| < \epsilon,$$

provided $n \geqq n_\epsilon$, and for all values of x in the given domain, so that n_ϵ is independent of x, the convergence of $\{s_n(x)\}$ is said to be uniform in the given domain.

The criterion of uniform convergence may be stated in the following form, in which the conditions of convergence and of uniform convergence are combined in one statement:

If, in a given domain, of one or more dimensions, the sequence $\{s_n(x)\}$ satisfies the condition that, corresponding to each arbitrarily chosen positive number ϵ, a number n_ϵ can be so determined that $|s_n(x) - s_{n'}(x)| < \epsilon$, provided n and n' are any pair of integers such that $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, whatever value x may have in the given domain, the sequence $\{s_n(x)\}$ is said to be uniformly convergent in the given domain.

In case the sequence consists of the partial sums of a series

$$u_1(x) + u_2(x) + u_3(x) + \dots,$$

where the terms are functions which have single definite values at each

point of a given linear or p-dimensional domain, the definition of uniform convergence of the series may be stated as follows:

If, corresponding to each arbitrarily assigned positive number ϵ, a value of n, independent of x, can be so determined that

$$|R_{n,1}(x)|,\ |R_{n,2}(x)| \dots |R_{n,s}(x)| \dots$$

are all less than ϵ, for every value of x, the series $u_1(x) + u_2(x) + \dots$ is said to converge uniformly in the given domain.

In case the convergence of the series at each point of the given domain is assumed, the condition of uniform convergence may be stated thus:

If the series $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ converge, for each value of x, in a given linear, or p-dimensional, domain, the series is said to converge uniformly in that domain provided that, corresponding to each arbitrarily assigned positive number ϵ, a number n, independent of x, can be determined such that all the remainders $R_n(x)$, $R_{n+1}(x)$, ... are, in absolute value, less than ϵ, for every value of x in the given domain.

In case a sequence $\{s_n(x)\}$ converges uniformly in a set E, it is clear that the sequence also converges uniformly in any part E_1, of E. But if $(E_1, E_2, \dots E_r, \dots)$ be a sequence of sets, each one of which is contained in the next, and of which E is the outer limiting set, a sequence $\{s_n(x)\}$, defined in E, may converge uniformly in each of the sets E_r, and yet may not converge uniformly in E. If $n(r, \epsilon)$ be the least integer such that, at every point of E_r, $|s(x) - s_n(x)| < \epsilon$, provided $n \geqq n(r, \epsilon)$, it may happen that, for some value of ϵ, $n(r, \epsilon)$ has no upper boundary for $r = 1, 2, 3, \dots$; in that case there exists no integer $n(\epsilon)$ such that $|s(x) - s_n(x)| < \epsilon$, for $n \geqq n(\epsilon)$, and for all points of E; the convergence is in that case not uniform in E. For example, let $s(x) - s_n(x) = \frac{x}{n}$, in the infinitely great semi-closed linear interval $(0 \leqq x)$. In any interval $(0, h)$, where $h > 0$, $|s(x) - s_n(x)| < \epsilon$, if $n > h/\epsilon$; but there is no value of n for which $|s(x) - s_n(x)| < \epsilon$ in the whole interval $0 \leqq x$. Thus, although the sequence converges uniformly in every finite interval $(0, h)$, it does not converge uniformly in the infinite interval $(0 \leqq x)$.

SIMPLY UNIFORM CONVERGENCE.

67. A mode of convergence of a series, or sequence, in a given domain, less stringent in character than that of uniform convergence, has been considered by Dini and by other writers. This mode of convergence has been termed by Dini "simple uniform convergence," and has been defined by him* as follows:

The series $u_1(x) + u_2(x) + u_3(x) + \dots$ which converges at each point x, of a given domain, to the value $s(x)$, is said to be simply-uniformly convergent

* See Dini's *Grundlagen*, by Lüroth and Schepp, p. 137.

in the domain if, corresponding to each arbitrarily assigned positive number ϵ, *and to each arbitrarily assigned integer* m', *at least one integer* m, *not less than* m', *can be so determined that, for all values of* x *in the domain,*

$$|R_m(x)| < \epsilon.$$

The condition of simple-uniform convergence is less stringent than that of uniform convergence, in that, in the latter case, all the remainders after a certain one are numerically less than ϵ, whereas in the former case, not necessarily all the remainders are, for all the values of x, numerically less than ϵ.

As regards the above definition, it should be remarked that, if there is one integer m ($\geqq m'$) such that $|R_m(x)| < \epsilon$, for all the values of x, there must be an infinite set of such integers. For we have only to ascribe to m' successively values which increase indefinitely, and for each of these there exists a corresponding value of m.

Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of diminishing positive numbers which converges to zero. If $\Sigma u(x)$ be a simply-uniformly convergent series, n_1 can be determined so that $|R_{n_1}(x)| < \epsilon_1$, for all the values of x; then an integer n_2 ($> n_1$) can be so determined that $|R_{n_2}(x)| < \epsilon_2$; then n_3 ($> n_2$) so that $|R_{n_3}(x)| < \epsilon_3$; and so on.

It follows that the sequence $s_{n_1}(x), s_{n_2}(x), s_{n_3}(x), \ldots$ converges uniformly to $s(x)$. If now the first n_1 terms of the series be amalgamated into one term, then those after the first n_1 up to, and including $u_{n_2}(x)$, and so on, the series is transformed into

$$s_{n_1}(x) + [s_{n_2}(x) - s_{n_1}(x)] + [s_{n_3}(x) - s_{n_2}(x)] \ldots;$$

and in this form the series is uniformly convergent.

It has thus been shewn that:

A simply-uniformly convergent series can be changed into one which is uniformly convergent, by bracketing the terms suitably, in accordance with some norm, and taking each bracket to constitute a term of the new series.

It thus appears that, if $\{s_n(x)\}$ is a convergent sequence, it is simply-uniformly convergent provided it contains $\{s_{n_p}(x)\}$, $(p = 1, 2, 3, \ldots)$, a sub-sequence which is uniformly convergent in the domain.

It should be observed that, when the sequence $\{s_n(x)\}$ does not converge everywhere in the domain of x, it may still be possible to determine a sub-sequence $\{s_{n_p}(x)\}$, $p = 1, 2, 3, \ldots$ which converges uniformly in the domain of x.

If each term $u_n(x)$ of a uniformly convergent series be replaced, in accordance with some norm, by the sum of r_n functions, such that

$$u_n(x) = U_{n,1}(x) + U_{n,2}(x) + \ldots + U_{n,r_n}(x),$$

then the new series

$$U_{1,1}(x) + U_{1,2}(x) + \dots + U_{1,r_1}(x) + U_{2,1}(x) + \dots$$

is not necessarily convergent, but may at any point of the domain be oscillatory. If, however, the series be convergent in the domain of x, it converges at least simply-uniformly. In any case the series is reducible to a uniformly convergent series by introducing a suitable set of brackets and amalgamating the terms in each bracket. It thus appears* that the distinction between uniform convergence and simply-uniform convergence is less fundamental than might at first sight have been supposed.

68. A series which converges for every value of x in a given domain is certainly simply-uniformly convergent in that domain in case there exist an infinite set of values of n such that $R_n(x) = 0$ for all the values of x.

Let us next suppose that there are at most a finite set of values of n such that $R_n(x) = 0$ for all these values of n, and for all values of x, in E. It will be shewn that the definition of simply-uniform convergence can, in this case, be reduced to a simpler form, viz. that, for each ϵ, a number n can be determined so that $|R_n(x)| < \epsilon$, for every value of x, and such that $R_n(x)$ does not vanish everywhere. Let us denote by $\bar{R}_n$ the upper boundary of $|R_n(x)|$ in the domain of x; $\bar{R}_n$ may be either infinite or finite. Let it be assumed that there exists one value of n, such that $|R_n(x)| < \epsilon$, and such that $R_n(x)$ does not vanish for all values of x in the domain; we have then $\bar{R}_n \leqq \epsilon$. Let us take a positive number ϵ_1 less than $\bar{R}_n$, and also less than all of those numbers $\bar{R}_1, \bar{R}_2, \dots \bar{R}_{n-1}$ which do not vanish. By hypothesis there exists an integer n_1, such that $|R_{n_1}(x)| < \epsilon_1 < \epsilon$, and such that $R_{n_1}(x)$ does not everywhere vanish. This number n_1 cannot be one of the numbers 1, 2, 3, ... n; for it is always possible to determine a value of x for which $|R_n(x)|$ is arbitrarily near its upper boundary, and is thus $> \epsilon_1$. Similarly it may be shewn that an integer n_2 ($> n_1$) exists which has the same property such that $|R_{n_2}(x)| < \epsilon$. Thus an indefinitely great set of values of n can be so determined, for which $|R_n(x)| < \epsilon$, for every x; and the condition in Dini's definition is satisfied. We have accordingly the following modified form of Dini's definition:

A series which converges for every value of x in a given linear, or p-dimensional, domain is said to converge simply-uniformly either, (1) *if there are at most a finite set of values of n for which $R_n(x) = 0$, for all the values of x, and if, corresponding to each arbitrarily assigned positive number ϵ, an integer n, independent of x, can be so determined that $|R_n(x)| < \epsilon$, for all the values of x, whilst $R_n(x)$ does not vanish for all the values of x, or* (2) *if there be an indefinitely great set of values of n for which $R_n(x) = 0$ for all the values of x.*

* See Arzelà, *Bologna Rendiconti* (5), vol. VIII (1899); also Hobson, *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 376.

A series which is uniformly convergent is also simply-uniformly convergent, but the converse does not hold.

If the series be simply-uniformly convergent, but be not uniformly convergent, there must, corresponding to each sufficiently small ϵ, be an indefinitely great set of values of n for which the condition $| R_n(x) | < \epsilon$, for all the values of x, is not satisfied; for if there were only a finite set of such values, n could be taken greater than the greatest of these, and thus the condition for uniform convergence would be satisfied, which is contrary to hypothesis.

If all the terms of a series $\Sigma u_n(x)$ be non-negative for all values of x in the domain of the variable, and if the series $\Sigma u_n(x)$ is simply-uniformly convergent, then it is necessarily uniformly convergent. For in this case $\{s_n(x)\}$ is a monotone non-diminishing sequence, for each value of x. If

$$| s(x) - s_n(x) | < \epsilon,$$

for any value of n, and for all the values of x, it follows that the inequality holds good for all greater values of n, and therefore the convergence is uniform.

UNIFORM DIVERGENCE AND UNIFORM APPROACH.

69. Let it be assumed that, in a set E_1, the sequence $\{s_n(x)\}$ diverges at each point, either to $+\infty$ or to $-\infty$.

If, corresponding to each arbitrarily assigned positive number N, an integer n_N can be so determined that, at each point of E_1, one or other of the conditions $s_n(x) > N$, $s_n(x) < -N$, according as the divergence is to $+\infty$ or to $-\infty$, is satisfied provided $n \geqq n_N$, the number n_N being independent of x, whatever point x may be, in E_1, the sequence is said to *diverge uniformly in* E_1.

If E_1 be a part of a domain E, for which the functions of the sequence $\{s_n(x)\}$ are defined, and that sequence converges uniformly in $E - E_1$, whilst it diverges uniformly in E_1, then the sequence is said to approach $s(x)$ uniformly in E. The term *uniform approach* may be taken to include uniform divergence and uniform convergence.

The definition of uniform approach may be stated as follows:

If, corresponding to each arbitrarily assigned pair of positive numbers A, ϵ, an integer $n(A, \epsilon)$, independent of x, can be so determined that, at each point of convergence of $\{s_n(x)\}$, $| s(x) - s_n(x) | < \epsilon$ and at each point of divergence $s_n(x) > A$, or $s_n(x) < -A$, according as the divergence is to ∞ or to $-\infty$, provided in each case $n \geqq n(A, \epsilon)$, the sequence is said to approach $s(x)$ uniformly in E, it being assumed that the sequence is not oscillatory at any point of E.

The justification for this terminology is to be found in the fact that, if the transformation $\sigma_n(x) = \dfrac{s_n(x)}{1 + |s_n(x)|}$ be employed, the sequence $\{\sigma_n(x)\}$ is uniformly convergent in E, in accordance with the definition of § 66, provided $\{s_n(x)\}$ approaches $s(x)$ uniformly in E.

To prove the theorem, let η be an arbitrarily chosen positive number, and let A and ϵ be such that $(1 + A)^{-1} < \eta$, $\epsilon < \frac{1}{3}\eta$. At a point at which $s_n(x) > A$, for $n \geqq m$ we have

$$|1 - \sigma_n(x)| = \left|1 - \frac{s_n(x)}{1 + s_n(x)}\right| < \frac{1}{1 + A} < \eta;$$

and similarly, at a point at which $s_n(x) < -A$, we have $|-1 - \sigma_n(x)| < \eta$, for $n \geqq m$. At a point at which $|s(x) - s_n(x)| < \epsilon$, for $n \geqq m$, we have $|\sigma(x) - \sigma_n(x)| < |s(x) - s_n(x)| < \epsilon < \eta$, provided $|s(x)| > \epsilon$, in which case $s(x)$ and $s_n(x)$ have the same sign. If, however, $|s(x)| \leqq \epsilon$, we have $|\sigma(x) - \sigma_n(x)| < |s_n(x)| + |s(x)| < 3\epsilon < \eta$. It thus appears that $\{\sigma_n(x)\}$ converges uniformly to $\sigma(x)$ in the set E, since, for the arbitrarily chosen number η, $|\sigma(x) - \sigma_n(x)| < \eta$, for $n \geqq m$, and for all points x, in E.

The converse of this theorem does not hold good. If it be assumed that $\{\sigma_n(x)\}$ converges uniformly to $\sigma(x)$, although it can be inferred (see I, § 219) that $\{s_n(x)\}$ converges or diverges, at every point x, to $s(x)$, the approach of the sequence to the limiting function is not necessarily uniform.

Uniform approach of the sequence $\{s_n(x)\}$ to $s(x)$ has been defined* otherwise by Hahn, as subsisting whenever $\{\sigma_n(x)\}$ converges uniformly to $\sigma(x)$. There is however a certain arbitrariness in this definition, as it depends upon the employment of a special transformation.

POINTS OF UNIFORM AND OF NON-UNIFORM CONVERGENCE.

70. Let the sequence $\{s_n(x)\}$ converge at each point of a domain E, of one or more dimensions, to the value of $s(x)$. Let $n(\epsilon, x)$ denote the least value which n can have, for a particular point x, such that

$$|s_n(x) - s(x)|,\ |s_{n+1}(x) - s(x)|,\ |s_{n+2}(x) - s(x)|,\ \ldots$$

are all $< \epsilon$; thus $n(\epsilon, x)$ has a definite value for each value of ϵ, and for each point x, of E. For a fixed value of ϵ that is sufficiently small, it may happen that $n(\epsilon, x)$ has no upper boundary in E; this will be the case when the convergence of the sequence is non-uniform in E.

In accordance with the theorem of I, § 213, there must be at least one point ξ, of E, in case E be closed, such that, in an arbitrarily small neighbourhood of ξ, $n(\epsilon, x)$ has no upper boundary. There may be a finite, or an infinite, set of such points ξ; and in an arbitrary neighbourhood of any point of this set, $n(\epsilon, x)$ has no upper boundary, and thus has values

* *Theorie der reellen Funktionen*, vol. I, p. 247.

greater than an arbitrarily chosen positive number A. Nevertheless $n(\epsilon, x)$ has a finite value at each point of E, for otherwise the sequence would not converge at the point.

A point for which, for each value of ϵ (>0), *there is some neighbourhood, dependent in general on* ϵ, *in which* $n(\epsilon, x)$ *has a finite upper boundary, is said to be a point of uniform convergence of the sequence.*

A point, in the neighbourhood of which $n(\epsilon, x)$ *has no finite upper boundary, provided* ϵ *be fixed sufficiently small, is said to be a point of non-uniform convergence of the sequence.*

If the domain E be not closed, the point ξ in the neighbourhood of which $n(\epsilon, x)$ has infinity for its upper boundary, need not belong to E, although it must then be a limiting point of E, and would thus belong to the closed set $E_0 \equiv M(E, E')$, obtained by adjoining to E those of its limiting points which do not belong to the set. Thus we should have to consider points of non-uniform convergence which belong to E_0 but not to E. Although the most important case is that in which the domain E, for which the functions of the sequence $\{s_n(x)\}$ are defined, is closed, being a closed linear interval, or a closed continuous domain of any number of dimensions, we shall, for generality, consider the case of any domain E which is not necessarily closed.

The above definitions are equivalent to the following:

If, for a point ξ, *of* E, *or of* E', *there exists, for each positive value of* ϵ, *a neighbourhood* $(\xi - d_\epsilon, \xi + d_\epsilon)$ *(linear or p-dimensional) such that, for* $n \geqq n_\epsilon$, *a number dependent on* ϵ, $|s(x) - s_n(x)| < \epsilon$, *for all points* x *in that neighbourhood, the point* ξ *is said to be a point of uniform convergence of the sequence* $\{s_n(x)\}$.

If, for a sufficiently small value of ϵ, *no such neighbourhood exists,* ξ *is said to be a point of non-uniform convergence of the sequence* $\{s_n(x)\}$. *A point* ξ, *of* E', *which does not belong to* E, *may be a point of non-uniform convergence.*

The definitions may also be stated in the following form:

At a point ξ, *of* E, *or of* E', *the convergence is uniform or non-uniform according as* $|R_n(x)|$ *has, or has not, the unique double limit zero, as* $x \sim \xi$, $n \sim \infty$.

71. It is convenient in this definition to take a neighbourhood $(\xi - d_\epsilon, \xi + d_\epsilon)$, of which ξ is the middle point, in the case of functions of a single variable; and it is convenient to take a square or cubic neighbourhood in the case of functions of two, or of three, variables. In general a neighbourhood

$$(\xi^{(1)} - d_\epsilon, \xi^{(2)} - d_\epsilon, \dots \xi^{(p)} - d_\epsilon;\ \xi^{(1)} + d_\epsilon, \xi^{(2)} + d_\epsilon, \dots \xi^{(p)} + d_\epsilon),$$

represented also by $(\xi - d_\epsilon, \xi + d_\epsilon)$, can be taken in the case in which ξ is a point $(\xi^{(1)}, \xi^{(2)}, \dots \xi^{(p)})$ of a p-dimensional set. If $|s(x) - s_n(\xi)| < \epsilon$, for $n \geqq n_\epsilon$, in a neighbourhood $(\xi - d_\epsilon, \xi + d_\epsilon)$, there will be an infinite set

of values of d_ϵ for which the condition is satisfied. This set will have a maximum value $\bar{d}_\epsilon$, such that, at any point *within* the neighbourhood $(\xi - \bar{d}_\epsilon, \xi + \bar{d}_\epsilon)$, the condition $| s_n(x) - s(x) | < \epsilon$ is satisfied for $n \geqq n_\epsilon$. If we give to d_ϵ the value $\frac{1}{2}\bar{d}_\epsilon$, for example, we have a rule for determining a neighbourhood, definite for each point ξ, in which (including its boundary) the condition is satisfied.

At a point ξ, of uniform convergence, the number $\bar{d}_\epsilon$ will in general depend upon the value of ϵ; and if $\epsilon \sim 0$, and n_ϵ increases indefinitely, the numbers $\bar{d}_\epsilon$ will converge to a number which is either positive (say $= d'$) or is zero. In the former case there exists a neighbourhood $(\xi - d'', \xi + d'')$, where $d'' < d'$, in which the convergence is uniform; points of convergence for which this is the case were considered by Weierstrass*, and spoken of as points in the neighbourhood of which the sequence converges uniformly. He proved that, for a closed domain E, if every point has this property, the sequence converges uniformly in E. In case $\bar{d}_\epsilon$ converges to zero with ϵ, the point ξ, of uniform convergence, has no neighbourhood in which the sequence converges uniformly; such a point has been termed by Pringsheim† a *singular point of uniform convergence.* Such a point is in general a limiting point of a set of points of non-uniform convergence. When the functions $s_n(x)$ are discontinuous, a point of uniform convergence may even be an isolated point of the set of all points of uniform convergence‡ (see § 95). The definition of a point of uniform convergence was given explicitly by W. H. Young§, and later by Van Vleck||. It was given implicitly by other writers, for example, in the first edition of this work.

In the case in which the domain E is linear, a distinction may be made between uniform continuity, at a point ξ, on the right and on the left. If the condition $| s(x) - s_n(x) | < \epsilon$, for $n \geqq n_\epsilon$, is satisfied for all points x in an interval $(\xi, \xi + d_\epsilon)$, and for all values of ϵ, the point ξ is one of uniform continuity on the right. By employing intervals $(\xi - d_\epsilon, \xi)$, uniform continuity on the left is defined. A point is of uniform convergence if it is uniformly convergent both on the right and on the left. A similar distinction might be made when the domain has two or more dimensions.

72. It has been shewn that if, for some sufficiently small value of ϵ, $n(\epsilon, x)$ has no upper boundary in a closed set E, there must be at least one point of non-uniform convergence, which belongs to E. It follows

* See *Werke*, vol. II, p. 202; also Du Bois-Reymond, *Crelle's Journal*, vol. C (1887), p. 335.

† *Münch. Sitzungsber.* for 1919, p. 419, where some remarks of a historical kind will be found.

‡ See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. I (1903), p. 90, but it is in agreement with Du Bois-Reymond's definition of "stetige Convergenz" at a point.

§ *Proc. Lond. Math. Soc.* (2), vol. I (1903), p. 89; see also (2) vol. VI (1908), p. 36.

|| *Trans. Amer. Math. Soc.* vol. VIII (1907), p. 204 footnote.

that, if every point of the closed domain E is a point of uniform convergence of the convergent sequence $\{s_n(x)\}$, the sequence is uniformly convergent in E. This does not hold good if E is not closed.

If a point ξ, of E, or of E', be such that a sequence $n_1, n_2, n_3, \ldots$ of increasing integers exists such that ξ is a point of uniform convergence of the sequence $s_{n_1}(x), s_{n_2}(x), s_{n_3}(x), \ldots$, without necessarily being a point of uniform convergence of the convergent sequence $\{s_n(x)\}$, the point ξ is said to be *a point of simply uniform convergence* of the sequence $\{s_n(x)\}$.

It is clear that, if the sequence $\{s_n(x)\}$ is simply-uniformly convergent in a closed domain E, every point of E is one of simply uniform convergence of the sequence. For we have only to apply the fact that, if $\{s_{n_p}(x)\}$ converges uniformly in E, every point of E is a point of uniform convergence of that sequence.

The converse theorem that, if every point of E is a point of simply uniform convergence, then the sequence converges simply uniformly in E, does not hold. For the datum only ensures the existence of an integer sequence $n_1, n_2, n_3, \ldots$, as in the definition, for each point of E, but there may exist no one such sequence which applies to all the points of E.

73. If $\{s_n(\xi)\}$ is divergent, say to $+\infty$, and the condition is satisfied that, for each positive number A, a neighbourhood of the point ξ, dependent in general on A, exists, such that at every point x in that neighbourhood, $s_n(x) > A$, for $n \geqq n_A$, the point ξ is said to be a point of *uniform divergence* of the sequence $\{s_n(x)\}$. It is seen at once that, when the transformation $\sigma_n(x) = \dfrac{s_n(x)}{1 + |s_n(x)|}$ is applied, a point of uniform divergence of $\{s_n(x)\}$ is a point of uniform convergence of $\{\sigma_n(x)\}$.

It now follows that, if every point of the closed set E is either a point of uniform convergence, or a point of uniform divergence, of $\{s_n(x)\}$, the sequence $\{\sigma_n(x)\}$ converges uniformly in E.

74. The definition of uniform convergence of a sequence $\{s_n(x)\}$ at a point ξ may be stated in the following form, in which the convergence of the sequence is not presupposed:

If the functions of a sequence $\{s_n(x)\}$ are defined in a domain E, the sequence is said to be uniformly convergent at a point ξ, of E, or of E', if, corresponding to each arbitrarily assigned positive number ϵ, a neighbourhood $(\xi - d_\epsilon, \xi + d_\epsilon)$ can be so determined that, for every point x, of E, in that neighbourhood, the condition $|s_n(x) - s_{n'}(x)| < \epsilon$, for $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, is satisfied; where n_ϵ is some integer dependent on ϵ.

That the definition*, in this form, implies the convergence of the sequence at the point ξ, in case ξ belongs to E, is seen by taking $x = \xi$ in the condition that is satisfied. It is, however, not necessarily the case

* This definition is given in Hahn's *Theorie der reellen Funktionen*, vol. I, p. 247.

that, when the condition is satisfied, the sequence should converge at any point in a neighbourhood of ξ, except at the point ξ itself. This definition is accordingly applicable to any sequence $\{s_n(x)\}$ not assumed to be convergent in E. It is thus more general than in the form, given in § 70, that the double limit of $|R_n(x)|$ at the point (ξ, ∞) should exist and have the value zero. For $R_n(x)$ need not exist except at ξ, in case ξ belongs to E.

This definition may be expressed in the following equivalent form:

If the functions of a sequence $\{s_n(x)\}$ are defined in the domain E, the sequence is said to be uniformly convergent at a point ξ, of E, or of E', if, corresponding to each arbitrarily assigned positive number ϵ, a neighbourhood Δ, of ξ, can be so determined that, for every point x, of E, in Δ, the conditions $|s_n(x) - \bar{s}(x)| < \epsilon$, $|s_n(x) - \underline{s}(x)| < \epsilon$ are satisfied, for $n > n_\epsilon$, where $\bar{s}(x)$, $\underline{s}(x)$ are the upper and lower limits of $s_n(x)$, as $n \sim \infty$.

To prove that this form follows from the first, choose Δ so that, in Δ,

$$|s_n(x) - s_{n'}(x)| < \tfrac{1}{2}\epsilon,$$

for $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$. By giving to n' the values in a properly chosen sequence, $s_{n'}(x)$ converges to $\bar{s}(x)$, and by a different sequence it converges to $\underline{s}(x)$; hence $|s_n(x) - \bar{s}(x)| < \epsilon$, $|s_n(x) - \underline{s}(x)| < \epsilon$. To prove that the first form of the definition follows from the second, choose Δ so that

$$|s_n(x) - \bar{s}(x)| < \tfrac{1}{2}\epsilon, \quad |s_n(x) - \underline{s}(x)| < \tfrac{1}{2}\epsilon,$$

for $n \geqq n_\epsilon$. It now follows that, if $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, $|s_n(x) - s_{n'}(x)| < \epsilon$.

75. A more stringent condition than the one contained in the above definition would be obtained by assuming that $|s_n(x) - s_{n'}(x')| < \epsilon$, for $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, and for every pair of points in the neighbourhood $(\xi - d_\epsilon, \xi + d_\epsilon)$ of the point ξ. When this condition is satisfied the sequence is said to be *continuously convergent* at the point ξ. This condition may be stated in the form that $s_n(x)$ should be continuous with respect to (x, n) at the point (ξ, ∞), so that $s_n(x)$ has a unique double limit, as $x \sim \xi$, $n \sim \infty$.

It is clear that, if the sequence is continuously convergent at the point ξ, it is also uniformly convergent at that point, but the converse does not in general hold good.

Consider, for example, the case of a discontinuous function $s_n(x)$ defined for a finite linear interval containing the point $x = 1$, by $s_n(1) = 1$, $s_n(x) = \dfrac{x}{n}$, for $x \neq 1$. We have then $s(1) = 1$, $s(x) = 0$, for $x \neq 1$. The condition of uniform convergence, that, in a sufficiently small neighbourhood of the point 1, $|s_{n'}(x) - s_n(x)| < \epsilon$, for $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, is satisfied, but the condition $|s_{n'}(x') - s_n(x)| < \epsilon$, is not satisfied, as is seen by taking $x = 1$. Thus the sequence converges uniformly, but not continuously, at the point 1. In fact the double limit of $R_n(x)$, at $(1, \infty)$ is zero, but that of

$s_n(x)$ is not $s(1)$. It can, however, be shewn that, if an infinite number of the functions $s_n(x)$ are continuous at the point ξ, and the convergence at that point is uniform, it is then also continuous. From the condition of uniform convergence it is seen that $s(\xi)$ exists and has a finite value.

A neighbourhood Δ', of ξ, and an integer n_ϵ, can be so chosen that both the inequalities $|s_n(x) - s_{n'}(x)| < \epsilon$, $|s_n(\xi) - s(\xi)| < \epsilon$, hold for $n \geqq n_\epsilon$, $n' \geqq n_\epsilon$, provided x is in Δ'. Let $n\ (\geqq n_\epsilon)$ have a fixed value such that $s_n(x)$ is continuous at ξ, then a neighbourhood Δ'', of ξ, contained in Δ', can be so chosen that, if x is in Δ'', for the fixed value of n, we have $|s_n(x) - s_n(\xi)| < \epsilon$. From the three inequalities it is seen that, in Δ'', $|s_{n'}(x) - s(\xi)| < 3\epsilon$, for $n' \geqq n_\epsilon$; and therefore $|s_{n'}(x) - s_{n''}(x')| < 6\epsilon$, for every pair of points x, x', belonging to E, and in Δ'', and for all values of n', n'' that are $\geqq n_\epsilon$. Since ϵ is arbitrary it follows that the convergence of the sequence is continuous.

If the sequence $\{s_n(x)\}$ is continuously convergent at the point ξ, the functions $\bar{s}(x)$, $\underline{s}(x)$ are both continuous at the point ξ, where they both have the value $s(\xi)$, in case ξ is a point of E.

Since, in a certain neighbourhood Δ, of ξ, $|s_n(\xi) - s_n(x)| < \epsilon$, for $n \geqq n_\epsilon$, by giving to n a sequence of values such that $s_n(x)$ converges to $\bar{s}(x)$, as n has the values in the sequence, we have $|s(\xi) - \bar{s}(x)| \leqq \epsilon$; thus $\bar{s}(x)$ is continuous at ξ; and in a similar manner it is seen that $\underline{s}(x)$ is continuous at ξ; in fact $\bar{s}(x) - \underline{s}(x)$ converges to zero, as $x \sim \xi$.

76. If the terms of a convergent series $\Sigma u_n(x)$ are all continuous at a point ξ, and consequently all the terms of the sequence $\{s_n(x)\}$ are continuous at that point, and if $s(x)$ be discontinuous at ξ, that point is one of non-uniform convergence of the series (see § 86), and may be said to be a *visible** point of non-uniform convergence. But if $s(x)$ is continuous at ξ, that point may still be a point of non-uniform convergence, and may be said to be an *invisible* point of non-uniform convergence. At every invisible point ξ, of non-uniform convergence, $R_n(x)$ is, for each value of n, continuous with respect to x, but $R_n(x)$, considered as a function of x and n, is discontinuous at (ξ, ∞) with respect to (x, n).

When some, or all of the functions $u_n(x)$ are discontinuous, there are still two classes of points of non-uniform convergence, the visible ones, at which one or more of the functions $s(x)$, $u_1(x)$, $u_2(x)$, ... are discontinuous, and invisible ones at which they are all continuous. That the discontinuity of a single function $u_r(x)$, at ξ, will entail the existence of a point of non-uniform convergence at ξ, if all the other functions are continuous, is seen from the consideration that $-u_r(x)$ is the sum-function of the series $u_1(x) + \ldots + u_{r-1}(x) - s(x) + u_{r+1}(x) + \ldots$.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. I, p. 93.

TESTS OF UNIFORM CONVERGENCE.

77. The following test, known as Weierstrass' test, is frequently sufficient to establish the fact that a series is uniformly convergent in a given domain of the variable. The domain may be of any number of dimensions.

If $\Sigma u_n(x)$ *denote a series of functions defined in a given domain of* x, *and if* $\bar{u}_n$ *denote the upper boundary of* $|u_n(x)|$ *in the domain, then if the series* $\bar{u}_1 + \bar{u}_2 + \ldots + \bar{u}_n + \ldots$ *is convergent, the series* $\Sigma u_n(x)$ *is uniformly convergent in the domain, and is absolutely convergent for each point* x. *Moreover* $\Sigma |u_n(x)|$ *is uniformly convergent.*

We have

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+m}(x)| \leq \bar{u}_{n+1} + \bar{u}_{n+2} + \ldots + \bar{u}_{n+m}$$

for all values of x in the given domain. From the condition of convergence of $\Sigma\bar{u}_n$, it follows that, if ϵ be an arbitrarily prescribed positive number, n may be so chosen that the sum $\bar{u}_{n+1} + \bar{u}_{n+2} + \ldots + \bar{u}_{n+m}$ is $< \epsilon$, for all values 1, 2, 3, ..., of m. Thus, with this value of n, $|R_{n,m}(x)| < \epsilon$, for all values of x in the given domain, and for $m = 1, 2, 3, \ldots$. Therefore, in accordance with the definition of § 66, the series $\Sigma u_n(x)$ is uniformly convergent in the given domain.

Since

$$|u_{n+1}(x)| + |u_{n+2}(x)| + \ldots + |u_{n+m}(x)| \leq \bar{u}_{n+1} + \bar{u}_{n+2} + \ldots + \bar{u}_{n+m}$$

the uniform convergence of the series $\Sigma |u_n(x)|$ can be established in the same manner.

78. *If all the terms of the series* $\Sigma u_n(x)$ *are* ≥ 0, *for all values of* x *in a given domain, of one or more dimensions, and if the series converge uniformly in that domain, then any series of type* ω, *obtained by rearranging the order of the terms, is also uniformly convergent in the domain of* x.

That the new series, obtained by the rearrangement of the order of the terms, is convergent at each point of the given domain, and has the same sum as the original series, has been proved in § 8. Considering the first n terms of the given series, and the remainder $R_n(x)$, there exists, corresponding to n, an integer n' such that the first n terms of the given series all occur amongst the first n' terms of the new series. If $R'_{n'}(x)$ denote the remainder after n' terms, of the new series, we have

$$0 \leq R'_{n'}(x) \leq R_n(x) < \epsilon,$$

provided n is so chosen that $R_n(x) < \epsilon$, for all values of x in the given domain. It is clear that, if $R'_{n'}(x) < \epsilon$, then also $R_{n'+m}(x) < \epsilon$, for $m = 1$, 2, 3, ..., since the remainders clearly cannot increase as the index increases. It follows that the new series converges uniformly.

If the series $|u_1(x)| + |u_2(x)| + \ldots + |u_n(x)| + \ldots$ *converges uniformly in a given domain of* x, *then the series* $u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$ *also converges uniformly in the same domain. Moreover any other series of type* ω, *obtained by rearranging the order of terms of the latter series, is uniformly convergent.*

In accordance with the theorem of § 25, the second series converges at each point of the domain of x. If n be so chosen that the remainder, after n terms, of the first series is $< \epsilon$, the absolute value of the remainder, after n terms, of the second series is also less than ϵ. Therefore the second series is uniformly convergent. Since, from the last theorem, a rearrangement of the order of the terms of the first series does not affect its uniform convergence, it follows that a corresponding rearrangement of the terms of the second series does not affect its uniform convergence.

The converse of this theorem has been established by Birkhoff*, and may be stated as follows:

If the series $\Sigma u_n(x)$ *be uniformly convergent in the domain of* x, *and if all the series obtained by systematic rearrangement of the order of the terms be also uniformly convergent, then the series* $\Sigma\,|u_n(x)|$ *is uniformly convergent in the domain.*

79. It may be shewn that: *If* $\Sigma u_n(x)$ *is uniformly convergent in a given domain, the terms of the series may be so bracketed that the resulting series is absolutely convergent for all values of* x *in the domain.*

It is easily seen that a sequence $n_1, n_2, \ldots n_r, \ldots$ of increasing integers may be so determined that

$$|s_{n_r}(x) - s_{n_{r-1}}(x)| < \frac{1}{2^{r-1}},$$

for $r = 1, 2, 3, \ldots$, and for all the values of x. It then follows that the series

$$s_{n_1}(x) + \{s_{n_2}(x) - s_{n_1}(x)\} + \ldots + \{s_{n_r}(x) - s_{n_{r-1}}(x)\} + \ldots$$

is absolutely convergent for all the values of x.

The following theorem is sometimes useful:

If the terms of the series $u_1(x) + u_2(x) + \ldots$ *be continuous in a perfect domain of* x, *either linear, or in any number of dimensions, and if the terms are all* $\geqq 0$, *for all the values of* x, *then if the series converge throughout the perfect domain to a continuous sum-function* $s(x)$, *the series converges uniformly in the domain.*

Let x_1 be any point of the domain, then

$$s(x) - s(x_1) = \{s_n(x) - s_n(x_1)\} + \{R_n(x) - R_n(x_1)\}.$$

The point x_1 being fixed, corresponding to an arbitrarily assigned positive number ϵ, n can be so determined that $R_n(x_1) < \frac{1}{3}\epsilon$. This value of n being

* See *Annals of Math.* (2), vol. VI (1905), p. 90.

fixed, a neighbourhood $(x_1 - \delta, x_1 + \delta)$, of x_1 can be so determined that, if x be in this neighbourhood, both $| s(x) - s(x_1) |$ and $| s_n(x) - s_n(x_1) |$ are $< \frac{1}{3}\epsilon$; this follows from the continuity of $s(x)$ and $s_n(x)$ at x_1. We now see that, if x is in this neighbourhood, the condition $R_n(x) < \epsilon$ is satisfied; and since the terms of the series are never negative, it follows that $R_{n+m}(x) < \epsilon$, for $m = 1, 2, 3, \ldots$. It has thus been shewn that x_1 is a point of uniform continuity of the series; and since x_1 may be any point whatever of the given perfect domain, the convergence of the series is uniform in (a, b).

It is clear that, if the sequence $\{s_n(x)\}$ of partial sums is monotone non-diminishing, $s(x)$ is the sum-function of a series of which all the terms are $\geqq 0$. Thus the theorem may be stated as follows:

A sequence of functions $\{s_n(x)\}$ which are all continuous in a perfect domain, and which sequence is monotone non-diminishing (or non-increasing), and converges to a continuous function $s(x)$, converges uniformly in the perfect domain to $s(x)$.

80. The following theorem* provides a test of uniform convergence which can be frequently employed:

Let $u_1(x), u_2(x), \ldots u_n(x), \ldots$ be defined in a given perfect domain, of one or more dimensions, and $u_n(x) \geqq 0$, for all values of n and x, and further $u_n(x) \geqq u_{n+1}(x)$, for all values of n and x. Also let it be assumed that $u_1(x)$, and consequently $u_n(x)$, is bounded in the given domain. Then, if $\sum\limits_{n=1} a_n$ be any convergent numerical series, the series $\sum\limits_{n=1} a_n u_n(x)$ is uniformly convergent in the given domain.

Moreover, if $\sum\limits_{n=1} a_n$ do not converge, but oscillate finitely, then, provided the additional condition is satisfied that the functions $u_n(x)$ are all continuous, and that $\lim\limits_{n \sim \infty} u_n(x) = 0$, for each value of x in the domain, the series $\Sigma a_n u_n(x)$ is uniformly convergent in the perfect domain, and its sum is consequently continuous (see § 86).

In the usual statement of the second part of the theorem it is unnecessarily presupposed that $u_n(x)$ converges uniformly to zero in the domain. It will be seen that this uniform convergence is a consequence of the conditions stated.

When the domain of x contains one point only, the theorem reduces to a theorem given in § 24, of which one part is due† to Abel and the other to Diriçhlet.

* See Hardy, *Proc. Lond. Math. Soc.* (2), vol. IV (1907), pp. 250, 251.

† See Whittaker and Watson's *Course of Modern Analysis*, 3rd ed. (1910), pp. 17, 50.

In case the series Σa_n is convergent, the partial remainder $R_{n,m}(x)$ of the series $\Sigma a_n u_n(x)$ being

$$(a_{n+1} + a_{n+2} + \dots + a_{n+m})\, u_{n+m+1}(x)$$
$$+ \sum_{r=1}^{r=m} (a_{n+1} + a_{n+2} + \dots + a_{n+r}) \{u_{n+r}(x) - u_{n+r+1}(x)\},$$

we see that, by choosing n so great that all the partial remainders of the series Σa_n, after the nth term, are numerically less than the arbitrarily prescribed positive number ϵ, the condition $|R_{n,m}(x)| < \epsilon u_{n+1}(x)$ is satisfied. Therefore, since all the differences $u_{n+r}(x) - u_{n+r+1}(x)$ are $\geqq 0$, for every value of x in the domain, we have

$$|R_{n,m}(x)| < \epsilon U,$$

where U is the upper boundary of $u_1(x)$ in the domain. Since ϵU is arbitrarily small, the condition of uniform convergence of $\Sigma a_n u_n(x)$ is satisfied.

When Σa_n oscillates between finite limits, K can be so determined that $|a_{n+1} + a_{n+2} + \dots + a_{n+r}| < K$, for all values of n and r. Then we have

$$|R_{n,m}(x)| < K\, |u_{n+1}(x)|.$$

Since the sequence $\{u_n(x)\}$ is monotone, non-increasing, and converges to the continuous limit zero, in the perfect domain of x, and $u_n(x)$ is continuous, it follows from the third theorem of § 79 that the convergence of the sequence to zero is uniform; and thus, if n be sufficiently large,

$$|u_{n+1}(x)| < \epsilon/K,$$

for all the values of x; and therefore $|R_{n,m}(x)| < \epsilon$. The uniform convergence of the series has thus been established.

The first part of the theorem can be extended to the case in which $a_1, a_2, \dots a_n, \dots$ are functions of x, provided Σa_n converges uniformly in the given domain. The second part can also be extended to the case in which $a_1, a_2, \dots a_n, \dots$ are functions of x, provided the partial sum s_n is numerically less than some fixed number K, for all values of n and x.

EXAMPLES.

(1) Let* $u_n(x) = x^n(1-x)$, $0 \leq x \leq 1$. In this case $s(x) = x$, for $0 \leq x < 1$; but $s(x) = 0$, for $x = 1$; and the series converges non-uniformly in the neighbourhood of the point $x = 1$.

(2) Let* $u_n(x) = x^n(1 - x^n)$. If $|x| < 1$, we find $s(x) = \dfrac{x}{1-x^2}$; also $s(1) = 0$; whereas $\lim\limits_{x=1} s(x) = \infty$. The series converges non-uniformly in the neighbourhood of the point 1, and its sum-function has an infinite discontinuity at that point.

(3) Let $u_n(x) = -2(n-1)^2 xe^{-(n-1)^2x^2} + 2n^2xe^{-n^2x^2}$. Here $s(x) = 0$, for every value of x; $R_n(x) = -2n^2xe^{-n^2x^2}$; and at $x = \dfrac{1}{n}$, $R_n\left(\dfrac{1}{n}\right) = -\dfrac{2n}{e}$. The series converges non-uniformly

* Arzelà, *Memorie di Bologna* (5), vol. VIII, p. 139.

in every neighbourhood of $x = 0$, since arbitrarily large values of $|R_n(x)|$ exist in such neighbourhood; but the sum-function is continuous at $x = 0$.

(4) Let* $$s_n(x) = \phi_n(x) + \frac{1}{2!}\phi_n(2!\,x) + \dots + \frac{1}{k!}\phi_n(k!\,x) + \dots$$

where $\phi_n(x) = \sqrt{2e}\,.\,n\sin^2 \pi x\,.\,e^{-n^2\sin^4 \pi x}$. The series which defines $s_n(x)$ converges uniformly, since $|\phi_n(k!\,x)| \leqq 1$; and thus $s_n(x)$ is a continuous function of x. The sum-function $s(x)$ is also a continuous function of x; but the convergence of the functions $s_n(x)$ to $s(x)$ is non-uniform in every sub-interval of the interval $(0, 1)$.

(5) Let† $u_{2n-1}(x) = x^{n+1}$, $u_{2n}(x) = -x^{n+1}\left\{1 - \frac{1}{(n+1)!}\right\}$, where $0 \leqq x < 1$, and

$$u_n(1) = \frac{1}{(n+1)!}.$$

The series $\Sigma u(x)$ is simply-uniformly convergent in $(0, 1)$, but it is not uniformly convergent.

(6) Consider‡ the series

$$\frac{1 + 5x}{2(1 + x)} + \dots + \frac{x(x+2)n^2 + x(4 - x)n + 1 - x}{n(n+1)\{(n-1)x + 1\}(nx + 1)} + \dots.$$

Here $u_n(x) = \left[\frac{1}{n} + \frac{2}{(n-1)x + 1}\right] - \left[\frac{1}{n+1} + \frac{2}{nx + 1}\right]$; thus $s(x) = 3$, unless $x = 0$, when $s(0) = 1$; and the sum-function is therefore discontinuous at the point 0.

Since $R_n(x) = \frac{1}{n+1} + \frac{2}{nx + 1}$, we find on equating this to ϵ, and solving for n,

$$n = \{x + 2 - \epsilon(x + 1) + \sqrt{[\{x + 2 - \epsilon(x+1)\}^2 + 4\epsilon x(3 - x)]}\}/2\epsilon x;$$

thus, for a fixed ϵ, the value of n increases indefinitely as x approaches the value 0.

(7) The series§

$$x^2 - x^2 + \frac{x^2}{1 + x^2} - \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} - \frac{x^2}{(1 + x^2)^2} + \frac{x^2}{(1 + x^2)^3} - \dots$$

is uniformly and absolutely convergent in any interval $(-A, B)$. For $s_{2n}(x) = 0$, $s_{2n+1}(x) = \frac{x^2}{(1 + x^2)^n}$, and hence $s_{2n+1}(x) < \frac{1}{n}$; therefore the series converges uniformly to the sum zero. The series

$$x^2 - x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} - \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^3} + \frac{x^2}{(1 + x^2)^4} - \frac{x^2}{(1 + x^2)^2} + \dots,$$

obtained by rearranging the terms of the given series, is however non-uniformly convergent in $(-A, B)$, the point 0 being a point of non-uniform convergence. For

$$s_{3n-1}(x) = \frac{(1 + x^2)^{n-1} - 1}{(1 + x^2)^{2n-2}};$$

and for $$x = \pm(2^{\frac{1}{n-1}} - 1)^{\frac{1}{2}},\quad s_{3n-1}(x) = \tfrac{1}{4}.$$

The given series does not satisfy the condition stated in the second theorem of § 78,

* Osgood, *Bulletin of the American Math. Soc.* (2), vol. III (1896), p. 70.

† Volterra, *Gior. di Mat.* vol. XIX (1881), p. 79.

‡ Stokes, *Math. and Phys. Papers*, vol. I, p. 280.

§ Bôcher, *Annals of Math.* (2), vol. IV (1904), p. 159.

that the series whose terms are the absolute values of those of the given series should uniformly convergent. For the series

$$x^2 + x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

has its sum discontinuous at the point $x = 0$, and therefore does not converge uniformly in an interval $(-A, B)$.

(8) Let* $u_{2n-1}(x) = \frac{x}{nx^2 + (1-nx)^2}$, $u_{2n}(x) = \frac{-x}{(n+1)x^2 + \{1-(n+1)x\}^2}$.

In this case, the series converges for all values of x, and

$$s(x) = \frac{x}{x^2 + (1-x)^2}; \qquad R_{2n-1}(x) = 0, \qquad R_{2n-2}(x) = u_{2n-1}(x).$$

In an interval (α, β), which contains the point $x = 0$, the series converges simply-uniformly, but it does not converge uniformly, since $R_{2n-2}\left(\frac{1}{n}\right) = 1$, however great n may be, and thus $R_n(x)$ has not the unique double limit zero, at $x = 0$, and that point is accordingly one of non-uniform convergence.

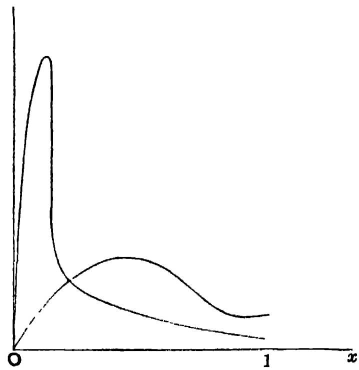


Fig. 1.

(9) Let† $s_n(x) = \frac{n^2 x}{1 + n^3 x^2}$, $s(x) = 0$, for $0 \le x \le 1$. This series converges non-uniformly in the neighbourhood of the point $x = 0$. The approximation curves $y = s_n(x)$ have peaks of height $\frac{1}{2}n^{\frac{1}{2}}$, which increase indefinitely in height as n is increased. At the same time, the point $\frac{1}{n^{\frac{3}{2}}}$, at which the ordinate is a maximum, continually approaches the point 0; and thus, in any neighbourhood of the point 0, x and n may be so chosen that $s_n(x)$ is arbitrarily great. At the point $x = 0$, we have $s_n(x) = 0$, for every n.

(10) Let $s_n(x) = \frac{nx}{1 + n^2 x^2}$, $s(x) = 0$, $0 \le x \le 1$. The curves $y = s_n(x)$ have peaks all of the same height $\frac{1}{2}$, at the points $x = \frac{1}{n}$. As in the last example the point $x = \frac{1}{n}$, below the peak, continually approaches the origin as n is increased. The convergence is non-uniform in the neighbourhood of $x = 0$.

* Tannery, *Théorie des fonctions*, p. 134.

† Osgood, *Amer. Journal of Math.* vol. XIX (1897), p. 156; also G. Cantor, *Math. Annalen*, vol. XVI (1880), p. 269.

Let $\phi_k(x) = \dfrac{n \sin^2 k\pi x}{1 + n^2 \sin^4 k\pi x}$; $s_n(x) = \phi_{1!}(x) + \dfrac{1}{2!}\phi_{2!}(x) + \dfrac{1}{3!}\phi_{3!}(x) + \ldots = \sum\limits_1^\infty \dfrac{1}{\iota!}\phi_{\iota!}(x)$.
The series which defines $s_n(x)$ converges uniformly, and thus $s_n(x)$ is a continuous function of x. In the neighbourhood of any rational point $x = p/q$, the curve $y = s_n(x)$ has peaks arising from the term $\dfrac{1}{k!}\phi_{k!}(x)$, where k is the smallest integer such that $k!$ is divisible by q. The series converges to the limit $s(x) = 0$, non-uniformly in any interval whatever (a, b), taken in the interval $(0, 1)$.

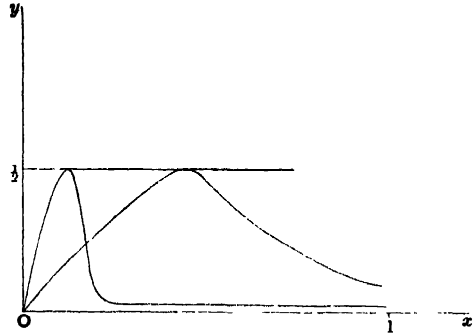


Fig. 2.

81. The following special theorem, which follows from the general theory given in § 85, can sometimes be usefully employed*:

If $\sum\limits_{n=1} u_n(m)$ *be a series which converges uniformly to* $f(m)$, *for all positive integral values of* m (*or for all continuously varying positive values of* m) *and if each limit* $\lim\limits_{m\sim\infty} u_n(m)$ *exists, then* $\lim\limits_{m\sim\infty} f(m)$ *exists, and the series* $\sum\limits_{n=1}\lim\limits_{m\sim\infty} u_n(m)$ *is convergent, and has* $\lim\limits_{m\sim\infty} f(m)$ *for its limiting sum.*

For, if ϵ be an arbitrarily chosen positive number, we have

$$\left| f(m) - \sum_{n=1}^{N} u_n(m) \right| < \epsilon,$$

provided the integer N is large enough. It follows that

$$\left| \overline{\lim_{m\sim\infty}} f(m) - \sum_{n=1}^{n=N} v_n \right| \leqq \epsilon, \quad \left| \underline{\lim_{m\sim\infty}} f(m) - \sum_{n=1}^{n=N} v_n \right| \leqq \epsilon,$$

where v_n denotes $\lim\limits_{m\sim\infty} u_n(m)$. It is then seen that

$$\left| \overline{\lim_{m\sim\infty}} f(m) - \underline{\lim_{m\sim\infty}} f(m) \right| \leqq 2\epsilon;$$

and since ϵ is arbitrary, $\overline{\lim\limits_{m\sim\infty}} f(m) = \underline{\lim\limits_{m\sim\infty}} f(m)$; or $\lim\limits_{m\sim\infty} f(m)$ exists as a definite number.

* See Osgood, *Lehrbuch der Funktionentheorie*, vol. I, p. 521.

Further, since $\left|\lim_{m\sim\infty} f(m) - \sum_{n=1}^{n=N} v_n\right| \leq \epsilon$, for all values of N greater than an integer N_ϵ, dependent on ϵ, it is seen that $\sum_{n=1} v_n$ is convergent, and has $\lim_{m\sim\infty} f(m)$ for its limiting sum.

In applying the theorem, in case m has positive integral values only*, it frequently happens that, for each value of m, the series $\sum_{n=1} u_n(m)$ has only a finite number of terms, that number being dependent on m, and increasing indefinitely as m does so. Such a finite series is, of course, a particular case of an infinite series, which arises when all the terms after a fixed one are zero.

EXAMPLES.

(1) If x be any fixed real number, and m a positive integer, we have

$$\left(1+\frac{x}{m}\right)^m = 1 + x + \frac{1\left(1-\frac{1}{m}\right)}{2!}x^2 + \ldots + \frac{1\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{r-1}{m}\right)}{r!}x^r + \ldots$$

the series having $m+1$ terms.

For a fixed value of r, the $(r+1)$th term of the series is numerically less than $\frac{1}{r!}|x|^r$, and this is the $(r+1)$th term of a convergent series. Thus the condition of uniform convergence in the above theorem is satisfied. The series formed by taking the limit, as $m \sim \infty$, of each term of the above series, is the convergent series

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots.$$

It follows that $\lim_{m\sim\infty}\left(1+\frac{x}{m}\right)^m$ exists, and is the sum of the convergent series

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots.$$

(2) It can be shewn by an elementary process that

$$\cos x = \cos^m\left(\frac{x}{m}\right) - \frac{\left(1-\frac{1}{m}\right)}{2!}x^2\left(\frac{\sin x/m}{x/m}\right)^2\cos^{m-2}\left(\frac{x}{m}\right) + \ldots$$
$$+ (-1)^r\frac{\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{2r-1}{m}\right)}{(2r)!}x^{2r}\left(\frac{\sin x/m}{x/m}\right)^{2r}\cos^{m-2r}\left(\frac{x}{m}\right) + \ldots,$$

where m is a positive integer, and the series stops after a finite number of terms, so that $2r-1 < m$. The general term of the series is numerically less than $\frac{x^r}{(2r)!}$, which, for each fixed value of m, is the general term of a convergent series of positive terms. Assuming the known theorem that $\lim_{m\sim\infty}\cos^m\left(\frac{x}{m}\right) = 1$, and the theorem that $\lim_{m\sim\infty}\frac{\sin x/m}{x/m} = 1$, we obtain at once, by applying the above theorem, the result that, for each value of x,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

converges to $\cos x$.

The series for $\sin x$ may be obtained in a similar manner.

* In this case the theorem is practically equivalent to a theorem given by Tannery, *Fonctions d'une variable*, 2nd ed., vol. I, p. 292.

THE CONTINUITY OF A SUM-FUNCTION AT A POINT.

82. Let the series $\sum_{n=1} u_n(x)$ converge in a domain E, of one or more dimensions, to the sum-function $s(x)$. Let ξ be a point of E at which all the functions $u_n(x)$ are continuous; a sufficient condition will be obtained that $s(x)$ may be continuous at the point ξ. We have, since

$$s(x) = s_n(x) + R_n(x),$$

at every point of E,

$$\begin{aligned} |s(x) - s(\xi)| &\leqq |s_n(x) - s_n(\xi)| + |R_n(x) - R_n(\xi)| \\ &\leqq |s_n(x) - s_n(\xi)| + |R_n(x)| + |R_n(\xi)|. \end{aligned}$$

Let it be assumed that an integer n exists such that a neighbourhood D_1, of ξ, can be so determined that, at every point of it that belongs to E, $|R_n(x)| < \frac{1}{3}\epsilon$. Since $s_n(x)$ is continuous at ξ, a neighbourhood D_2, of ξ, can be so determined that, in it, at every point that belongs to E,

$$|s_n(x) - s_n(\xi)| < \tfrac{1}{3}\epsilon.$$

A neighbourhood D, of ξ, can be determined, all the points of which belong both to D_1 and to D_2. It follows that, at every point x of E, that is in D, the condition $|s(x) - s(\xi)| < \epsilon$, is satisfied. If D can be determined, corresponding to any value of ϵ (> 0) whatever, $s(x)$ is continuous at ξ. The following theorem has thus been established:

If a series $\sum_{n=1} u_n(x)$ converge to a function $s(x)$ at the points of a domain E, of one or more dimensions, and the point ξ, of E, be a point of continuity of all the functions $u_n(x)$, it is a sufficient condition for the continuity of $s(x)$ at the point ξ, that, corresponding to any arbitrary η, an integer n should exist and also a neighbourhood of ξ, such that

$$|s_n(x) - s(x)| < \eta,$$

at every point x, of E, in that neighbourhood.

It will be observed that the neighbourhood depends upon both n and η.

It should be noticed that the condition in the theorem is satisfied when ξ is a point of uniform convergence of the series, and also when it is a point of simply uniform convergence; but either of these latter conditions is more stringent than that in the theorem.

When η is prescribed, the condition in the theorem asserts that it is sufficient for one value of n and a neighbourhood Δ, dependent on η, to exist, in which $|s_n(x) - s(x)| < \eta$. But when the point ξ is one of uniform convergence, a neighbourhood $\Delta(\eta)$, dependent on η, exists in which

$$|s_n(x) - s(x)| < \eta,$$

for every value of n greater than some integer n_η, dependent only on η.

When the point ξ is a point of simply uniform convergence, a neighbourhood $\Delta(\eta)$, dependent on η, exists in which $|s_n(x) - s(x)| < \eta$, for a divergent sequence $n_1, n_2, \ldots$ of values of n, only.

It thus appears that the following criterion is sufficient for the continuity of a sum-function at a point:

If a series $\sum\limits_{n=1} u_n(x)$ *converge to a function* $s(x)$ *at the points of a domain* E, *of one or more dimensions, and the point* ξ, *of* E, *be a point of continuity of all the functions* $u_n(x)$, *it is a sufficient condition for the continuity of* $s(x)$ *at the point* ξ, *that the point* ξ *be a point of simply uniform convergence of the series. A fortiori, it is sufficient that* ξ *be a point of uniform convergence of the series.*

83. In order to determine necessary conditions for the continuity of $s(x)$ at ξ, we see that, since $\{s_n(\xi)\}$ converges to $s(\xi)$, an integer N exists such that $|s(\xi) - s_n(\xi)| < \frac{1}{3}\epsilon$, for every value of n that is $> N$. Taking any one such value of n, a neighbourhood of ξ can be determined for which $|s_n(x) - s_n(\xi)| < \frac{1}{3}\epsilon$, where x is any point of E, in that neighbourhood. If $s(x)$ is continuous at ξ, a neighbourhood of ξ can be determined such that $|s(x) - s(\xi)| < \frac{1}{3}\epsilon$, for all points of E in that neighbourhood. It follows that a neighbourhood of ξ can be determined, in which both $|s(x) - s(\xi)|$ and $|s_n(x) - s_n(\xi)|$ are $< \frac{1}{3}\epsilon$, for all points of E in that neighbourhood. It now follows from the three inequalities that

$$|s(x) - s_n(x)| < \epsilon,$$

in that neighbourhood. Taking this result in conjunction with the first theorem of § 82, we have the following:

If a series $\Sigma u_n(x)$ *converges to* $s(x)$ *in a domain* E, *of one or more dimensions, the necessary and sufficient condition that* $s(x)$ *should be continuous at a point* ξ, *of* E, *at which the functions* $u_n(x)$ *are all continuous, is that, having assigned an arbitrarily chosen positive number* ϵ, *an integer* N_ϵ *should exist, such that, for each value of n that is* $> N_\epsilon$, *a neighbourhood* $(\xi - d_{n,\epsilon}, \xi + d_{n,\epsilon})$ *of* ξ *exists so that at every point of it that is in* E, *the condition*

$$|s(x) - s_n(x)| < \epsilon$$

is satisfied.

It will be observed that the neighbourhood of ξ depends not only upon the value of ϵ, but also upon that of n; it may accordingly be denoted by $\Delta(\xi, \epsilon, n)$, where $n > N_\epsilon$. This mode of convergence is accordingly less stringent than that in which the point ξ is a point of uniform convergence, and in which the neighbourhood depends only upon ϵ, provided n be $> N_\epsilon$, and may be denoted by $\Delta(\xi, \epsilon)$.

Another formulation of necessary and sufficient conditions for the continuity of $s(x)$ at ξ is the following:

It is necessary and sufficient for the continuity of $s(x)$ *at the point* ξ *that, if* N_1 *be an arbitrarily chosen integer, and* ϵ *an arbitrarily chosen positive*

number, an integer $n_1(\xi, \epsilon, N_1)\ (> N_1)$ *can be determined, and also a neighbourhood of* ξ, $\Delta(\xi, \epsilon, N_1)$, *such that*

$$|s(x) - s_{n_1}(x)| < \epsilon,$$

for all points of E in that neighbourhood.

Since the neighbourhood depends not only upon ϵ but also upon N_1, and may thus be denoted by $\Delta(\xi, \epsilon, N_1)$, this mode of convergence is less stringent than that in which the point ξ is a point of simply uniform convergence of the series, and in which accordingly the neighbourhood depends only on ϵ, and may thus be denoted by $\Delta(\xi, \epsilon)$, and can be taken to be the same for all values of n in some infinite sequence.

The sufficiency of the condition follows from the first theorem of § 82. Its necessity follows from the last theorem.

Conditions of continuity substantially identical with those formulated above were given* by Dini. It may be remarked that the term simply uniform convergence is, by some writers†, applied to the mode of convergence indicated in this theorem.

84. It is frequently convenient to transform the function $R_n(x)$, of (n, x), into the function $R(x, y)$, of (x, y), where $y = 1/n$. In $R(x, y)$, the field of y consists then of the set of reciprocals of the positive integers. At a point of convergence of the sequence or series to which $R(x, y)$ is related, we have $\lim_{y\sim 0} R(x, y) = 0$. For a prescribed ϵ, there is, for a point x, of convergence, a certain range of values of y, for all of which, without a gap, $|R(x, y)| < \epsilon$; and the upper boundary of these values of y may be denoted by $\phi_\epsilon(x)$: but there may be other greater values of y separated from $\phi_\epsilon(x)$ by values of y for which the condition is not satisfied, for which the condition $|R(x, y)| < \epsilon$, is also satisfied. At a point x_1 of non-uniform convergence of the series, the lower limit of $\phi_\epsilon(x)$, for the values of x in any neighbourhood of x_1, is zero, provided ϵ be chosen sufficiently small; whereas, for a point x_1 of uniform convergence, a neighbourhood of x_1, in general dependent on ϵ, can be found for which the lower limit of $\phi_\epsilon(x)$ is greater than zero.

The distinction between the three classes of points in the interval (a, b), viz. (1), those at which the series is uniformly convergent, (2), those at which the series is non-uniformly convergent, but at which the sum-function is continuous, and (3), those points at which the sum-function is discontinuous, may be illustrated by means of figures‡ which indicate the regions of (x, y) in the neighbourhood of $(x_1, 0)$, at which $|R(x, y)|$ is less than an arbitrarily chosen ϵ.

* See *Grundlagen*, pp. 143–146.

† See Hahn's *Theorie der reellen Funktionen*, pp. 282, 283.

‡ See Hobson, "On modes of convergence of an infinite series of functions of a real variable," *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 378.

Fig. 3 represents the neighbourhood of a point P at which the convergence of the series is uniform. The blackened lines represent those portions of the lines whose ordinates are $1/n$, $1/(n+1)$, $1/(n+2)$, ... at which $|R_n(x)|$, $|R_{n+1}(x)|$ ... are $\leqq \epsilon$. These portions consist of all those parts of the lines which are bounded by the curve $y = \phi_\epsilon(x)$, there being also possibly such pieces outside the curve. An area, for example semicircular, can be drawn, bounded by a portion of the x-axis containing P, and such that for every point within it $|R(x, y)| < \epsilon$; and that this should be possible for every value of ϵ is the condition that $R(x, y)$ be continuous at the point P with regard to the domain of (x, y).

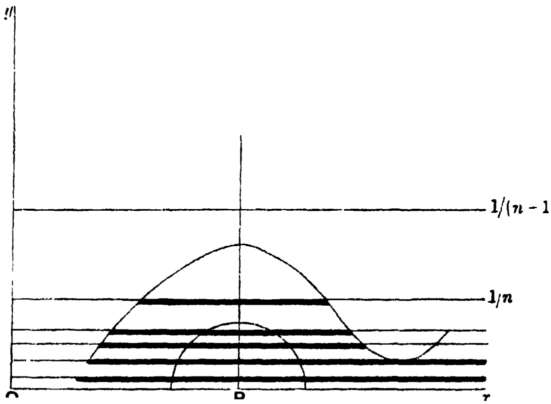


Fig. 3.

Fig. 4 represents the neighbourhood of a point P at which the function $s(x)$ is continuous, but at which the series is non-uniformly convergent. In this case the function $\phi_\epsilon(x)$ is for all values of ϵ, less than some number ϵ_0, discontinuous at P. The value of $\phi_\epsilon(x)$ at P is itself finite; but the functional limits $\phi_\epsilon(x_1 + 0)$, $\phi_\epsilon(x_1 - 0)$ at P are both zero. The breadth of

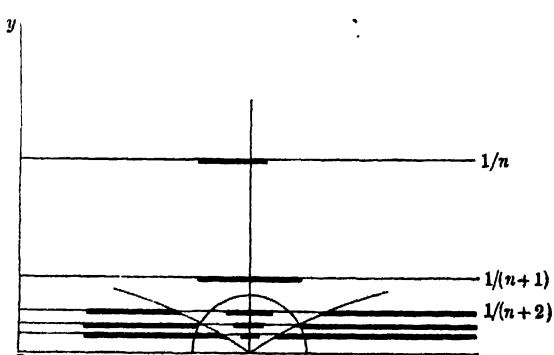


Fig. 4.

the blackened portions of the straight lines parallel to the x-axis, which represent the portions of those lines at which $|R_n(x)| \leqq \epsilon$, diminishes indefinitely as y approaches the value zero at P. In this case no semi-circle can be drawn with P as centre, for all internal points of which $|R(x, y)| < \epsilon$; and thus the point P is one of non-uniform continuity. In the figure, the convergence is non-uniform on both sides of P; it is clear however in what manner the figure must be modified for the case in which the convergence is non-uniform on one side only of P. In case the measure of non-uniform convergence (see § 90) be indefinitely great the figure will be essentially similar to the above figure, whatever value of ϵ be chosen; otherwise the figure applies to an ϵ which is less than the measure ϵ_0 of non-uniform convergence, viz. the saltus at P of $|R(x, y)|$ in the two-dimensional continuum.

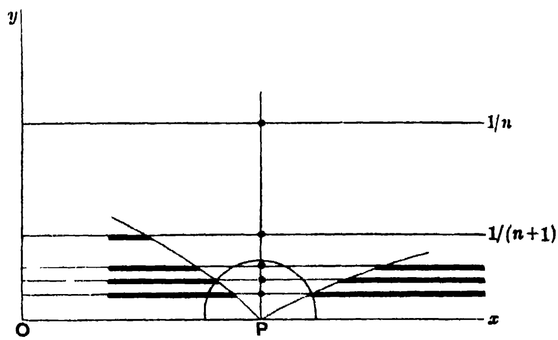


Fig. 5.

Fig. 5 represents the neighbourhood of a point P at which $s(x)$ is discontinuous, the value of ϵ being less than the measure of non-uniform convergence of the series at P (see § 90). In this case, as before, $\phi_\epsilon(x)$ is finite at P, and $\phi_\epsilon(x_1 + 0)$, $\phi_\epsilon(x - 0)$ are zero; but, on the parallels to Ox intersecting the ordinate at P, there are no intervals near P intersecting the ordinate, at which $|R(x, y)| < \epsilon$, but only points on the ordinate through P itself.

EXAMPLE.

As an example we may take the case in § 80, Ex. 10, $R_n(x) = \dfrac{nx}{1 + n^2x^2}$, and thus

$$R(x, y) = \frac{xy}{x^2 + y^2};$$

and we may suppose the domain of x to be the interval $(0, 1)$. In this case, the point $x = 0$ is a point of discontinuity of $R(x, y)$, and we find that if $\epsilon < \frac{1}{2}$, the condition

$$|R(x, y)| < \epsilon,$$

is satisfied for the space bounded by the x-axis, and by the straight line

$$y = \phi_\epsilon(x) = x\left[\frac{1}{2\epsilon} - \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right].$$

The same condition is also satisfied for the space between the y-axis and the straight line

$$y = x\left[\frac{1}{2\epsilon} + \left(\frac{1}{4\epsilon^2} - 1\right)^{\frac{1}{2}}\right],$$

and thus the point $x = 0$ is a point of continuity of the function $s(x)$, although the convergence is non-uniform at that point. If $\epsilon > \frac{1}{2}$, then $|R(x, y)| < \epsilon$, for the whole space between the axes; and thus the measure of non-uniform convergence at the point $x = 0$ is $\frac{1}{2}$, the upper double limit of $R(x, y)$ having the value $\frac{1}{2}$ (see § 90).

85. In case the domain E is linear, the two sides of a point ξ may be considered separately, the neighbourhoods of the point being taken on the two sides separately. It is sufficient to consider the case of a point ξ which is a limiting point of the domain on its right, and to assume that the functions $s_n(\xi)$ are continuous on the right. Further we may consider the functions $s_n(\xi + 0)$ instead of $s_n(\xi)$, it being assumed that the point ξ is excluded from the domain of which it is a limiting point on the right.

We thus obtain necessary and sufficient conditions that $s(\xi + 0)$ may exist and that the series $\Sigma u_n(\xi + 0)$ may converge to $s(\xi + 0)$. The following theorem contains these conditions:

A necessary and sufficient condition that the sum $s(x)$ of the convergent series $\Sigma u(x)$ may have a definite limit $s(\xi + 0)$ at the limiting point ξ of the linear domain of x to which the series $\Sigma u_n(\xi + 0)$ may converge, the terms of this series being assumed to have definite values, is that, corresponding to each arbitrarily chosen positive number ϵ, and to each integer n, which is greater than some fixed integer N_ϵ, dependent on ϵ, a positive number $\theta(\xi, \epsilon, n)$ can be determined, such that, for every value of x in the domain and in the interval $(\xi, \xi + \theta)$, the condition $|R_n(x)| < \epsilon$ is satisfied; the number θ being dependent in general upon n as well as ϵ.

In case θ is, for each value of ϵ, independent of n $(> N_\epsilon)$, the point ξ is a point of uniform convergence on the right, therefore *uniform convergence at ξ on the right is a sufficient condition that $s(\alpha + 0)$ may have a definite value and that the series $\Sigma u_n(\alpha + 0)$ may converge to $s(\alpha + 0)$.*

This theorem is a particular case of the first theorem of § 83, but it may also be obtained from that of I, § 305. For, let $n = 1/y$, then $s_n(x)$ becomes a function $s(x, y)$ of the two variables x and y, and the condition in the theorem is equivalent to the condition that the repeated limits

$$\lim_{x \sim \xi}\lim_{y \sim 0} s(x, y), \quad \lim_{y \sim 0}\lim_{x \sim \xi} s(x, y)$$

should both exist and have the same value.

By employing the last theorem of § 83, or that of I, § 306, we obtain the following theorem:

Necessary and sufficient conditions that the sum $s(x)$ of the convergent series $\Sigma u(x)$ may have a definite limit $s(\xi + 0)$ at the limiting point ξ of its

linear domain are (1), *that* $s_n(\xi + 0)$ *should converge to a definite limit as* $n \sim \infty$ *and* (2), *that, corresponding to each arbitrarily chosen positive number* ϵ, *and to each arbitrarily chosen integer* N_1, *there should exist a value of* n ($> N_1$), *and also a positive number* $\theta(\xi, \epsilon, n)$, *such that* $|R_n(x)| < \epsilon$ *for every value of* x *belonging to the domain which is in the interval* $(\xi, \xi + \theta)$.

In this formulation the condition (1) is not included in (2), and must therefore be stated separately. In case the number θ depends only on ϵ, and not also on n, the point ξ would be one of simply uniform convergence. However, in general θ will depend upon the value of n as well as upon ϵ, and thus the condition is less stringent than that of simply uniform convergence at the point.

THE CONTINUITY OF A SUM-FUNCTION IN A DOMAIN

86. It will now be assumed that the functions $u_1(x), u_2(x), u_3(x), \ldots$, of one or more variables, are all continuous in E. The following theorem will be established:

If the series $\Sigma u(x)$ *converge simply-uniformly in the domain* E, *the sum-function* $s(x)$ *is continuous in* E. *A fortiori, the condition that the series converges uniformly in* E *is sufficient to secure that the sum-function may be continuous in* E.

It should be observed that the condition in the theorem is sufficient, but not necessary, for the continuity of the sum-function.

Since the convergence of the series is simply-uniform, a value n_ϵ of n, corresponding to an arbitrarily chosen positive number ϵ, can be so determined that $|R_{n_\epsilon}(x)| < \frac{1}{4}\epsilon$, for all points x, in E. Consider a point ξ, of E; a neighbourhood $(\xi - \delta, \xi + \delta)$ of ξ, can be so determined that for every point x, of E, in that neighbourhood, $|s_{n_\epsilon}(x) - s_{n_\epsilon}(\xi)| < \frac{1}{2}\epsilon$, since $s_{n_\epsilon}(x)$ is continuous at ξ.

Since

$$|s(\xi) - s(x)| \leq |s_{n_\epsilon}(\xi) - s_{n_\epsilon}(x)| + |R_{n_\epsilon}(\xi)| + |R_{n_\epsilon}(x)| < \epsilon;$$

provided x is in $(\xi - \delta, \xi + \delta)$. Since ϵ is arbitrary, it follows that $s(x)$ is continuous at ξ.

The above proof suffices to establish the following more general theorem:

If the functions $u_n(x)$ *be all continuous at the point* ξ, *but not necessarily elsewhere, the condition of simply uniform convergence of the series in some neighbourhood of* ξ *is sufficient to ensure that* $s(x)$ *is continuous at* ξ.

It has already been proved, in § 82, that, if the sum-function is anywhere discontinuous, the convergence cannot be either uniform or simply

uniform, but the following additional proof of this important fact may be given.

If the function $s(x)$ be discontinuous at the point ξ, there exists a positive number α such that points x exist in every neighbourhood of ξ, however small, for which $|s(x) - s(\xi)| > \alpha$, or

$$|R_n(x) - R_n(\xi) + s_n(x) - s_n(\xi)| > \alpha.$$

It is impossible to choose n so that $|R_n(x)| < \frac{1}{4}\epsilon$, for all values of x, consisting of all points of E in some neighbourhood of ξ, provided ϵ is sufficiently small; for we should then have

$$\tfrac{1}{2}\epsilon > |R_n(x) - R_n(\xi)| > \alpha - |s_n(x) - s_n(\xi)|.$$

For any value of n that might be chosen, ξ could be so taken that

$$|s_n(x) - s_n(\xi)| < \tfrac{1}{2}\epsilon,$$

and thus $\epsilon > \alpha$. Since ϵ can be chosen to be $< \alpha$, the impossibility of choosing n so that $|R_n(x)| < \frac{1}{4}\epsilon$, for all points x in a neighbourhood of ξ, is demonstrated. Therefore, in this case, the convergence of the series is neither uniform nor simply uniform, and the point ξ is not a point of uniform convergence.

87. It has long been known that the sum of a convergent series of which all the terms are continuous is not necessarily itself continuous. The statement has often been made that the important discovery that discontinuity, when it occurs, is due to non-uniform convergence of the series was made by Stokes*, Seidel†, and Weierstrass‡, independently of one another. A critical discussion has been given by Hardy§ of the treatment of the matter, undoubtedly independently of one another, by these three Mathematicians. Hardy shows that the above statement requires considerable modification; he points out that the conception defined by Seidel, in 1848, is that which has been called in § 71, uniform convergence in the neighbourhood of a particular point, whereas Stokes, in 1847, defined a mode of convergence equivalent to what is here described as simply

* *Camb. Phil. Trans.* vol. VIII (1847), pp. 533–583; also *Mathematical and Physical Papers*, vol. I, pp. 236–313.

† *Münch. Abhand.* vol. VII (1848), pp. 381–394; also Ostwald's *Klassiker der exacten Wissenschaften*, no. 116.

‡ *Abhandlungen aus der Funktionenlehre*, pp. 69–101.

§ *Proc. Camb. Phil. Soc.* vol. XIX (1918), p. 148. It should be observed that, although what is there called quasi-uniform convergence in an interval is identical with what is called in § 67, simply uniform convergence in the interval, and what is there called quasi-uniform convergence in the neighbourhood of a point agrees with what is here understood by simply uniform convergence in the neighbourhood of a point, nevertheless what is called quasi-uniform convergence at a point is not equivalent to simply uniform convergence at a point, as defined in § 72, but is equivalent to the mode of convergence given in § 83, in the second form of the necessary and sufficient condition of continuity. On the history of the discovery see also Reiff's *Gesch. der unendl. Reihen*, p. 207, and also Pringsheim's article in the *Encykl. d. Math. Wissensch.* II, A I.

uniform convergence in a fixed neighbourhood of the particular point. Stokes gave a valid demonstration that his condition is sufficient to ensure continuity of the sum-function at the point, but his attempted proof that the condition is necessary for continuity is invalid because he failed to distinguish his condition from that given in the second statement of necessary and sufficient conditions in § 83. Both Stokes and Seidel confined their attention to a fixed neighbourhood of a particular point, whereas Weierstrass was familiar with the conceptions of uniform convergence in a linear interval and of uniform convergence in the neighbourhood of a point, as early as 1841 or 1842. That uniform convergence in the neighbourhood of every point of a linear interval involves uniform convergence in the interval was first proved* by Weierstrass in 1880. Under the influence of Weierstrass, the great importance of the notion of uniform convergence in the Theory of Functions became fully recognized.

The question whether non-uniform convergence necessarily implies discontinuity in the sum-function remained for some time an open one. It was decided in the negative sense when Darboux and Du Bois-Reymond constructed examples in which the series are non-uniformly convergent, and yet nevertheless have continuous sum-functions.

88. In order to determine necessary and sufficient conditions for the continuity of the sum-function of a series of continuous functions in the whole domain E of the convergent series, it is sufficient to consider the case in which E consists of a closed set. Let ϵ be an arbitrarily chosen positive number, then, if n be sufficiently large, there exist points of E at which $| R_n(x) | < \epsilon$. If we assume that $s(x)$ is continuous in E, since $s_n(x)$ is by hypothesis continuous in E, it follows that $| R_n(x) |$ is continuous in E, and therefore the set of points at which $| R_n(x) | \geqq \epsilon$ is closed relatively to E, and the set at which $| R_n(x) | < \epsilon$ is consequently open relatively to E; let this set be denoted by O_n. Each point of E belongs to all the sets of the sequence $\{O_n\}$, from and after some value of n dependent on the particular point, since $R_n(x)$ converges to zero, as $n \sim \infty$, for each value of x. If m be an integer chosen arbitrarily, employing de la Vallée Poussin's extension of the Heine-Borel theorem (I, § 75), a finite set of the open sets $O_m, O_{m+1}, \ldots$ exists such that every point of E belongs to one at least of these open sets, which we may denote by $O_{m+i_1}, O_{m+i_2}, \ldots O_{m+i_r}$. On the assumption that $s(x)$ is continuous in E, it thus appears that $| R_s(x) | < \epsilon$ at every point of E, provided that s has one of the values $m + i_1, m + i_2, \ldots m + i_r$, that value being dependent on the particular point.

Conversely, if it be assumed that this last condition is satisfied for every value of ϵ, then, remembering that m is arbitrary, any particular

* *Loc. cit.* pp. 71, 72.

it is everywhere zero, and it is continuous with respect to y, for $x = \xi$, but it is not necessarily continuous at $(\xi, 0)$ with respect to (x, y).

In accordance with the definition in § 70, the sequence $\{s_n(x)\}$ is uniformly convergent at the point ξ, if, corresponding to each arbitrarily chosen positive number ϵ, a neighbourhood of the point $(\xi, 0)$ exists defined by the two-dimensional, or $(p + 1)$-dimensional, cell

$$(\xi - d_\epsilon, 0;\ \xi + d_\epsilon, y_\epsilon)$$

exists such that $|R(x, y)| < \epsilon$, for every point (x, y) in this neighbourhood. If we denote by $U(\xi, d, y)$ the upper boundary of $|R(x, y)|$ in the neighbourhood $(\xi - d, 0;\ \xi + d, y)$ of $(\xi, 0)$ this function U is monotone non-increasing as the number d is diminished, and also as the number y is diminished. Consequently it has a lower limit as d, y converge to zero in any manner, independent of the particular mode in which the convergence takes place. In case the point ξ is one of uniform convergence, this limit is zero. When the limit is not zero, the point is one of non-uniform convergence.

The limit $\lim\limits_{\substack{d \sim 0 \\ y \sim 0}} U(\xi, d, y)$ which may have a finite value, or may be ∞, when it is not zero, is said to be the *measure of non-uniform convergence* of the sequence $\{s_n(x)\}$ at the point ξ. Denoting this measure by $\beta(\xi)$, the function $\beta(x)$ is a function of x which may be termed the *convergence-function*.

The definition of $\beta(\xi)$ may be stated as follows:

If the series $\Sigma u_n(x)$ converge to $s(x)$ in the linear, or p-dimensional, domain E, and the upper boundary of $|R_n(x)|$, for all values of $n \geqq m$, in a neighbourhood $(\xi - d, \xi + d)$ of the point ξ be determined, the limit of this upper boundary, as the numbers d converge to zero, and $m \sim \infty$, defines the measure $\beta(\xi)$, of non-uniform convergence of the sequence at the point ξ. In fact $\beta(\xi) = \overline{\lim\limits_{n \sim \infty, x \sim \xi}} |R_n(x)|$.

In case $\beta(\xi)$ is finite, a neighbourhood $(\xi - d, \xi + d)$ of ξ, can be determined, and an integer n_ϵ, such that, in that neighbourhood, $|s(x) - s_n(x)| < \beta(\xi) + \epsilon$, for all points x in $(\xi - \delta, \xi + \delta)$, and for all values of $n > n_\epsilon$, where ϵ is an arbitrary positive number.

Moreover there must exist in every neighbourhood of ξ, points at which $|s(x) - s_n(x)| > \beta(\xi) - \epsilon$, for some value of n $(> n_\epsilon)$, whatever positive number ϵ may be. In case $\beta(\xi)$ is infinite, corresponding to an arbitrarily chosen positive number N, a neighbourhood $(\xi - d, \xi + d)$, and an integer n_N, can be so determined that there exist in that neighbourhood points at which $|R_n(x)| > N$, for values of n that are $> n_N$.

91. In the case of a linear domain the measure of non-uniform convergence may be defined separately for the right and the left of the point ξ. The neighbourhoods employed in the definition will be taken to be neighbourhoods $(\xi, \xi + d)$, $(\xi - d, \xi)$ on the right and left respectively. Thus two functions $\beta^+(x)$, $\beta^-(x)$ will be defined, which have the values, at any point ξ, of the measures of non-uniform convergence at ξ, on the right and on the left respectively. If $\beta^+(\xi) = 0$, $\beta^-(\xi) > 0$, the point is one of uniform convergence on the right; a corresponding definition holds for the left. The measure $\beta(\xi)$ is the greater of the two numbers $\beta^+(\xi)$, $\beta^-(\xi)$; and at a point of uniform convergence, $\beta^+(\xi) = \beta^-(\xi) = 0$.

The earliest definition of the measure of non-uniform convergence was given by Osgood* for the case of a linear interval. The term "Grad der ungleichmässigen Convergenz" was employed by Schoenflies† who uses Osgood's definition. The term "Convergence function" was also employed by Schoenflies.

THE DISTRIBUTION OF POINTS OF NON-UNIFORM CONVERGENCE

92. Assuming that $\{s_n(x)\}$ is convergent in a domain E, it will first be shewn that:

The convergence function is upper semi-continuous in the domain of convergence of the series.

It must be shewn that a neighbourhood of a point ξ exists such that, at every point in it that belongs to the domain E, $\beta(x) < \beta(\xi) + \eta$, where η is an arbitrarily chosen positive number, and $\beta(\xi)$ is supposed to be finite. For, let it be supposed that in every such neighbourhood there exists a point at which $\beta(x) \geqq \beta(\xi) + \eta$. In an arbitrarily small neighbourhood of such a point there are points at which $| R_n(x) | \geqq \beta(\xi) + \eta$, for sufficiently large values of n, and such neighbourhood can be chosen so as to be interior to any assigned neighbourhood of the point ξ. Therefore in an arbitrarily small neighbourhood of ξ there are points at which $| R_n(x) | \geqq \beta(\xi) + \eta$, for sufficiently large values of n, and this is inconsistent with the fact that the measure of non-uniform convergence at ξ is $\beta(\xi)$. In case $\beta(\xi)$ is infinite, ξ is certainly a point of upper semi-continuity.

Employing a theorem given in I, § 230, which is applicable to any closed domain of any number of dimensions, we have the following theorem:

If the domain E, of the functions be a closed set of points, and σ be any positive number, the set of points of E at which $\beta(x) \geqq \sigma$ is closed, relatively to E, and therefore absolutely.

* *Amer. Journal of Math.* vol. XIX (1897), p. 166.

† See *Bericht,* vol. I, p. 226.

If we denote this closed set by G_σ, and assign to σ the values in a diminishing sequence $\{\sigma_n\}$ which converges to zero, each point of non-uniform convergence belongs to all the sets G_{σ_n}, from and after some value of n. Thus the set of all points of non-uniform convergence of the series is the outer limiting set G, of the sequence $\{G_{\sigma_n}\}$ of closed sets relatively to E. The set G_σ may be non-dense in E, or it may be dense in the whole, or, in a part, of E. Therefore the points of non-uniform convergence may be either non-dense in E, or may be dense in the whole, or in a part, of E. It will, however, be shewn that in case all the terms of the series or sequence are continuous in E, the set G_σ is necessarily non-dense in E. This is formulated in the following theorem:

If, in the closed domain E, of any number of dimensions, the functions $s_n(x)$ which converge in E to $s(x)$ are all continuous in E, the closed set of points G_σ, for which the measure of non-uniform convergence is $\geqq \sigma$, a positive number, is non-dense in E.

Let it be supposed that, if possible, G_σ is not non-dense in E; there must then exist a closed part E_1, of E, such that every point of E_1 belongs to G_σ. The set E_1 can contain no isolated points, because an isolated point is one of uniform convergence; thus E_1 must be a perfect set. Let ξ be a point of E_1; we take a neighbourhood D, of ξ, such that every point of E, in D, belongs to E_1, and also an arbitrarily chosen integer N. If σ' be a positive number $< \sigma$, there exists in D a point ξ', of E_1, such that

$$|s(\xi') - s_{n_1}(\xi')| > \sigma',$$

for some value of n_1 that is $> N$. Since the sequence $\{s_n(\xi')\}$ is convergent at ξ', an integer $n_2 > n_1$ exists such that $|s_{n_2}(\xi') - s_{n_1}(\xi')| > \sigma'$.

On account of the continuity of $s_{n_2}(x)$, $s_{n_1}(x)$ at ξ', a neighbourhood D_1 contained in D, can be determined so that, at every point of E_1 in D_1 we have $|s_{n_2}(x) - s_{n_1}(x)| > \sigma'$.

Taking a point ξ'', of E_1, interior to D_1, in a similar manner a neighbourhood D_2, contained in D_1, can be so determined that at every point of E_1, in D_2 we have $|s_{n_4}(x) - s_{n_3}(x)| > \sigma'$, where $n_4 > n_3 > n_2 > n_1 > N$.

Proceeding indefinitely in this manner, we obtain a sequence $\{D_m\}$ of neighbourhoods, all containing points of E_1, and such that, in D_m, we have $|s_{n_{2m}}(x) - s_{n_{2m-1}}(x)| > \sigma'$, at every point of E_1 in D_m; where $n_{2m} > n_{2m-1} > n_{2m-2} \dots > N$. These neighbourhoods $\{D_m\}$ can be so determined that only one point $\bar{\xi}$ of E_1 is in all of them; at this point $\bar{\xi}$, we have $|s_{n_{2m}}(\bar{\xi}) - s_{n_{2m-1}}(\bar{\xi})| > \sigma'$, for all values of m; this is contrary to the condition that the sequence $\{s_n(\bar{\xi})\}$ is convergent. It has thus been shewn that G_σ must be non-dense in E.

93. Since G_σ is non-dense in E, it follows that the set G, of all the points of non-uniform convergence of the sequence $\{s_n(x)\}$ of continuous functions,

is of the first category in E, and consequently (I, § 96) that the set of points of uniform continuity is everywhere dense in E. We have thus established the theorem that:

If the functions of a sequence $\{s_n(x)\}$ which converges everywhere in a closed set E to the function $s(x)$, are all continuous in E, the set of points of non-uniform convergence is of the first category in E, and the convergence is accordingly uniform at points of a set which is everywhere-dense in E.

This theorem was first established by Osgood*, for the case in which $s(x)$ is continuous.

Since $s(x)$ is certainly continuous at every point of uniform convergence, it follows that†:

If $\{s_n(x)\}$ converges, in a closed set E, to $s(x)$, and the functions $s_n(x)$ are all continuous in E, the function $s(x)$ is at most point-wise discontinuous (I, § 238) *with respect to E, and the points of continuity are accordingly everywhere-dense in E.*

94. Those points of the closed domain E at which the measure of non-uniform convergence is infinite, when such points exist, are of special importance in some parts of the theory of series. It can be shewn that:

When the convergence function $\beta(x)$ is unbounded in the closed domain E, there exists at least one point at which $\beta(x)$ is infinite.

For, in accordance with I, § 213, there must be at least one point ξ, of E, in whose arbitrarily small neighbourhood $\beta(x)$ is unbounded. The value of $\beta(\xi)$ cannot be finite, because this would be inconsistent with the fact that $\beta(x)$ is upper semi-continuous.

It can now be shewn that:

The set of points of infinite measure of non-uniform convergence is closed.

For, let $\sigma_1, \sigma_2, \ldots \sigma_n, \ldots$ denote a divergent sequence of increasing positive numbers, and let e_n denote the closed set of points at which $\beta(x) \geqq \sigma_n$. The set of points at which the measure of non-uniform convergence is infinite is the inner limiting set of the sequence $\{e_n\}$ of closed sets each of which contains the next; and in accordance with I, § 67, this inner limiting set is closed.

It is necessary and sufficient in order that there may be no points, in the closed set E, at which the measure of non-uniform convergence is infinite, that, from and after some fixed value of n, $|R_n(x)|$ should be bounded as a function of (n, x) for all such values of n, and for all points x, in E.

* *Amer. Journal of Math.* vol. XIX (1897). A proof, free from this restriction, was given by Hobson, *Proc. Lond. Math. Soc.* (1), vol. XXXIV (1902). Other proofs were given by W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. I (1904) and by Dell' Agnola, *Rend. Lincei*, vol. XIX (1910).

† This theorem was first established in a different manner, by Baire, *Ann. di mat.* (3), vol. III (1899). For another proof see W. H. Young, *Mess. of Math.* (2), vol. XXXVII (1907); see also Dell' Agnola, *Rend. Lomb.* vol. XLI (1908).

That the condition is sufficient is clear, since if $|R_n(x)|$ is bounded, so also is $\beta(x)$. To shew that the condition is necessary, we observe that, in the $(p+1)$-dimensional domain, a neighbourhood of each point x, of E, can be determined such that $|R(x, y)| < \beta(x) + \epsilon$, in that neighbourhood. These neighbourhoods form an infinite set of cells, each of which contains one or more of the points of the closed set E, in its interior. Employing the Heine-Borel theorem in its generalized form (I, § 74) a finite set of these neighbourhoods exists, such that every point of E is interior to one or more of them. It follows that there exists a value y_1 of y, the least linear dimension in y of the cells of the finite set, such that $|R(x, y)|$ is bounded, provided $y < y_1$. Hence there exists a value n_1, of n such that $|R_n(x)|$ is bounded as a function of (n, x) for $n \geqq n_1$, and for all values of x in E.

EXAMPLE

Let* $u_1(x) = 0$, at all points of the interval $(0, 1)$, except at the point $\frac{1}{2}$, where $u_1(x) = 1$. Let $u_2(x) = -1$, at $x = \frac{1}{2}$, and $u_2(x) = 1$ at $x = \frac{1}{4}, \frac{3}{4}$, and $u_2(x) = 0$, everywhere else; let $u_3(x) = -1$, at $x = \frac{1}{4}, \frac{3}{4}$, and $u_3(x) = 1$ at $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, and $u_3(x) = 0$ at all other points of the interval; and so on. Thus $s_1(x)$ is zero, except at $x = \frac{1}{2}$, where $s_1(\frac{1}{2}) = 1$; $s_2(x)$ is zero, except that $s_2(\frac{1}{4}) = s_2(\frac{3}{4}) = 1$; $s_3(x)$ is zero, except that $s_3(\frac{1}{8}) = s_3(\frac{3}{8}) = s_3(\frac{5}{8}) = 1$; and so on. The function $s(x)$ is everywhere zero, and therefore is continuous in $(0, 1)$; but the series converges non-uniformly at every point of the interval, since, in the neighbourhood of every assigned point, there are discontinuities of $R_n(x)$, of measure 1.

It has hitherto been assumed that $s(x)$ is everywhere finite; this restriction may be removed by employing the transformation

$$\sigma(x) = \frac{s(x)}{1 + |s(x)|},$$

of § 62. A point of continuity of $\sigma(x)$ corresponds to a point of continuity of $s(x)$, or to a point of continuity in the extended sense, according as $|\sigma(x)| < 1$, or $|\sigma(x)| = 1$. We have accordingly the theorem:

If the sequence $\{s_n(x)\}$ is convergent or divergent (to $+\infty$, or to $-\infty$) at every point of the closed set E, and the functions $s_n(x)$ are all continuous, at least in the extended sense, at every point of E, the points of discontinuity of $s(x)$ form a set of the first category relative to E, and the points of continuity of $s(x)$, at least in the extended sense, are everywhere-dense in E.

95. If the functions $s_n(x)$ are not all continuous in the closed set E, the closed set G_σ, of points at which the measure of non-uniform is $\geqq \sigma$ may be everywhere-dense in the whole, or in a part of E; and then this is also true of the set of all the points of discontinuity of $s(x)$. In any case the set of points of continuity of $s(x)$, when it exists, forms an inner limiting set.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. I (1903), p. 94.

It has been shewn by W. H. Young* that, for the case in which E is a linear interval, when the functions $u_n(x)$ are point-wise discontinuous functions, the visible points of non-uniform convergence may be dense in the whole or in a part of E, and the invisible points of non-uniform convergence form a set of the first category relative to E. He has shewn how to construct a series of point-wise discontinuous functions, for a given linear interval, which have an assigned inner limiting set F for the set of its points of uniform convergence of the sum-function, and such that the sum-function is non-uniformly convergent at all points of $C(F)$.

96. The following general theorem has been established by Hahn†:

If E be a set in any number of dimensions it is necessary and sufficient in order that a sequence $\{s_n(x)\}$, convergent in E, may exist which converges non-uniformly at all points of E_1, a part of E, and uniformly at all points of $E - E_1$, that E_1 should be the outer limiting set of a sequence of sets which are closed relatively to E.

The proof of this theorem requires the theorem or assumption‡ that every set of points which is dense in itself can be normally ordered. Without assuming this, the following less general theorem, also given by Hahn, can be established:

If E be a set of points in any number of dimensions, and F be a part of E which is of the first category relatively to E, a sequence $\{f_n(x)\}$ of functions such that $0 \leq f_n(x) \leq 1$ can be defined, which converges everywhere in E to the limit zero, the convergence being non-uniform at every point of F, and uniform at every point of $E - F$.

A part E_1 of a set E, of points in any number of dimensions is said to be *closed, relatively to E*, if every limiting point of E_1 that is in E, is a point of E_1 itself. This is a generalization of the definition in I, § 55. The set E_1 is closed in the absolute sense in case E is closed.

The complement $E - E_1$, relatively to E, of a set E_1, closed relatively to E, is said to be *open relatively to E*. If F is a part of E which is the outer limiting set of a sequence of sets, each of which is closed relatively to E, and non-dense in E, the set F is said to be of the *first category relatively to E*. This is a generalization of the definition given in I, § 93.

First, let F be closed relatively to E, and non-dense in E. Let the continuous function $\chi_n(t)$ be defined for all non-negative values of the continuous variable t, by the prescriptions, $\chi_n(t) = 0$, when $t = 0$, and when $t \geq \frac{2}{n}$; $\chi_n(t) = 1$, when $t = \frac{1}{n}$; $\chi_n(t) = nt$ in the interval $\left(0, \frac{1}{n}\right)$; $\chi_n(t) = -nt + 2$, in the interval $\left(\frac{1}{n}, \frac{2}{n}\right)$.

* *Loc. cit.*; see also *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 356.

† See *Theorie der reellen Funktionen*, vol. I, p. 274.

‡ *Ibid.* p. 200.

If x be any point of E, and $d(x, F)$ be its distance from F (see I, § 104), then $s_n(x)$ is defined to be $\chi_n\{d(x, F)\}$. The function $s_n(x)$ is continuous with respect to E, and vanishes at all points of F; also $0 \leqq s_n(x) \leqq 1$, and $\lim\limits_{n \sim \infty} s_n(x) = 0$.

Let ξ be a point of E that does not belong to F, then $d(\xi, F) > 0$; a neighbourhood Δ, of ξ, can be so determined that, for all points of E in that neighbourhood, $d(x, F) > \alpha$, where α is some fixed positive number. In Δ, we have $s_n(x) \equiv \chi_n\{d(x, F)\} = 0$, provided $d(x, F) > \dfrac{2}{n}$, which condition is satisfied if $n \geqq \dfrac{2}{\alpha}$; therefore the convergence of $s_n(x)$ to zero is uniform at the point ξ.

If ξ be a point of F, in any neighbourhood Δ, of ξ, there are points x, of E, which do not belong to F, for which $d(x, F)$ has some value β (> 0), and for each such point there is a value of n for which $s_n(x) > \frac{1}{2}$; for there is at least one value of n, such that $\dfrac{1}{2\beta} \leqq n \leqq \dfrac{3}{2\beta}$, or $\dfrac{1}{2n} \leqq \beta \leqq \dfrac{3}{2n}$, provided β be sufficiently small. Such a value of n increases indefinitely as the distance of x from ξ is diminished. It follows that the convergence of $\{s_n(x)\}$ at the point ξ is non-uniform. Thus the theorem has been established for the case in which F is closed relatively to E, and non-dense in E.

Next let F be the outer limiting set of a sequence $\{F_m\}$ of sets F_m, each of which is closed relatively to E, and non-dense. Let $s_{mn}(x)$, where $n = 1, 2, 3, \ldots$, is the sequence, defined as above, corresponding to F_m; and let us consider the double sequence of functions $\dfrac{1}{m} s_{mn}(x)$, where m and n have all integral values. This double sequence can be arranged as a single sequence $\{f_p(x)\}$; and it will be shewn that this single sequence has the required properties. If ϵ be any prescribed positive number, it is impossible that, at any point ξ, $f_p(\xi) \geqq \epsilon$ for an infinite set of values of p. For we have $0 < \dfrac{1}{m} s_{mn}(\xi) < \epsilon$, if $m > \dfrac{1}{\epsilon}$, and for all values of n; if α be the greatest integer which does not exceed $\dfrac{1}{\epsilon}$, the only values of m for which $\dfrac{1}{m} s_{mn}(\xi) \geqq \epsilon$, are $1, 2, 3, \ldots \alpha$; for each of these values there are only a finite number of values of n for which $\dfrac{1}{m} s_{mn}(\xi) \geqq \epsilon$, and therefore only a finite number of values of n for which this condition is satisfied for any of the values $1, 2, \ldots \alpha$, of m. It follows that, except for a finite set of values of p, we have $f_p(\xi) < \epsilon$. Since ϵ is arbitrary, $\{f_p(\xi)\}$ converges to zero.

Since the sequence $\frac{1}{m} s_{m1}(x)$, $\frac{1}{m} s_{m2}(x)$, ... is a part of the sequence $\{f_p(x)\}$, the upper boundary of $f_p(x)$, in the neighbourhood Δ of a point ξ, of E, cannot be less than the upper boundary of the first sequence. When ξ is a point of F_m this upper boundary converges, as $\Delta \sim 0$, to a value greater than zero, hence the same must hold for the sequence $\{f_p(x)\}$; and this must be the case for each value of m. It is therefore the case for any point ξ, of F; the convergence of $\{f_p(x)\}$ at ξ is therefore non-uniform.

If ξ be a point of $E - F$, in a properly chosen neighbourhood Δ, of ξ, we have $f_p(x) < \epsilon$, for all values of p greater than some fixed number dependent on ϵ. Since $\frac{1}{m} s_{mn}(x) < \epsilon$, for all values of n, provided $m > \frac{1}{\epsilon}$, we need only consider those values 1, 2, 3, ... α, of m, for which $\alpha \leqq \frac{1}{\epsilon}$. For each of these values r, of m, $\frac{1}{m} s_{mn}(x) < \epsilon$, for $n > n_\epsilon^{(r)}$, provided Δ is properly chosen.

Hence if n_ϵ be the greatest of the numbers $n_\epsilon^{(r)}$, we have $\frac{1}{m} s_{mn}(x) < \epsilon$, for $m = 1, 2, \dots \alpha$, and for $n > n_\epsilon$. Thus $f_p(x) < \epsilon$, in Δ, for all values of p, except a finite number; therefore ξ is a point of uniform convergence of the sequence. The theorem has now been completely established.

FUNCTIONS INVOLVING A PARAMETER

97. The theory of uniform and non-uniform convergence of sequences, or series, may be extended, by a slight modification, to apply to the case of a function $f(x, \alpha)$, defined in a domain of x, of any number of dimensions, for each value of the parameter α which is in an assigned linear interval, closed or open. Let it be assumed that $f(x, \alpha)$ is defined for values of α such that $\alpha_0 < \alpha \leqq \alpha_0 + c$. If at a point ξ, of the domain of x, there is a definite value of $\lim_{\alpha \sim \alpha_0} f(\xi, \alpha)$, α varying continuously, the function $f(x, \alpha)$ is said to be convergent at the point ξ, as α converges to α_0.

In case α diverges to ∞, we shall suppose $f(x, \alpha)$ to be defined for all values of α greater than, or equal to, some fixed number C.

The following definitions of uniform convergence in the domain E, of x, are precisely similar to those in which α is confined to have values in a sequence, given in § 66.

If, as $\alpha \sim \alpha_0$, $f(x, \alpha)$ converges to a definite number $\phi(x)$, the convergence is said to be uniform in E, provided that, if ϵ be any arbitrarily chosen positive number, a number h_ϵ can be so determined that $|f(x, \alpha) - \phi(x)| < \epsilon$, for all values of x, in E, provided α lies in the interval $(\alpha_0, \alpha_0 + h_\epsilon)$.

If a_0 is infinite, and $f(x, a)$ converges to a definite number $\phi(x)$, for each value of x, in E, and if ϵ be an arbitrarily chosen positive number, then $f(x, a)$ is said to converge uniformly to $\phi(x)$, as $a \sim \infty$, provided a number C_ϵ can be so determined that $|\phi(x) - f(x, a)| < \epsilon$, for $a > C_\epsilon$, and for all values of x.

The definition of a point ξ of uniform convergence of $f(x, a)$ to $\phi(x)$ is then precisely similar to that in § 70, a neighbourhood of ξ being employed.

The theorems of §§ 82, 83, relating to the continuity of $\phi(x)$ can readily be stated so as to apply to the functions $f(x, a)$.

THE UNIFORM CONVERGENCE OF INFINITE PRODUCTS

98. If $u_1(x), u_2(x), \dots u_n(x), \dots$ be a sequence of functions defined in a domain of one or more dimensions, the infinite product $\prod_{n=1} \{1 + u_n(x)\}$ is, in accordance with § 39, convergent at a particular point x if, corresponding to each prescribed positive number ϵ, the condition

$$|\{1 + u_{n+1}(x)\}\{1 + u_{n+2}(x)\} \dots \{1 + u_{n+m}(x)\} - 1| < \epsilon$$

is satisfied for $m = 1, 2, 3, \dots$, provided n is greater than some integer n_ϵ, dependent on ϵ.

In case the infinite product converges at each point x of a domain E, if the integer n_ϵ corresponding to each value of ϵ, can be so chosen as to be independent of x, then the infinite product is said to *converge uniformly* in the domain E.

The theory of uniform and non-uniform convergence of infinite products is similar to that of infinite series, and indeed may be deduced from it by the method of taking logarithms. It is the particular case of the theory of uniform and non-uniform convergence of a sequence $\{s_n(x)\}$ which arises when $s_n(x)$ has the special form

$$\{1 + u_1(x)\}\{1 + u_2(x)\} \dots \{1 + u_n(x)\}.$$

On account of the importance which this method of representation of functions has in Analysis, a short statement will here be given of the properties of infinite products in regard to their uniform convergence.

In analogy with Weierstrass' test for the uniform convergence of series, the following test may be applied to the case of infinite products:

If, in the domain E, of x, $|u_n(x)| \leqq v_n$, where the infinite product $\prod_{n=1} (1 + v_n)$ is convergent, then the infinite product $\prod_{n=1} \{1 + u_n(x)\}$ converges uniformly in E.

For

$$|\{1+u_{n+1}(x)\}\{1+u_{n+2}(x)\}\dots\{1+u_{n+m}(x)\}-1|$$
$$\leqq (1+v_{n+1})(1+v_{n+2})\dots(1+v_{n+m})-1.$$

Since n_ϵ can be so chosen that, for $n > n_\epsilon$, the expression on the right hand side is $< \epsilon$, it is seen that the condition for the uniform convergence of the infinite product $\prod\limits_{n=1}\{1+u_n(x)\}$ is satisfied.

It is easily seen, as in § 78, that:

If the infinite product $\prod\limits_{n=1}\{1+|u_n(x)|\}$ *converges uniformly in a domain of* x, *so also does the infinite product* $\prod\limits_{n=1}\{1+u_n(x)\}$.

The definition of a point of uniform convergence of a product

$$\prod_{n=1}\{1+u_n(x)\}$$

at a point ξ of a closed domain E is similar to that of § 70 for the case of a series:

If, for a point ξ *of the closed domain* E, *linear or p-dimensional, a neighbourhood* $(\xi-d_\epsilon,\ \xi+d_\epsilon)$ *(linear or p-dimensional) exists such that, for* $n \geqq n_\epsilon$, *a number dependent on* ϵ,

$$|\{1+u_n(x)\}\{1+u_{n+1}(x)\}\dots\{1+u_{n+m}(x)\}-1|<\epsilon$$

for all values of m, *for all points* x *in* $(\xi-d_\epsilon,\ \xi+d_\epsilon)$, *the point* ξ *is said to be a point of uniform convergence of the infinite product* $\prod\limits_{n=1}\{1+u_n(x)\}$.

The following theorem corresponds to the theorem of § 81:

If the infinite product $\prod\limits_{n=1}\{1+u_n(m)\}$ *is uniformly convergent for all positive integral values of* m *(or for all positive continuously varying values of* m*), and if* $\lim\limits_{m\sim\infty}u_n(m)$ *has a definite value* v_n, *for each value of* n, *then if* $f(m)$ *denotes the limiting value of* $\prod\limits_{n=1}\{1+u_n(m)\}$, $\lim\limits_{m\sim\infty}f(m)$ *has a definite value, and the infinite product* $\prod\limits_{n=1}(1+v_n)$ *is convergent and has* $\lim\limits_{m\sim\infty}f(m)$ *for its value.*

We have, for all the values of m, $\left|f(m)-\prod\limits_{n=1}^{n=N}\{1+u_n(m)\}\right|<\epsilon$, provided N is not less than some fixed integer N_ϵ, dependent on ϵ. It follows that $\left|\overline{\lim}f(m)-\prod\limits_{n=1}^{n=N}(1+v_n)\right|\leqq\epsilon$, from whence we deduce that $\left|\overline{\lim\limits_{m\sim\infty}}f(m)-\underline{\lim\limits_{m\sim\infty}}f(m)\right|\leqq 2\epsilon$. Since ϵ is arbitrary, it follows that $\lim\limits_{m\sim\infty}f(m)$ exists. Again we see that $\left|\lim\limits_{m\sim\infty}f(m)-\prod\limits_{n=1}^{n=N}(1+v_n)\right|\leqq\epsilon$, for $N\geqq N_\epsilon$; hence $\prod\limits_{n=1}(1+v_n)$ converges to $\lim\limits_{m\sim\infty}f(m)$.

EXAMPLE

It can be shewn that, if m is an even integer,

$$\frac{\sin x}{m \sin \frac{x}{m} \cos \frac{x}{m}} = \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{\pi}{m}}\right)\left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{2\pi}{m}}\right) \dots \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{r\pi}{m}}\right)$$

where $r = \frac{1}{2}(m - 2)$. Let $0 < x < (s + 1)\pi$, and s is a fixed integer $< r$. We can write the above result in the form

$$\frac{\sin x}{m \sin \frac{x}{m} \cos \frac{x}{m}} \Big/ \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{\pi}{m}}\right) \dots \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{s\pi}{m}}\right) = \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{\overline{s+1}\pi}{m}}\right) \dots \left(1 - \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{r\pi}{m}}\right).$$

Since $\frac{\sin \theta}{\theta}$ diminishes as θ increases from 0 to $\frac{\pi}{2}$, $\frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{p\pi}{m}} < \frac{x^2}{p^2\pi^2}$, where $s + 1 \leq p \leq r$.

Keeping x and s fixed, the product on the right hand side is less than the convergent product

$$\left(1 + \frac{x^2}{(s+1)^2\pi^2}\right)\left(1 + \frac{x^2}{(s+2)^2\pi^2}\right) \dots \left(1 + \frac{x^2}{r^2\pi^2}\right) \dots .$$

Thus, for the fixed value of x, the above theorem is applicable; hence the infinite product

$$\left(1 - \frac{x^2}{(s+1)^2\pi^2}\right)\left(1 - \frac{x^2}{(s+2)^2\pi^2}\right) \dots$$

converges to the limit, as $m \sim \infty$, of the expression on the left hand side, which is

$$\frac{\sin x}{x} \Big/ \left(1 - \frac{x^2}{\pi^2}\right) \dots \left(1 - \frac{x^2}{s^2\pi^2}\right).$$

Therefore the infinite product $x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right) \dots$ converges to $\sin x$.

In a similar manner it can be shewn that $\left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right) \dots$ converges to $\cos x$.

THE CONVERGENCE OF A SEQUENCE IN A MEASURABLE DOMAIN

99. If the domain E, of any number of dimensions, be measurable, and have its measure finite, and the functions of the sequence $\{s_n(x)\}$ be all measurable, the following theorem expresses the condition that $\{s_n(x)\}$ should converge to a function $s(x)$ almost everywhere in E, that is, at every point with the possible exception of the points of a set of which the measure is zero.

If E be a measurable set (in any number of dimensions) of finite measure l, the necessary and sufficient condition that a sequence $\{s_n(x)\}$, of measurable finite functions, should converge to a finite function $s(x)$, almost everywhere in E, is that a set H_ζ, contained in E, and of measure greater than $l - \zeta$, where ζ is an arbitrarily chosen positive number, exists, such that $\{s_n(x)\}$ converges uniformly in H_ζ to $s(x)$.

This theorem was given by Egoroff*. That the condition is sufficient is clear, for if, however small ζ may be chosen, there is a set of measure $> l - \zeta$ in which the sequence converges, the set of all points of convergence has its measure $> l - \zeta$, for all positive values of ζ, and therefore its measure is l.

To prove that the condition is necessary, let $\{\epsilon_r\}$ be a monotone sequence of positive numbers which converges to zero. Let e_n denote the set of points of E, at each of which $| s(x) - s_{n+s}(x) | < \epsilon_r$, for $s = 0, 1, 2, 3, \ldots$. Considering the sets $e_n, e_{n+1}, \ldots$, it is clear that each of these sets is contained in the next. Each point of E, with the exception of a set of measure zero, belongs to all the sets of the sequence $\{e_n\}$, from and after some value of n dependent on the particular point. The set of all such points of E, of measure l, is the outer limiting set of the sequence $\{e_n\}$. Let $\eta_1, \eta_2, \eta_3, \ldots$ be a diminishing sequence of positive numbers whose sum converges to the arbitrarily chosen number ζ. There exists a least value of n such that $m(e_n) > l - \eta_r$ (see I, § 131); and let this set e_n be denoted by F_r. The sets $F_1, F_2, \ldots$ denote the sets F_r which correspond to $\eta_1, \eta_2, \ldots$ respectively. We have $m(F_1) > l - \eta_1$; $m(F_2) > l - \eta_2$, $m(F_3) > l - \eta_3, \ldots$. There exists a set of measure

$$> l - \eta_1 - \eta_2 - \eta_3 - \ldots,$$

or $> l - \zeta$, of points each of which belongs to all the sets $F_1, F_2, \ldots$. In this set H_ζ, of measure $> l - \zeta$, we have $| s(x) - s_m(x) | < \epsilon_r$, for each value of r, provided m is not less than some integer dependent on the value of ϵ_r. Therefore the sequence converges uniformly in the set H_ζ; and thus the necessity of the condition in the theorem has been established.

100. In the above theorem it has been established that, if $\{\zeta_n\}$ be a diminishing sequence of positive numbers which converges to zero, there exists a sequence $\{H_{\zeta_n}\}$ of sets, such that $m(H_{\zeta_n}) > l - \zeta_n$, so that

$$\lim_{n \sim \infty} m(H_{\zeta_n}) = l,$$

in each of which sets the convergence of the sequence is uniform. This mode of convergence has been termed by Weyl†, essentially uniform convergence (*wesentlich gleichmässige Convergenz*). Thus the theorem of Egoroff establishes the equivalence of the convergence of the sequence almost everywhere in the measurable set with essentially uniform convergence in that set. It can be shewn that the sets H_{ζ_n} can be so determined that each one is contained in the next. For let $\xi_n = \zeta_n - \zeta_{n+1}$, for all values of n, and consider a sequence $\{K_{\xi_n}\}$ of sets, such that $m(K_{\xi_n}) > l - \xi_n$, and such that the convergence is uniform in K_{ξ_n}. Let H_{ζ_n}

* *Comptes Rendus*, vol. CLII (1911), p. 244.
† *Math. Annalen*, vol. LXVII (1909), p. 225.

be the set of points common to all the sets $K_{\xi_n}, K_{\xi_{n+1}}, \ldots$. It is then clear that the convergence of the sequence is uniform in H_{ζ_n}; also

$$m(H_{\zeta_n}) > l - \xi_n - \xi_{n+1} - \ldots > l - \zeta_n,$$

and H_{ζ_n} is contained in $H_{\zeta_{n+1}}$. Thus the sequence $\{H_{\zeta_n}\}$ has the required property.

Egoroff's theorem cannot be applied when the domain is measurable but of infinite measure. To see this, let such a domain E be the outer limiting set of a sequence $\{E_n\}$, where $m(E_n)$ is finite, and E_n is contained in E_{n+1}. Let $s_n(x) = 1$ in the set $E_{n+1} - E_n$, and $s_n(x) = 0$ for all points not in that set. The function $s(x)$ exists at every point of E, and has the value zero, but it does not converge uniformly in any set of infinite measure contained in E.

Egoroff's theorem has been extended by W. H. Young* to the case in which the sequence $\{s_n(x)\}$ is non-convergent in the measurable set E of finite measure. The main results of his investigations may be stated as follows:

If $\{s_n(x)\}$ *be a sequence of functions defined for all points of a measurable set* E, *of finite measure, and the upper (lower) function of the sequence is finite at almost all points of* E, *then* (1), *there exists in* E, *a set* E_1 *of measure* $> m(E) - \zeta$, *so that in* E_1 *the sequence* $\{s_n(x)\}$ *has uniform oscillations of the first kind* (§ 114); *and* (2), *there exists in* E *a set* E_2, *of measure* $> m(E) - \zeta$, *in which the sequence has uniform oscillations of the second kind; also* (3), *there exists a set* E_3, *of measure* $> m(E) - \zeta$, *in which the sequence has uniform oscillations both of the first and of the second kinds.*

101. Egoroff's theorem may be applied to obtain the following property† of double sequences:

If E *be a measurable set of points, of finite, or of infinite, measure, and* $\{s_{mn}(x)\}$ *a double sequence such that* $\lim\limits_{n\sim\infty} s_{mn}(x)$ *exists and has a value* $s_m(x)$ *almost everywhere in* E, *for each value of* m, *and such that* $\lim\limits_{m\sim\infty} s_m(x)$ *exists, and has a value* $s(x)$, *almost everywhere in* E, *then two increasing sequences* $\{m_i\}$, $\{n_i\}$ *of integers can be so determined that* $\{s_{m_i n_i}(x)\}$ *has the unique limit* $s(x)$, *almost everywhere in* E, *as* i *is indefinitely increased.*

By employing, when necessary, the transformation

$$\sigma_{mn}(x) = \frac{s_{mn}(x)}{1 + |s_{mn}(x)|},$$

the theorem can be reduced to the case in which all the functions $s_m(x)$, $s(x)$ are bounded; thus it will be sufficient to assume this to be the case.

* *Quart. Journ. Math.* vol. XLIV (1913), p. 129; and *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 363.

† See Fréchet, *Rend. di Palermo*, vol. XXII (1906), p. 15, where the theorem is established for the case in which E is a finite linear interval.

Let E be the outer limiting set of a sequence $\{E_i\}$, of measurable sets, each of which is contained in the next, and each of which is of finite measure. If E has finite measure, we may take all the sets E_i to be identical with E.

In E_i there exists a set K_i of measure $> m(E_i) - \zeta_i$, where ζ_i is arbitrarily fixed, in which $\{s_m(x)\}$ converges uniformly; thus there exists a least integer m_i, such that $| s(x) - s_{m_i}(x) | < \epsilon_i$, in the set K_i; where $\{\epsilon_i\}$ is a diminishing sequence converging to zero.

The integer m_i being fixed, there exists a set L_i, in K_i, of measure $> m(E_i) - 2\zeta_i$, and a least integer n_i $(> n_{i-1})$, such that

$$| s_{m_i}(x) - s_{m_i n_i}(x) | < \epsilon_i,$$

in L_i. Therefore, in the set L_i, we have $| s(x) - s_{m_i n_i}(x) | < 2\epsilon_i$. Now let the sequence $\{\zeta_r\}$ be so chosen that $\sum_{r=1}^{\infty} \zeta_r < \zeta$, where ζ is an arbitrarily chosen positive number. The part $D(L_i, E_p)$ of the set L_i that is in E_p, where $i \geqq i_1 > p$, and i_1 is chosen arbitrarily, has its measure $> m(E_p) - 2\zeta_i$; and all these sets have, for a fixed value of i_1, a common part, of measure $> m(E_p) - 2 \sum_{i=i_1}^{\infty} \zeta_i$. In this set, contained in E_p, we have $| s(x) - s_{m_i n_i}(x) | < 2\epsilon_i$, for $i \geqq i_1$: the measure of this set is less than $m(E_p)$ by an amount which can be made arbitrarily small by taking i_1 sufficiently large. It follows that, in E_p, the sequence $\{s_{m_i n_i}(x)\}$ converges to $s(x)$ almost everywhere. Since this is the case for each value of p, it follows that $\{s_{m_i n_i}(x)\}$ converges to $s(x)$ almost everywhere in E.

The following theorem for double sequences is a simplification of the above theorem, of less generality:

If $\{s_{mn}(x)\}$ be a double sequence, defined in a set E, of any number of dimensions, then if, for each value of m, the sequence $\{s_{mn}(x)\}$ converges uniformly in E to a function $s_m(x)$, and the sequence $\{s_m(x)\}$ converges uniformly in E to a function $s(x)$, sequences $\{m_i\}$, $\{n_i\}$, can be so determined that the single sequence $\{s_{m_i n_i}(x)\}$ converges uniformly in E to $s(x)$.

If $\{\epsilon_i\}$ denote a diminishing sequence of positive numbers which converges to zero, a least integer m_i can be determined, such that

$$| s(x) - s_{m_i}(x) | < \epsilon_i,$$

at all points of E.

Again, when m_i has been fixed, a least integer n_i can be determined, such that $| s_{m_i}(x) - s_{m_i n_i}(x) | < \epsilon_i$, at all points of E. We have then $| s(x) - s_{m_i n_i}(x) | < \epsilon_i$, at all points of E; and since $| s(x) - s_{m_j n_j}(x) | < \epsilon_i$, for $j \geqq i$, at all points of E, it follows that the sequence $\{s_{m_i n_i}(x)\}$ converges uniformly in E, to the value $s(x)$.

MONOTONE SEQUENCES OF FUNCTIONS

102. A monotone sequence of functions $\{s_n(x)\}$ defined for a domain E, of any number of dimensions, is such that, for every point x, of E,

$$s_{n+1}(x) \geqq s_n(x)$$

for every value of n; or else such that $s_{n+1}(x) \leqq s_n(x)$ for every value of n. In the former case the sequence is said to be monotone non-diminishing, and in the second case it is said to be monotone non-increasing. It is clear that, at every point ξ, there is a single-valued sum-function, since, the sequence $\{s_n(\xi)\}$, being monotone, either converges to a definite limit, as $n \sim \infty$, or diverges. Thus the sequence cannot oscillate at any point of E. We need only consider the case in which the set E is dense in itself, so that every point is a limiting point. We shall here consider monotone sequences of functions which are either continuous, or semi-continuous, functions in the set E, and also certain less simple types of functions which occur in this connection. The following preliminary proposition* is of fundamental importance in this theory:

A monotone non-increasing sequence $\{s_n(x)\}$ of functions which are all upper semi-continuous at the point ξ, of the domain E, dense in itself, is such that $s(x)$ is upper semi-continuous at ξ.

In case $s(\xi)$ is finite, n can be so determined that $s_n(\xi) < s(\xi) + \frac{1}{2}\epsilon$; and since $s_n(x)$ is upper semi-continuous at ξ, a neighbourhood D, of ξ, can be so determined that, for every point x, of E, that is in D, the condition $s_n(x) < s_n(\xi) + \frac{1}{2}\epsilon < s(\xi) + \epsilon$ is satisfied. Since $s(x) \leqq s_n(x)$, we have $s(x) < s(\xi) + \epsilon$, for all points x, of E, in D. Therefore $s(x)$ is upper semi-continuous at ξ.

In case $s(\xi) = -\infty$, for a sufficiently large value of n we have $s_n(\xi) < -N$, and a neighbourhood of ξ can be so determined that, in that neighbourhood, $s_n(x) < s_n(\xi) + \eta < -N + \eta$; where η is arbitrarily chosen. It follows that, in that neighbourhood, $s(x) < -N + \eta$; and since N and η are both arbitrary, $s(x)$ is upper semi-continuous at ξ.

If $s(\xi) = \infty$, which involves $s_n(\xi) \doteqdot \infty$, for all values of n, the point ξ is regarded as one of upper semi-continuity of $s(x)$.

If we consider the sequence $\{-s_n(x)\}$, we observe that it is a non-diminishing sequence of functions all of which are lower semi-continuous at the point ξ, and its sum-function $-s(\xi)$ is lower semi-continuous at ξ; we thus deduce the theorem that:

A monotone non-diminishing sequence $\{s_n(x)\}$ of functions which are all lower semi-continuous at the point ξ is such that $s(x)$ is lower semi-continuous at ξ.

* See Baire, *Bull. Soc. Math. de France*, vol. XXXII (1904), p. 125; also W. H. Young, *Mess. of Math.* vol. XXXVII (1908), p. 148.

103. If the functions $s_n(x)$ be all continuous at the point ξ, we deduce from the above theorems, since a continuous function is both upper semi-continuous and lower semi-continuous, that:

If the functions of the monotone non-increasing sequence $\{s_n(x)\}$ be all continuous at the point ξ, the function $s(x)$ is upper semi-continuous at ξ. If the sequence be monotone non-diminishing, and all the functions $s_n(x)$ are continuous at ξ, $s(x)$ is lower semi-continuous at ξ.

From the theorems that have been established, the following at once follow:

If the sequence $\{s_n(x)\}$ be monotone non-increasing, and the functions $s_n(x)$ are all upper semi-continuous in their domain E, which is dense in itself, the limiting function is upper semi-continuous.

If the sequence be monotone non-diminishing, and the functions $s_n(x)$ are all lower semi-continuous in their domain, the limiting function is lower semi-continuous.

The limiting function of a non-increasing monotone sequence of continuous functions is upper semi-continuous, and that of a non-diminishing monotone sequence of continuous functions is a lower semi-continuous function.

104. If the domain E be perfect, the following theorem may be established:

If a monotone non-increasing sequence of upper semi-continuous functions converges in the perfect domain E (of any number of dimensions) to a continuous function, the convergence is uniform in E.

The corresponding result holds for a non-diminishing sequence of lower semi-continuous functions, if the limiting function be continuous.

Let $s(x)$ be continuous, and suppose the sequence to consist of upper semi-continuous functions, and to be non-increasing. If ξ be any point of E, an integer n can be so chosen that $s_n(\xi) - s(\xi) < \frac{1}{3}\epsilon$; and a neighbourhood D_1 of ξ can be so determined, that $s_n(x) < s_n(\xi) + \frac{1}{3}\epsilon$, for all points of E that are in D_1, where ϵ is an arbitrarily chosen positive number. Again, a neighbourhood D_2, of ξ, can be so determined that $s(x) > s(\xi) - \frac{1}{3}\epsilon$, for all points of E in D_2, since $s(x)$ is continuous at ξ. If a neighbourhood D, of ξ, be chosen that is interior both to D_1 and D_2; we have $s_n(x) < s_n(\xi) + \frac{1}{3}\epsilon$, and $s(x) > s(\xi) - \frac{1}{3}\epsilon$, for all points x, of E, that are in D_2; hence $s_n(x) - s(x) < s_n(\xi) - s(\xi) + \frac{2}{3}\epsilon < \epsilon$, for all points of E that are in D: and this must hold for all greater values of n. Thus the point ξ is a point of uniform convergence of the sequence $\{s_n(x)\}$, and since it is an arbitrary point of E, the sequence is uniformly convergent in E.

105. The converse of the theorem that a monotone sequence of continuous functions has for its limiting function a semi-continuous function may be stated as follows:

Every function defined for a set E, of one or more dimensions, which is lower semi-continuous in E, is the limit of a non-diminishing sequence of continuous functions. If the function is upper semi-continuous, it is the limit of a monotone non-increasing sequence of continuous functions.

This theorem was first established by Baire*. Proofs have also been given by W. H. Young†, Tietze‡, and Hahn§ and by Carathéodory||. The proof here given is essentially that which has been given by Hausdorff¶.

It is sufficient to prove the first part of the theorem, as the second part is then immediately deducible.

It will in the first instance be assumed that the lower semi-continuous function $s(x)$ is bounded in E. At a point x, of E, let $s_n(x)$ be defined as the lower boundary of the function $s(x') + nD(x, x')$, with respect to x', for all points in E, where $D(x, x')$ denotes the distance between the points x and x'. It will be shewn that the function $s_n(x)$ is continuous in E. Let x_1 be a point so chosen that $D(x, x_1) < h$.

We have then $s_n(x_1) \leqq s(x') + nD(x_1, x')$, for every point x', in E; also $D(x_1, x') \leqq D(x, x') + D(x, x_1) < h + D(x, x')$. Thus

$$s_n(x_1) < nh + \{s(x') + nD(x, x')\};$$

and since x' may be so chosen that $s(x') + nD(x, x') < s_n(x) + h$, we have $s_n(x_1) < s_n(x) + (n+1)h$. Since it may be shewn in the same way that $s_n(x) < s_n(x_1) + (n+1)h$, we see that $|s_n(x_1) - s_n(x)| < (n+1)h$, provided $D(x, x_1) < h$. Since h is arbitrary, the continuity of $s_n(x)$ at the point x has been proved.

It is clear from the definition of $s_n(x)$ that $s_n(x) \leqq s_{n+1}(x)$, and that $s_n(x)$ cannot be less than the lower boundary of $s(x)$ in E. Thus $\{s_n(x)\}$ forms a non-diminishing monotone sequence, and it has a limiting function $\psi(x)$. Also, since $s_n(x) \leqq s(x') + nD(x, x')$, we have $s_n(x) \leqq s(x)$.

If a point x_n satisfies the condition $s(x_n) + nD(x, x_n) < s_n(x) + \frac{1}{n}$, we have $D(x, x_n) < \frac{1}{n}\left(\frac{1}{n} + s(x) - l\right)$ where l is the lower boundary of $s(x)$ in E. It follows that $D(x, x_n)$ converges to zero as $n \sim \infty$, or the sequence $\{x_n\}$ converges to x. Since $s(x)$ is lower semi-continuous, we

* *Bull. Soc. Math. de France*, vol. XXXII (1904), p. 125.
† *Proc. Camb. Phil. Soc.* vol. XIV (1908), p. 523.
‡ *Crelle's Journal*, vol. CXLV (1914), p. 9.
§ *Wien. Sitzungsber.* vol. CXXVI (1917), p. 100.
|| *Vorlesungen über reelle Funktionen*, p. 401.
¶ *Math. Zeitschr.* vol. V (1919), p. 293.

have $s(x) \leqq \varliminf_{n\sim\infty} s(x_n) \leqq \lim_{n\sim\infty}\left\{s_n(x)+\frac{1}{n}\right\} \leqq \psi(x)$; but since $s_n(x) \leqq s(x)$, we have $\psi(x) \leqq s(x)$; and from the two inequalities we have $s(x) = \psi(x)$, and thus $\{s_n(x)\}$ converges to $s(x)$. If x is an isolated point of E, x_n will coincide with x, from and after some value of n, and $s(x) < s_n(x) + \frac{1}{n}$, and then, as before, $s(x) \leqq s_n(x)$.

The above proof is applicable even if $s(x)$ have no upper boundary, provided it have a finite lower boundary. In order to remove this restriction we employ the transformation $\sigma(x) = \frac{s(x)}{1+|s(x)|}$; then $\sigma(x)$ is in the interval $(-1, 1)$, and is lower semi-continuous provided $s(x)$ be so. Applying the result already obtained, $\sigma(x)$ is the limit of a sequence of continuous functions $\sigma_n(x)$, where $|\sigma_n(x)| \leqq 1$ in the set E. The functions $s_n(x) = \frac{\sigma_n(x)}{1-|\sigma_n(x)|}$, which are all continuous, at least in the extended sense, converge as $n \sim \infty$, to $s(x)$.

If the function $s(x)$ be finite, although unbounded, it is possible to determine $s_n(x)$ so that it is finite for each value of n, and consequently continuous in the ordinary sense; whereas when $\sigma_n(x)$ has one of the values $1, -1$, $s_n(x)$ has the value ∞ or $-\infty$, and is therefore continuous only in the extended sense.

If any of the functions $\sigma_n(x)$ have the value 1 or -1, so that $s_n(x)$ has the value ∞, or $-\infty$, we can modify the transformation so as to ensure that $s_n(x)$ shall be finite for each value of n. Let $\{e_n\}$ be a sequence of increasing positive numbers which converge to 1. Instead of the sequence $\{\sigma_n(x)\}$ we may employ the sequence $\{e_n\sigma_n(x)\}$ which converges to $\sigma(x)$, and is monotone increasing; then $s_n(x)$, being defined by $\frac{e_n\sigma_n(x)}{1-e_n|\sigma_n(x)|}$, lies between finite boundaries $\frac{\pm e_n}{1-e_n}$, and has the required property.

106. In case the set E consists of a closed linear interval (a, b), a simple proof of the above theorem can be given which includes the fact that the continuous functions can be so chosen as to be polygonal, and thus possess derivatives on the right and on the left which are continuous except for a finite set of values of the variable. Thus:

If a function $f(x)$ be upper semi-continuous in the interval (a, b), and have a finite upper boundary, it is the limit of a monotone non-increasing sequence of continuous polygonal functions.

Also if the function be lower semi-continuous, and have a finite lower boundary, it is the limit of a monotone non-diminishing sequence of continuous polygonal functions.

It will be sufficient to establish the first of these theorems. Let a system of symmetrical nets with closed meshes be fitted on to (a, b); we may suppose the breadth of each mesh of the net of order n to be $(b-a)/2^n$. Let the values of $\phi_n(x)$ at the ends of the meshes of the nth net be defined thus: Let $\phi_n(a)$ be the upper boundary of $f(x)$ in the mesh with a at its end; let the value of $\phi_n(x)$ at the point $a + \frac{r(b-a)}{2^n}$ be the upper boundary of $f(x)$ in the interval $\left(a + \frac{(r-1)(b-a)}{2^n}, a + \frac{(r+1)(b-a)}{2^n}\right)$; and let the value of $\phi_n(x)$ at the point b be the upper boundary of the function $f(x)$ in the mesh $\left(b - \frac{b-a}{2^n}, b\right)$. Let the continuous polygonal function $\phi_n(x)$ be defined by its values at the end-points of the meshes, as thus specified. It is clear that $\phi_n(x) \geqq \phi_{n+1}(x)$; so that $\{\phi_n(x)\}$ is a non-increasing sequence of continuous functions.

If ξ be any point in (a, b), a neighbourhood of ξ can be so determined that $f(x) < f(\xi) + \epsilon$, for every point x, of that neighbourhood. For a sufficiently large value of n, the interval

$$\left(a + \frac{(r-2)(b-a)}{2^n}, a + \frac{(r+2)(b-a)}{2^n}\right)$$

is contained in this neighbourhood of ξ; where ξ is contained in

$$\left(a + \frac{(r-1)(b-a)}{2^n}, a + \frac{(r+1)(b-a)}{2^n}\right).$$

The values of $\phi_n(x)$ at the points

$$a + \frac{(r-1)(b-a)}{2^n},\quad a + \frac{r(b-a)}{2^n},\quad a + \frac{(r+1)(b-a)}{2^n},$$

are between $f(\xi)$ and $f(\xi) + \epsilon$; hence also $\phi_n(\xi)$ lies between $f(\xi)$ and $f(\xi) + \epsilon$. Thus the function $\phi_n(\xi)$ differs from $f(\xi)$ by less than ϵ. It now follows, by considering a sequence of values of ϵ, converging to zero, that

$$\lim_{n\sim\infty} \phi_n(\xi) = f(\xi);$$

and thus the theorem is established.

107. The following theorem was established by Hahn*:

If $s^{(l)}(x)$, $s^{(u)}(x)$ be functions, defined for a domain E, of any number of dimensions, $s^{(l)}(x)$ being lower semi-continuous, and $s^{(u)}(x)$ upper semi-continuous, and such that $s^{(l)}(x) \geqq s^{(u)}(x)$, there exists a continuous function $s(x)$, such that $s^{(l)}(x) \geqq s(x) \geqq s^{(u)}(x)$.

Let $\chi(t) = t$, when $t > 0$, and $\chi(t) = 0$, when $t \leqq 0$; this function $\chi(t)$ is a continuous, monotone non-diminishing function of the real variable t, as t increases in the indefinite interval $(-\infty, \infty)$.

* *Wien. Sitzungsber.* vol. CXXVI (II *a*) (1917), p. 103. The proof in the text was given by Hausdorff, *Math. Zeitschr.* vol. V (1919), p. 295.

The functions $s^{(l)}(x)$, $s^{(u)}(x)$ are, in accordance with the theorem of § 105, the limits of sequences $\{s_n^{(l)}(x)\}$, $\{s_n^{(u)}(x)\}$ of continuous functions; the first sequence being monotone non-diminishing, and the second monotone non-increasing.

Since $s_n^{(u)}(x) - s_n^{(l)}(x) \geqq s_n^{(u)}(x) - s_{n+1}^{(l)}(x) \geqq s_{n+1}^{(u)}(x) - s_{n+1}^{(l)}(x)$, for $n = 1, 2, 3, \ldots$, we have

$$\chi\{s_n^{(u)}(x) - s_n^{(l)}(x)\} \geqq \chi\{s_n^{(u)}(x) - s_{n+1}^{(l)}(x)\} \geqq \chi\{s_{n+1}^{(u)}(x) - s_{n+1}^{(l)}(x)\}.$$

The series

$$\begin{aligned} s_1^{(l)}(x) &+ \chi\{s_1^{(u)}(x) - s_1^{(l)}(x)\} - \chi\{s_1^{(u)}(x) - s_2^{(l)}(x)\} \\ &+ \chi\{s_2^{(u)}(x) - s_2^{(l)}(x)\} - \chi\{s_2^{(u)}(x) - s_3^{(l)}(x)\} \\ &+ \ldots \end{aligned}$$

of which the terms are continuous in E, and of alternate signs, after the first term, is such that each term, after the first, is numerically not less than the next, for each fixed value of x; and it is seen that the general term converges to zero. The series is accordingly convergent at every point x, of E; the partial sums of even order form a monotone non-diminishing sequence which accordingly converges to a lower semi-continuous function, and the partial sums of odd order form a monotone non-increasing sequence which must converge to an upper semi-continuous function. The sum-function of the series, being both upper and lower semi-continuous, is a continuous function, $s(x)$. It will be shewn that $s(x)$ satisfies the conditions of the theorem.

At a point at which $s^{(l)}(x) = s^{(u)}(x)$, we have

$$s_n^{(u)}(x) - s_n^{(l)}(x) \geqq 0, \quad s_n^{(u)}(x) - s_{n+1}^{(l)}(x) \geqq 0;$$

and thus $s(x) = s_1^{(l)}(x) + \{s_1^{(u)}(x) - s_1^{(l)}(x)\} - \{s_1^{(u)}(x) - s_2^{(l)}(x)\} + \ldots$,

whence $\quad s(x) = \lim\limits_{n\sim\infty} s_n^{(l)}(x) = \lim\limits_{n\sim\infty} s_n^{(u)}(x) = s^{(l)}(x) = s^{(u)}(x).$

At a point x, at which $s^{(l)}(x) > s^{(u)}(x)$, let the first term with negative argument, of the series which defines $s(x)$, be $\chi\{s_m^{(u)}(x) - s_m^{(l)}(x)\}$; then $s(x)$ is given by the finite series

$$s_1^{(l)}(x) + \{s_1^{(u)}(x) - s_1^{(l)}(x)\} - \{s_1^{(u)}(x) - s_2^{(l)}(x)\} + \ldots - \{s_{m-1}^{(u)}(x) - s_m^{(l)}(x)\},$$

or $s(x) = s_m^{(l)}(x)$; then $s^{(l)}(x) \geqq s_m^{(l)}(x) \geqq s_m^{(u)}(x) \geqq s^{(u)}(x)$, and thus

$$s^{(l)}(x) \geqq s(x) \geqq s^{(u)}(x).$$

Similarly, if the first term with a negative argument in the series which defines $s(x)$ be $\chi\{s_m^{(u)}(x) - s_{m+1}^{(u)}(x)\}$, it can be seen that $s(x) = s_m^{(u)}(x)$, and thence, as before, that $s^{(l)}(x) \geqq s(x) \geqq s^{(u)}(x)$.

Thus the theorem has been established.

THE EXTENSION OF FUNCTIONS

108. From the last theorem, the following theorem, due to Tietze, may be deduced:

If H be a closed set contained in the set E, and $s(x)$ a function defined in H, and continuous relative to H, then a function exists in E which is continuous in E, and has the value $s(x)$ at every point of H. In particular, a function exists which is continuous in the whole linear, or p-dimensional space, and has at all points of the closed set H, the values of an assigned function $s(x)$, continuous relative to H.

Let U and L denote the upper and lower boundaries of $s(x)$ in H; U and L will at first be assumed to be finite. The function $s^{(l)}(x)$ defined in E by the conditions $s^{(l)}(x) = s(x)$, in H, and $s^{(l)}(x) = U$, in $E - H$, is lower semi-continuous in E; also the function $s^{(u)}(x)$ defined by the conditions $s^{(u)}(x) = s(x)$, in H, and $s^{(u)}(x) = L$, in $E - H$, is upper semi-continuous in E. Since $s^{(l)}(x) \geqq s^{(u)}(x)$, in E, a function $f(x)$, continuous in E, exists, such that $s^{(l)}(x) \geqq f(x) \geqq s^{(u)}(x)$. This function $f(x)$ has the value of $s(x)$ at every point of H; and is the function which has the required properties.

In case U and L are not both finite, so that $s(x)$ is continuous, in H, only in the extended sense, we employ the transformation $\sigma(x) = \dfrac{s(x)}{1 + |s(x)|}$. Then $\sigma(x)$ is bounded and continuous in H; if $F(x)$ be the function which is continuous in E and $= \sigma(x)$ in H; the required function $f(x)$ may be defined to be $\dfrac{F(x)}{1 - |F(x)|}$.

109. A direct method of constructing a function $f(x)$ which satisfies the conditions of the last theorem has been given by Hausdorff (*loc. cit.*). The method may be applied to the construction of a function which satisfies a less restricted condition as regards its values in the closed set H. The following general theorem* is relevant to the theory of Jordan curves:

If E be any set of points in one or more dimensions, and H be a closed set, contained in E; then, if $s(x)$ be any function defined in H, a function $f(x)$ can be defined in E which has the value $s(x)$ at each point of H, and is continuous at all points of $E - H$, and also at each point of H at which $s(x)$ is continuous relatively to H.

It will in the first instance be assumed that $s(x)$ is bounded in H, so that $U \geqq s(x) \geqq L$, in H. Let $f(x) = s(x)$, in H, and let $f(x)$ at each point x, of $E - H$, have the value of the lower boundary of

$$s(x') + \frac{D(x, x')}{d(x)} - 1,$$

* See Pal, *Crelle's Journal*, vol. CXLIII (1913), p. 294; also Brouwer, *Math. Annalen*, vol. LXXIX (1918), p. 209.

for all points x', of H; where $D(x, x')$ denotes the distance between the points x, x', and $d(x)$ is the lower boundary of $D(x, x')$ for all points x', of H. Consider a point x, of the set $E - H$, which is open relative to E; it will be shewn that $f(x)$ is continuous at x. We need only consider points x', of H, for which

$$s(x') + \frac{D(x, x')}{d(x)} - 1 < f(x) + 1, \text{ or } \frac{D(x, x')}{d(x)} < U - L + 2,$$

which may be written $D(x, x') < kd(x)$. Let ξ be a point of $E - H$ in a neighbourhood of x for which $D(x, \xi)$ is less than an arbitrarily chosen positive number h, and which contains no points of H. The point x', of H, can be so chosen that

$$s(x') + \frac{D(x, x')}{d(x)} - 1 < f(x) + h,$$

and that $D(x, x') < kd(x)$.

We have

$$\begin{aligned} f(\xi) &\leq s(x') + \frac{D(\xi, x')}{d(\xi)} - 1 \\ &< f(x) + h + \frac{D(\xi, x')}{d(\xi)} - \frac{D(x, x')}{d(x)} \\ &< f(x) + h + \frac{D(\xi, x') - D(x, x')}{d(\xi)} + kd(x)\left\{\frac{1}{d(\xi)} - \frac{1}{d(x)}\right\} \\ &< f(x) + h + \frac{h}{d(\xi)} + k\frac{d(x) - d(\xi)}{d(\xi)} \\ &< f(x) + h\left(1 + \frac{k+1}{d(\xi)}\right) < f(x) + h\left(1 + \frac{k+1}{\alpha}\right), \end{aligned}$$

where α is a positive number dependent on the neighbourhood of x in which ξ is taken. This holds for all points ξ in that neighbourhood. It may similarly be shewn that $f(x) < f(\xi) + h\left(1 + \frac{k+1}{d(x)}\right)$. Since h is arbitrary, it follows from the two inequalities that $f(x)$ is continuous at the point x, of $E - H$.

Next, let x be a point of H which is on the boundary of H relative to $E - H$; consider a point ξ, of $E - H$, such that $D(x, \xi) < h < 1$. Let x' be a point of H such that $\frac{D(\xi, x')}{d(\xi)} < 1 + h$, then

$$f(\xi) \leq s(x') + \frac{D(\xi, x')}{d(\xi)} - 1 < s(x') + h.$$

Since $d(\xi) \leq D(x, \xi) < h$, we have

$$D(x, x') \leq D(x, \xi) + D(\xi, x') < h + (1 + h)h < 3h.$$

If now x is a point of continuity of $s(x)$ relatively to H, we have

$$s(x') < s(x) + \eta,$$

where η and h converge to zero together; therefore $f(\xi) < s(x) + \eta + h$. As before, in the definition of $f(\xi)$ we need only consider points x'', of H, for which $D(\xi, x'') < kd(\xi) < kh$, or for which

$$D(x, x'') < D(x, \xi) + D(\xi, x'') < (1 + k)h,$$

and for all such points $s(x'') > s(x) - \eta'$, where η' converges to zero with h. Then $f(\xi)$ is the lower limit of $s(x'') + \dfrac{D(\xi, x'')}{d(\xi)} - 1$ which is $\geqq$ the lower limit of $s(x'')$, and this is $> s(x) - \eta'$. Since $f(\xi)$ lies between $s(x) - \eta'$ and $s(x) + \eta + h$, where η, η', h converge together to zero, it follows that $f(x)$ is continuous at the point x, of H, at which $s(x)$ is continuous relatively to H.

If x be a point of H that is not a limiting point of $E - H$, $f(x)$ is continuous at x if $s(x)$ is continuous at x relatively to H. The theorem has now been established.

The theorem can be extended to the case in which $s(x)$ is unbounded by employing the transformation $\sigma(x) = \dfrac{s(x)}{1 + |s(x)|}$, and applying the theorem to the bounded function $\sigma(x)$; continuity will then be understood in the extended sense.

110. The method employed in § 109 can be used to establish the following theorem due to Tietze (*loc. cit.*):

If $s(x)$ be defined in a set E, and is continuous with respect to E at every point of the closed set H contained in E, a function $f(x)$, continuous in E, can be defined such that, in E, $f(x) \leqq s(x)$, and in H, $f(x) = s(x)$.

When $s(x)$ is bounded, the required function can be defined at a point x, of $E - H$, as the lower boundary of $s(x') + \dfrac{D(x, x')}{d(x)}$, for all points x', of E, and $f(x) = s(x)$, at every point x, of H. That this function is continuous can be shewn as in § 109. The extension to the case in which $s(x)$ is unbounded can be made as before.

The following theorem is also due to Tietze:

Every function $s(x)$ defined in the set E, and lower semi-continuous in E, but continuous with respect to E at every point of the closed set H, contained in E, is the limit of a sequence of continuous functions $\{s_n(x)\}$ such that $s_n(x) = s(x)$, in H.

In the first instance $s(x)$ may be taken to be bounded in E, and the result may afterwards be extended. In accordance with the last theorem, there exists a continuous function $f(x) \leqq s(x)$, such that $f(x) = s(x)$

in H. By the theorem of § 105, $s(x)$ is the limit of a non-diminishing monotone sequence $\{f_n(x)\}$ of continuous functions. Let $s_n(x)$ at each point, denote the greater of the two numbers $f_n(x)$, $f(x)$; the sequence $\{s_n(x)\}$ is then monotone non-diminishing, and its limit at each point must be the greater of the two numbers $s(x)$, $f(x)$; that is $s(x)$. At a point x, of H, we have $s_n(x) = s(x)$.

CLASSES OF MONOTONE SEQUENCES

111. With a view to application to the theory of integration the properties of functions formed by taking a succession of monotone sequences, the first of which sequences is a monotone sequence of continuous functions, have been investigated by W. H. Young*.

In a domain E, of one or more dimensions, which may be taken to be dense in itself, a function which is upper semi-continuous in E is termed a u-function; as has been shewn in § 105, a u-function is always representable as the limit of a monotone non-increasing sequence of continuous functions. Similarly a lower semi-continuous function is termed an l-function, and is the limit of a non-diminishing sequence of continuous functions.

It is easily seen that the sum of two u-functions is a u-function, and that the sum of two l-functions is an l-function. A similar statement may be made for the product of two u-functions, or of two l-functions, provided both functions are $\geqq 0$.

It is also easily seen that, if we have two semi-continuous functions of the same type, the function which has at each point the value of the greater of the two functions is also semi-continuous, and of the same type. Also the function which has at each point the value of the lesser of the two functions is semi-continuous, and of the same type. It is clear that this statement may be extended to apply to any finite set of semi-continuous functions, all of the same type.

A monotone non-diminishing sequence of upper semi-continuous, or u-functions, converges to a function which may be termed a lower-upper semi-continuous function, or shortly an lu-function. Similarly, a monotone non-increasing sequence of lower semi-continuous functions converges to a function which may be termed an upper-lower semi-continuous function, or ul-function. Thus an *ascending* sequence is spoken of as *lower*, and a *descending* sequence as *upper*, in consonance with the fact that an ascending sequence of continuous functions converges to a lower semi-continuous function, and that a descending sequence of continuous functions converges to an upper semi-continuous function.

* *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 15.

In accordance with § 103, an upper-upper semi-continuous function, or *uu*-function, is a *u*-function; and similarly an *ll*-function is an *l*-function.

The *ul*-functions and the *lu*-functions are both functions of new types, but each of them includes both *u*-functions and *l*-functions as sub-classes.

By considering monotone sequences of *ul*-functions and of *lu*-functions, we appear to obtain functions of the four types *uul, lul, ulu, llu.*

It can however be shewn that only the *lul*-functions, and the *ulu*-functions are of new types; in fact it may be shewn that *a uul-function is a ul-function, and that an llu-function is an lu-function.*

Let $s(x) = \lim_{n\sim\infty} s_n(x)$, where $s_n(x) \leq s_{n+1}(x)$, for all values of n, and where $s_n(x) = \lim_{m\sim\infty} s_{nm}(x)$, where $s_{n,m}(x) \leq s_{n,m+1}(x)$, for all values of n and m, and the functions $s_{nm}(x)$ are all u-functions, so that $s_n(x)$ is an *lu*-function, and $s(x)$ is an *llu*-function.

Let $\sigma_n(x)$ denote the function which has at each point the value of the greatest of the functions $s_{1n}(x)$, $s_{2n}(x)$, ... $s_{nn}(x)$, then $\sigma_n(x)$ is a u-function.

Since $s_{rn}(x) \leq s_{r,n+1}(x)$, for $r = 1, 2, 3, \ldots$, it is clear that $\sigma_n(x) \leq \sigma_{n+1}(x)$. Thus the limit of the monotone sequence $\{\sigma_n(x)\}$ is an *lu*-function, and it will be shewn that this limit $\sigma(x) = s(x)$. Since $s_{nm}(x) \leq s_n(x) \leq s(x)$, for all values of n and m, it follows that $\sigma_n(x) \leq s(x)$, and therefore $\sigma(x) \leq s(x)$. A value n_1, of n, can be so determined that $s_{n_1}(x) > s(x) - \epsilon$, and a value m_1, of m, can be so determined that

$$s_{n_1 m_1}(x) > s_{n_1}(x) - \epsilon > s(x) - 2\epsilon.$$

If $m_1 \leq n_1$, $s_{n_1 m_1}(x) \leq \sigma_{n_1}(x)$, hence $\sigma_{n_1}(x) \geqq s_{n_1 m_1}(x) > s(x) - 2\epsilon$. If $m_1 > n_1$, $\sigma_{m_1}(x) \geqq s_{n_1 m_1}(x) > s(x) - 2\epsilon$. In either case an index n can be so determined that $\sigma_n(x) > s(x) - 2\epsilon$; since ϵ is arbitrary, it follows that $\sigma(x) \geqq s(x)$. It now follows that $\sigma(x) = s(x)$, and thus that $s(x)$ is an *lu*-function. Similarly it may be shewn that a *uul*-function is a *ul*-function.

It is easily seen that the sum of two functions of the same type *lu*, *ul*, *lul*, or *ulu* is a function of the same type, and that the product of two functions, both of which are $\geqq 0$, and of the same type, is also of that type.

It can also be shewn that the function which has at each point the value of the greater (or of the lesser) of two functions of the same type, is also of that type.

It is clear that, proceeding from the *lul*-functions and from the *ulu*-functions, new classes of functions may be obtained, but these are not of importance in the theory of integration, although they are of interest in connection with the classification of functions.

112. Let $\{s_n(x)\}$ be any sequence of lower semi-continuous functions, not necessarily convergent, defined in a set E. Let $V_n(x)$ denote the function which has as its value at each point x the greatest of the numbers $s_1(x), s_2(x), \dots s_n(x)$.

The sequence $\{V_n(x)\}$ is a monotone non-diminishing sequence of lower semi-continuous functions, and it must converge to an l-function $W_1(x)$. At each point x, $W_1(x)$ has the value of the upper boundary of all the numbers $s_1(x), s_2(x), \dots$. The function $W_m(x)$ is formed in a similar manner from the functions $s_m(x)$, (x), The sequence $\{W_m(x)\}$ is a monotone non-increasing sequence of l-functions, and it converges to the upper function $\bar{s}(x)$, which is therefore a *ul*-function. If the functions $s_n(x)$ were all u-functions, the functions $V_n(x)$ would all be u-functions, and the function $W_n(x)$ would be an *lu*-function; it then follows that $\bar{s}(x)$ would be a *ulu*-function.

In a similar manner, a non-diminishing monotone sequence $\{w_n(x)\}$ can be formed, which converges to $\underline{s}(x)$; and it can be seen that, when the functions $s_n(x)$ are all u-functions, the function $\underline{s}(x)$ which is the limit of the non-diminishing monotone sequence $\{w_n(x)\}$, is an *lu*-function; and when the functions $\{s_n(x)\}$ are l-functions, $\underline{s}(x)$ is an *lul*-function.

The monotone descending sequence $\{W_n(x)\}$, which converges to $\bar{s}(x)$, and the monotone ascending sequence $\{w_n(x)\}$ which converges to $\underline{s}(x)$ may be termed the *monotone sequences associated with any sequence* $\{s_n(x)\}$, whether the functions $s_n(x)$ are semi-continuous or not.

Since a continuous function is both an l-function and a u-function, it follows that:

The upper function $\bar{s}(x)$, of a sequence $\{s_n(x)\}$ of continuous functions, is a ul-function, and the lower function $\underline{s}(x)$ is an lu-function. If the sequence $\{s_n(x)\}$ of continuous functions is convergent (even in the extended sense which includes divergence), the limiting function $s(x)$ is both a ul-function and an lu-function.

113. *If a sequence $\{s_n(x)\}$ of ul-functions converges uniformly to a finite limiting function $s(x)$, the function $s(x)$ is also a ul-function. If a sequence of lu-functions converges uniformly to $s(x)$, then $s(x)$ is also an lu-function.*

It will be sufficient to prove the first part of the theorem. It will be shewn that $s(x)$ is the limit of a monotone non-increasing sequence of *ul*-functions, and is therefore a *uul*-function, that is a *ul*-function.

If $\{\epsilon_p\}$ be a diminishing sequence of positive numbers which converges to zero, integers $\{n_p\}$ can be so determined that $|s(x) - s_{n_p}(x)| < \epsilon_p$ for all values of p, and for all points x, in E. We have now

$$s_{n_p}(x) + 2\epsilon_p > s(x) + \epsilon_p; \quad s_{n_{p+1}}(x) + 2\epsilon_{p+1} < s(x) + 3\epsilon_{p+1};$$

if now $\epsilon_p > 3\epsilon_{p+1}$, we have $s_{n_p}(x) + 2\epsilon_p > s(x) + 3\epsilon_{p+1}$. Choosing the sequence $\{\epsilon_p\}$ so that this condition is satisfied, the monotone decreasing sequence $\{s_{n_p}(x) + 2\epsilon_p\}$ has the limit $s(x)$, and $s_{n_p}(x) + 2\epsilon_p$ is a *ul*-function. Therefore $s(x)$ is also a *ul*-function.

This theorem is a particular case* of the general theorem that:

The limiting function of a uniformly convergent sequence of functions all of the same type, is also of that type.

UNIFORM OSCILLATION OF A SEQUENCE OF FUNCTIONS

114. The theory of uniform convergence may be generalized so as to apply to a sequence $\{s_n(x)\}$ of functions defined in a set E, of any number of dimensions, the sequence being in general, at least, non-convergent. If p be the number of dimensions of the domain E, we may regard $s_n(x)$ as a single valued function in the $(p+1)$-dimensional domain $\mathcal{E}$, which is constituted by x in E, and n in the integer sequence 1, 2, 3, ... If the transformation $n = 1/y$ be employed, $s_n(x)$ becomes $s(x, y)$, which is defined for the domain $\mathcal{D}$, defined by

$$\{x \text{ in } E,\ y \text{ in the sequence } (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots)\}.$$

Denoting by $s(\xi)$ the multiple-valued function defined, as in § 61, as having the values of the limiting points of the linear sequence $\{s_n(\xi)\}$, the values of $s(\xi)$ consist of the values of $\lim\limits_{n\sim\infty} s_n(\xi)$, or of $\lim\limits_{y\sim 0} s(\xi, y)$, and they form a closed linear set, of which $\bar{s}(\xi)$, $\underline{s}(\xi)$ are the upper and lower boundaries, either of which may be either finite or infinite. We may suppose that $s(\xi, 0)$ has for its values all the values of $s(\xi)$.

Let us consider the associated functions, defined, as in § 64, with reference to the function $s(x, y)$ at the point $(\xi, 0)$, or of $s_n(x)$ at the point (ξ, ∞).

These functions $A[\{s_n\}, x]$, $a[\{s_n\}, x]$, associated with the sequence $\{s_n(x)\}$, will be defined, at each point ξ, as the upper and lower double limits of $s(x, y)$, or of $s_n(x)$ at the point $(\xi, 0)$ of $\mathcal{D}$, or the point (ξ, ∞) of $\mathcal{E}$.

Stated more explicitly, the numbers $A[\{s_n\}, \xi]$, $a[\{s_n\}, \xi]$ are defined as follows:

If Δ be a neighbourhood of the point ξ, and n_1 a value of n, and the upper boundary of $s_n(x)$, for all points x, of E, in Δ, except the point ξ, for all values of n that are $\geqq n_1$ be considered, the lower limit of this upper boundary, as Δ converges to ξ, and n_1 diverges to ∞, defines the value of $A[\{s_n\}, \xi]$.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 357, where a different proof is given. In Hahn's *Theorie der reellen Funktionen*, vol. I, p. 334, a proof similar to that above is given.

If the lower boundary of $s_n(x)$ in the neighbourhood Δ, with the same restriction, for all values of $n \geqq n_1$ be considered, the upper limit of this lower boundary, as Δ converges to ξ, and $n_1 \sim \infty$, defines the value of $a\,[\{s_n\}, \xi]$.

In case, in the above definition, the point ξ itself were not excluded, functions $M\,[\{s_n\}, x]$, $m\,[\{s_n\}, x]$ would be defined, which, at any point ξ, would have the values of the maximal and minimal functions associated with the sequence $\{s_n(x)\}$.

The two functions $A\,[\{s_n\}, x]$, $a\,[\{s_n\}, x]$ were defined by W. H. Young, who gave to them the names peak function and chasm function respectively. It is accordingly frequently convenient to denote them by $p(x)$ and $c(x)$.

It is clear that, at ξ, the value of the maximal function for the sequence is the greater of the numbers $p(\xi)$, $\bar{s}(\xi)$, and that of the minimal function is the lesser of $c(\xi)$ and $\underline{s}(\xi)$.

The function $p(x)$ is upper semi-continuous, and $c(x)$ is lower semi-continuous. A similar statement applies to the maximal or minimal functions.

That this is the case can be established as in § 65. If we denote by $\phi(x)$ the function $A\,\{\bar{s}(x), x\}$ associated with the function $\bar{s}(x)$, and by $\psi(x)$ the function $a\,\{\underline{s}(x), x\}$ associated with the function $\underline{s}(x)$, it is easily seen that $p(\xi) \geqq \phi(\xi) \geqq \psi(\xi) \geqq c(\xi)$. For, at each point x of the neighbourhood Δ, of ξ, the upper boundary of $s_n(x)$ for all values of n that are $\geqq n_1$ is $\geqq \bar{s}(x)$, and the lower boundary of $s_n(x)$ is $\leqq \underline{s}(x)$.

115. More than one mode of generalizing the conception of uniform convergence at a point, or in a set E, so as to apply to a sequence which oscillates, is possible.

In accordance with the definition of uniform convergence of a sequence $\{s_n(x)\}$ at the point ξ, at which $\bar{s}(\xi) = \underline{s}(\xi)$, given in § 70, having given a positive number ϵ, a neighbourhood Δ and an integer n_ϵ exist, such that, at every point x, of E, in Δ, the two inequalities

$$s_n(x) < \bar{s}(x) + \epsilon, \qquad s_n(x) > \underline{s}(x) - \epsilon$$

are satisfied provided $n \geqq n_\epsilon$. In case the first condition is satisfied for every value of ϵ, but not necessarily the second, the convergence of the sequence at ξ may be said to be *uniform above*, it being assumed that the sequence is convergent at ξ. If the second condition is satisfied, but not necessarily the first, the convergence may be said to be *uniform below* Uniform convergence at ξ, where $s(\xi)$ exists, occurs when the convergence is uniform both above and below.

If the sequence be oscillatory at ξ, it is said to be *uniformly oscillatory above*, in case, for each ϵ, a neighbourhood Δ exists, and an integer n_ϵ, such that $s_n(x) < \bar{s}(x) + \epsilon$, when $n \geqq n_\epsilon$, for all points of E, in Δ. Similarly if the condition $s_n(x) > \underline{s}(x) - \epsilon$ is satisfied, for $n \geqq n_\epsilon$, the sequence is

said to be *uniformly oscillatory below.* If both conditions are satisfied the sequence is said to be uniformly oscillatory at ξ.

The sequence $\{s_n(x)\}$ *is said to oscillate uniformly above in the set* E, *if,* ϵ *being an arbitrarily chosen positive number, an integer* n_ϵ *exists such that* $s_n(x) < \bar{s}(x) + \epsilon$, *at all points of* E, *provided* $n \geqq n_\epsilon$. *If* $s_n(x) > \underline{s}(x) - \epsilon$, *at all points of* E, *the sequence is said to oscillate uniformly below in* E. *When both conditions are satisfied the sequence is said to oscillate uniformly in* E.

This definition was given by W. H. Young, who developed the theory of uniform oscillation. In view of a different definition to be considered below*, this mode of oscillation was termed by him uniform oscillation of the second kind. It will here be spoken of simply as uniform oscillation, as it appears to be the simplest and most direct generalization of uniform convergence as defined in § 66. In case the sequence is convergent, the uniform oscillation becomes uniform convergence.

It can be shewn, as in § 70, that, if the set E be closed, and the sequence be not uniformly oscillatory in E above (below), there must be at least one point ξ, of E, at which the sequence is not uniformly oscillatory above (below).

For, at each point of E, we have $s_n(x) < \bar{s}(x) + \epsilon$, provided n is $\geqq$ some integer dependent on x; let $\psi_1(\epsilon, x)$ denote this integer for the point x. If $\psi_1(\epsilon, x)$ is bounded in E, for each value of ϵ, it is clear that the function is uniformly oscillatory above at every point of E. If, however, there exists a value of ϵ, such that $\psi_1(\epsilon, x)$ is unbounded in E, there exists at least one point ξ, of E, in the arbitrarily small neighbourhood of which $\psi_1(\epsilon, x)$ is unbounded. At this point ξ, there exists no neighbourhood Δ, such that $s_n(x) < \bar{s}(x) + \epsilon$ in it, for all values of n greater than some fixed integer; thus the oscillation above, at ξ, is not uniform.

It has been assumed that $\bar{s}(x)$, $\underline{s}(x)$ are both finite at each point of E. If, in a part E_1, of E, we have $\bar{s}(x) = +\infty$, whilst $\underline{s}(x)$ is finite, the sequence is regarded as uniformly oscillatory in E_1, in an extended sense, if it is uniformly oscillatory below in E_1. When this condition is satisfied, employing the transformation in § 62, we see that $\{\sigma_n(x)\}$ is uniformly oscillatory in E_1.

A similar definition applies to the case in which E contains a part E_2, in which $\bar{s}(x)$ is finite and $\underline{s}(x) = -\infty$. If in a part E_3, of E, we have $\bar{s}(x) = +\infty$, $\underline{s}(x) = -\infty$, the sequence is uniformly oscillatory in E_3, in

* *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 346, where, however, the statement of the definition requires amendment. The correct definition is given in *Quart. Journ.* vol. XLIV (1913), p. 132. Earlier investigations of uniform oscillation were given by W. H. Young in *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 398, and in *Camb. Phil. Trans.* vol. XXI (1909), p. 241.

the extended sense. In this case $\sigma_n(x) < \bar{\sigma}(x) + \epsilon$, $\sigma_n(x) > \underline{\sigma}(x) - \epsilon$ for all values of n, since $\bar{\sigma}(x) = 1$, $\underline{\sigma}(x) = -1$.

The sequence is said to be uniformly oscillatory in E, in the extended sense, if it be uniformly oscillatory in each of the sets E_1, E_2, E_3, and also in $E - E_1 - E_2 - E_3$.

116. The following property of a sequence that is uniformly oscillatory either above or below, or both, will be established.

If the sequence $\{s_n(x)\}$ consist of functions which are lower (upper) semi-continuous at ξ, and the sequence oscillates uniformly above (below), the upper (lower) function is lower (upper) semi-continuous at ξ.

We have

$$\bar{s}(\xi) - \bar{s}(x) = \{\bar{s}(\xi) - s_n(\xi)\} + \{s_n(\xi) - s_n(x)\} + \{s_n(x) - \bar{s}(x)\}.$$

A neighbourhood Δ, of ξ, can be so chosen that $s_n(x) - \bar{s}(x) < \frac{1}{3}\epsilon$, if x is in Δ, for $n \geqq n_1$; a value of n $(\geqq n_1)$ can be so fixed that

$$\bar{s}(\xi) - s_n(\xi) < \tfrac{1}{3}\epsilon.$$

Also, a neighbourhood Δ_1, contained in Δ, can be so chosen that, for the fixed value of n, $s_n(\xi) - s_n(x) < \frac{1}{3}\epsilon$. It follows that, in Δ_1, we have $\bar{s}(x) > \bar{s}(\xi) - \epsilon$. Since ϵ is arbitrary, $\bar{s}(x)$ is lower semi-continuous at ξ.

It follows that:

If the functions $s_n(x)$ are continuous at ξ, and the sequence oscillates uniformly at ξ, $\bar{s}(x)$ is lower semi-continuous, and $\underline{s}(x)$ is upper semi-continuous at ξ.

A point ξ at which $p(\xi) = \bar{s}(\xi)$ will be said to be a point at which the sequence $\{s_n(x)\}$ is *continuously oscillatory above*. A point at which $c(\xi) = \underline{s}(\xi)$ will be said to be a point at which the sequence is *continuously oscillatory below*. The point ξ will be said to be a point of continuous oscillation of the sequence $\{s_n(x)\}$, if both the conditions $p(\xi) = \bar{s}(\xi)$, $c(\xi) = \underline{s}(\xi)$ are satisfied. This definition may be stated in the form that:

A point ξ, of continuous oscillation of the sequence $\{s_n(x)\}$, is one at which $\bar{s}(\xi)$ is the upper multiple limit at (ξ, ∞) of $s_n(x)$, and at which $\underline{s}(\xi)$ is the lower multiple limit of $s_n(x)$. The two conditions, taken separately denote continuous oscillation above and below respectively.

The condition of continuous oscillation at ξ may be also stated in the form that $M[\{s_n\}, \xi] = \bar{s}(\xi)$, $m[\{s_n\}, \xi] = \underline{s}(\xi)$. This is a generalization of the definition given in § 75, of continuous convergence, that

$$p(\xi) = c(\xi) = s(\xi).$$

117. *If, in a set E, $p(x) = \bar{s}(x)$, at all points, the sequence is said to oscillate continuously above in E, and if $c(x) = \underline{s}(x)$ the sequence is said to oscillate continuously below in E. If both conditions are satisfied the sequence is said to oscillate continuously in E.*

It can be shewn, as in § 115, that if a sequence does not oscillate continuously above (below) in a closed set E, there must be at least one point of E at which the sequence does not oscillate continuously above (below). It will be shewn that:

If the sequence $\{s_n(x)\}$ consist of functions which are lower (upper) semi-continuous at ξ, and the sequence oscillates continuously above (below), the upper (lower) function is upper (lower) semi-continuous at ξ.

For we have

$$\overline{\lim_{x\sim\xi}}\, \bar{s}(x) = \overline{\lim_{x\sim\xi}}\, \overline{\lim_{n\sim\infty}}\, s_n(x) \leqq \overline{\lim_{x\sim\xi,\, n\sim\infty}}\, s_n(x) \leqq p(\xi),$$

and since $p(\xi) = \bar{s}(\xi)$, we have $\overline{\lim\limits_{x\sim\xi}}\, \bar{s}(x) \leqq \bar{s}(\xi)$, and therefore $\bar{s}(x)$ is upper semi-continuous at ξ.

From this theorem it follows that:

If the functions $s_n(x)$ are all continuous at ξ, and the sequence $\{s_n(x)\}$ oscillates continuously at ξ, $\bar{s}(x)$ is upper semi-continuous, and $\underline{s}(x)$ is lower semi-continuous at ξ.

These results are in contrast with those of § 116, relating to uniform oscillation, as the resulting properties of $\bar{s}(x)$, $\underline{s}(x)$ are reversed in the two cases.

It follows that, if the functions $s_n(x)$ are all continuous at ξ, the sequence $\{s_n(x)\}$ cannot be both uniformly oscillatory and continuously oscillatory at ξ unless both $\bar{s}(x)$ and $\underline{s}(x)$ are continuous at ξ. In order to obtain a more precise knowledge of the relation between continuous and uniform oscillation of sequences, we require the following theorems:

If $\{s_n(x)\}$ oscillates uniformly above, at the point ξ, and $\bar{s}(x)$ is upper semi-continuous at ξ, then the sequence oscillates continuously above, at the point ξ.

If $\{s_n(x)\}$ oscillates continuously above, at the point ξ, and $\bar{s}(x)$ is lower semi-continuous at ξ, then the sequence oscillates uniformly above, at the point ξ.

To prove the first theorem, we have, for points of E in a neighbourhood Δ, of ξ, $s_n(x) < \bar{s}(x) + \epsilon < \bar{s}(\xi) + 2\epsilon$, for $n \geqq n_\epsilon$, provided Δ be taken sufficiently small. Hence the result, since ϵ is arbitrary.

To prove the second theorem, we have $s_n(x) < \bar{s}(\xi) + \epsilon < \bar{s}(x) + 2\epsilon$, for $n \geqq n_\epsilon$, in a sufficiently small neighbourhood of ξ. Thus the condition for uniform oscillation above, at ξ, is satisfied.

From these theorems it follows that, if $\bar{s}(x)$ is continuous at ξ, and $\{s_n(x)\}$ is either uniformly convergent above, or continuously oscillatory above, at ξ, it is both. A similar result holds as regards $\underline{s}(x)$ for uniform and continuous oscillation below. Thus we have the theorem:

If $\bar{s}(x)$ is continuous at ξ, uniform oscillation above, and continuous oscillation above, at ξ, are equivalent to one another in the sense that the existence of either entails that of the other. Similarly, if $\underline{s}(x)$ is continuous at ξ, uniform oscillation below, and continuous oscillation below, are equivalent to one another.

Even in case the functions $s_n(x)$ are all continuous, we cannot infer that a point at which the sequence is uniformly oscillatory is also one at which it is continuously oscillatory unless it is known that the functions $\bar{s}(x)$, $\underline{s}(x)$ are both continuous. In fact the condition satisfied at a point where the oscillation is continuous is more stringent than the condition satisfied when the oscillation is uniform.

118. We proceed now to connect with the theory the monotone sequences $\{W_n(x)\}$, $\{w_n(x)\}$, of § 112, associated with the sequence $\{s_n(x)\}$, which converge respectively to $\bar{s}(x)$, $\underline{s}(x)$. It will be shewn that:

The peak functions for the two sequences $\{s_n(x)\}$, $\{W_n(x)\}$ are identical, and the chasm functions for the two sequences $\{s_n(x)\}$, $\{w_n(x)\}$, are identical.

Denoting by $p'(x)$ the peak function for $\{W_n(x)\}$, since $s_n(x) \leq W_n(x)$, it follows that $p(x) \leq p'(x)$. Let $\{n_r\}$, $\{x_r\}$ be sequences of n and of x, where $\{x_r\}$ converges to x, and n_r increases indefinitely with r, such that

$$\lim_{r \sim \infty} W_{n_r}(x_r) = p'(x).$$

Since $W_{n_r}(x_r)$ is the upper boundary of all the numbers

$$s_{n_r}(x_r),\ s_{n_r+1}(x_r),\ \ldots,$$

an integer $n'_r\ (\geq n_r)$ exists such that $s_{n'_r}(x_r) > W_{n_r}(x_r) - \epsilon_r$, where $\{\epsilon_r\}$ is a descending sequence of positive numbers which converges to zero. The sequence $\{s_{n'_r}(x_r)\}$, as $r \sim \infty$, has all its limits $\geq p'(x)$; it thus follows that $p(x) \geq p'(x)$. Since also $p(x) \leq p'(x)$, it is seen that $p(x) = p'(x)$. The second part of the theorem can be proved in a similar manner.

The following theorem exhibits clearly the fact that uniform oscillation is an extension of uniform convergence:

It is necessary and sufficient in order that a sequence may oscillate uniformly above (below) at a point ξ, or in the whole domain E, that the descending (ascending) associated monotone sequence $\{W_n(x)\}$ should converge uniformly at the point, or in the domain, to the upper function $\bar{s}(x)$ (the lower function $\underline{s}(x)$).

To shew the necessity of the condition, we see that, if $s_n(x) < \bar{s}(x) + \epsilon$, for $n \geq n_\epsilon$, either in a neighbourhood Δ_ϵ of the point ξ, or in the whole domain E, then also $W_n(x) \leq \bar{s}(x) + \epsilon$, for $n \geq n_\epsilon$, since $W_n(x)$ is the upper boundary of $s_n(x)$, $s_{n+1}(x)$, ... This is the condition of uniform convergence of $\{W_n(x)\}$ to $\bar{s}(x)$. Conversely the first inequality follows from the second, since $s_n(x) \leq W_n(x)$; thus the condition is sufficient.

If we denote by $V_{n,m}(x)$ the function which has as its value at each point the largest of the numbers $s_n(x)$, $s_{n+1}(x)$, ... $s_{n+m-1}(x)$, the sequence $V_{n,1}(x)$, $V_{n,2}(x)$, ..., is monotone non-diminishing, and converges to $W_n(x)$. It will be shewn that:

It is necessary and sufficient in order that a sequence of continuous functions $\{s_n(x)\}$ in the domain E should oscillate continuously above, that the sequence $V_{n,1}(x)$, $V_{n,2}(x)$, ..., should, for each value of n, converge uniformly to $W_n(x)$. This condition may be applied either at a point ξ, of E, or in the whole domain.

To shew that the condition is sufficient, we observe that, as all the functions $s_n(x)$ are continuous at ξ, or in the domain E, the functions $V_{n,1}(x)$, $V_{n,2}(x)$, ..., are also continuous in the same sense. If the convergence of this sequence to $W_n(x)$ is uniform, it follows that $W_n(x)$ is continuous, either at ξ, or in the domain E, as the case may be. The function $\bar{s}(x)$ is accordingly upper semi-continuous, at ξ, or in E.

We have $W_n(\xi) - \bar{s}(\xi) < \epsilon$, for $n \geqq n_\epsilon$; and choosing a first value of n which is $\geqq n_\epsilon$, we have $W_n(x) - W_n(\xi) < \epsilon$, provided x is in a certain neighbourhood of ξ; therefore $W_n(x) - \bar{s}(\xi) < 2\epsilon$, for this value of n, and for all greater values. It now follows that $p'(\xi)$, the upper multiple limit of $W_n(x)$ as $n \sim \infty$, $x \sim \xi$, is $\leqq \bar{s}(\xi) + 2\epsilon$. Since ϵ is arbitrary, we have $p'(\xi) \leqq \bar{s}(\xi)$. Again, since $W_n(\xi) - \bar{s}(\xi) \geqq 0$, and

$$W_n(x) - W_n(\xi) > -\epsilon$$

in a certain neighbourhood of ξ, dependent on the value of n, we have $W_n(x) - \bar{s}(\xi) \geqq -\epsilon$; it now follows that $p'(\xi) - \bar{s}(\xi) \geqq -\epsilon$, or

$$p'(\xi) \geqq \bar{s}(\xi),$$

since ϵ is arbitrary. From the two inequalities we see that $p'(\xi) = \bar{s}(\xi)$. Since now $p'(\xi) = p(\xi)$, we have $p(\xi) = \bar{s}(\xi)$ and therefore, at ξ, the oscillation of $\{s_n(x)\}$ is continuous above.

To prove the necessity of the condition in the theorem, let it be assumed that $p(\xi) = \bar{s}(\xi)$. Then we have

$$s_n(x) < p(\xi) + \epsilon < W_m(\xi) + \epsilon,$$

provided x is in a neighbourhood Δ_ϵ, of ξ, and $n \geqq n_\epsilon$, m having any value.

A neighbourhood Δ'_ϵ, contained in Δ_ϵ can be so chosen that

$$s_n(x) < s_n(\xi) + \epsilon < W_r(\xi) + \epsilon,$$

provided $r \leqq n$, and n has one of the values 1, 2, 3, ... $n_\epsilon - 1$. From the two inequalities, it follows that, in Δ'_ϵ, $s_n(x) < W_r(\xi) + \epsilon$, for all values of n, provided $r \leqq n$. Since $W_r(x)$ is the upper boundary of $s_r(x)$, $s_{r+1}(x)$ ..., it follows that $W_r(x) \leqq W_r(\xi) + \epsilon$. Hence the function $W_r(\xi)$ is upper semi-continuous at ξ, or in the whole of E, as the case may be. But $W_r(x)$ is lower semi-continuous, since it is the limit of an ascending

sequence of continuous functions. Therefore $W_r(x)$ is continuous, either at ξ, or in E. Since the monotone sequence of continuous functions $V_{r,1}(x)$, $V_{r,2}(x)$, ... converges to the continuous function $W_r(x)$, either at one point, or in E, the convergence is uniform either at the point, or in the set E, as the case may be.

119. A mode of defining uniform oscillation of a sequence in a domain, or at a point of the domain, has been given* by W. H. Young, and has been termed by him uniform oscillation of the first kind, and by Hahn† secondarily uniform oscillation (sekundär-gleichmässig oscillirend).

Denoting by $V_{n,r}(x)$ the function which has, at each point x, the value of the greatest of the numbers $s_n(x)$, $s_{n+1}(x)$, ... $s_{n+r-1}(x)$, we have $W_n(x) = \lim_{r \sim \infty} V_{n,r}(x)$; where $W_n(x)$ has, as in § 111, the value of the upper boundary of the numbers $s_n(x)$, $s_{n+1}(x)$, ...

When the convergence of the sequence $\{V_{n,r}(x)\}$ to $W_n(x)$ is, for each value of n, uniform at a point ξ, or in the domain E, the sequence $\{s_n(x)\}$ is said to be uniformly oscillatory above at the point ξ, or in the domain E.

In the case of continuous functions, when this definition is satisfied, it has been shewn in § 118 that the sequence oscillates continuously above. Uniform oscillation below is defined in a similar manner, by employing the sequence $\{w_n(x)\}$ and the sequences $\{v_{n,r}(x)\}$, when $v_{n,r}(x)$ denotes at each point the least of the numbers $s_n(x)$, $s_{n+1}(x)$, ... $s_{n+r-1}(x)$.

When the sequence has both upper and lower uniform oscillation at a point, or in the domain, it is said to be uniformly oscillatory at the point, or in the domain.

In case the sequence converges uniformly, $V_{n,r}(x)$ converges for each value of n, uniformly to $W_n(x)$; and also $W_m(x)$ converges uniformly to $\bar{s}(x)$, or $s(x)$. It can however be seen that, when the sequence is convergent, but the functions $\{s_n(x)\}$ are not continuous, the sequence can satisfy the definition of uniform oscillation without necessarily converging uniformly.

FAMILIES OF EQUI-CONTINUOUS FUNCTIONS

120. Let a family of continuous functions $f(x)$ be defined in a given interval, or cell (a, b), the number of functions in the family being infinite, and not necessarily enumerable. If ϵ be an arbitrarily prescribed positive number, then in virtue of the theorem of I, § 217, the interval or cell can be divided into a finite number m_ϵ, of sub-intervals or sub-cells such that in each one of them, all of which may be taken to be closed, the fluctuation of a continuous function is less than ϵ. In case the family of continuous

* *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 359.

† *Theorie der reellen Funktionen*, vol. I (1921), p. 257.

functions is such that, for each value of ϵ, a single set of sub-intervals or sub-cells can be so determined that, for every function $f(x)$, of the family, the fluctuation in a sub-interval, or a sub-cell, of the set, is less than ϵ, the family is said to consist of *equi-continuous* functions. When $|f(x)| < K$ for every point x, and every function $f(x)$, the family is said to be bounded.

The following fundamental property of a family of bounded equi-continuous functions was given* by Arzelà, for the case of a linear interval:

If $\{f(x)\}$ be a bounded family of equi-continuous functions defined in a linear interval (a, b), or in a cell (a, b) of any number of dimensions, there exists a sequence $\{f_n(x)\}$ of functions, all of which belong to the family, which is uniformly convergent, and therefore converges to a continuous function $\phi(x)$.

The positive number ϵ_1 being arbitrarily chosen, let (a, b) be divided into m intervals or cells, such that, in each of them (taken to be closed) the fluctuation of every function $f(x)$, of the given family, is $< \epsilon_1$. Let $x_1, x_2, \dots x_m$ be the centres of the m cells or intervals, then, if $f(x)$ be any function belonging to the family, we may regard $(f(x_1), f(x_2), \dots f(x_m))$ as defining a point in m-dimensional space. Assuming the family to be bounded, the set of all such points, when all the functions $f(x)$ are taken into account, has at least one limiting point, and a sequence $\{f^{(n)}(x)\}$ of functions, all of which belong to the given family exists, such that the points $(f^{(n)}(x_1), f^{(n)}(x_2) \dots f^{(n)}(x_n))$ converge to a limiting point; it follows that $|f^{(n)}(x_r) - f^{(n')}(x_r)| < \epsilon_1$, for $r = 1, 2, 3, \dots m$; provided n and n' are both greater than some positive integer n_{ϵ_1}.

If x be any point in the cell, or interval, of which x_r is the centre, we have $|f^{(n)}(x_r) - f^{(n)}(x)| < \epsilon_1$, for all values of n; and it thus follows that $|f^{(n)}(x) - f^{(n')}(x)| < 3\epsilon_1$, provided $n > n_{\epsilon_1}$, $n' > n_{\epsilon_1}$.

Now let $\{\epsilon_n\}$ denote a sequence of decreasing numbers which converges to zero. It is convenient to denote the sequence $\{f^{(n)}(x)\}$ which has been determined, by $\{f_{1n}(x)\}$, where $n = 1, 2, 3, \dots$. By the same reasoning as before, a part $\{f_{p_2,n}(x)\}$ of the sequence $\{f_{1n}(x)\}$ can be so determined that $|f_{p_2,n}(x) - f_{p_2,n'}(x)| < 3\epsilon_2$, for all points x, in the given interval, or cell, provided n and n' are not less than some integer n_{ϵ_2}. Proceeding in the same manner, the sequence $\{f_{p_2,n}(x)\}$ contains a sequence $\{f_{p_3,n}(x)\}$, such that $|f_{p_3,n}(x) - f_{p_3,n'}(x)| < 3\epsilon_3$, provided n and n' are not less than some integer n_{ϵ_3}; and so on indefinitely.

Let us consider the sequence of functions

$$f_{11}(x),\ f_{p_2,2}(x),\ f_{p_3,3}(x) \dots f_{p_n,n}(x), \dots;$$

it will be shewn that this sequence converges uniformly in the given interval or cell. Taking a fixed integer m, if n is greater than m, $f_{p_n,n}(x)$

* *Memorie Acad. Bologna* (5), vol. v (1895), p. 55, and (5) vol. VIII (1899), p. 176.

is equal to $f_{p_{m},r}(x)$, for some value of r ($\leqq n$). Hence we have, if $n' > n$, $|f_{p_n,n}(x) - f_{p_{n'},n'}(x)| = |f_{p_m,r}(x) - f_{p_m,r'}(x)| < 3\epsilon_m$, provided r and r' are both $\geqq n_{\epsilon_m}$; that is provided n and n' are both not less than some integer dependent on ϵ_m. Since this holds good for every value of m it follows that the sequence $\{f_{p_n,n}(x)\}$ is uniformly convergent; and it therefore converges to a continuous function $\phi(x)$, which may, or may not, belong to the given family of equi-continuous functions.

HOMOGENEOUS OSCILLATION

121. If, not only the sequence $\{f_n(x)\}$ has a certain property, but also all the sub-sequences (see § 61) have the same property, the sequence $\{f_n(x)\}$ is said to be *homogeneous* in respect of that property. Thus, if $\{f_n(x)\}$ and all its sub-sequences oscillate uniformly at a point, or in a set of points E, the sequence $\{f_n(x)\}$ is said to oscillate *uniformly and homogeneously* at the point, or in the set E. In the same way, if $\{f_n(x)\}$ and all its sub-sets oscillate continuously, the sequence is said to oscillate *continuously and homogeneously*. A similar statement applies if the continuous oscillation, or the uniform oscillation, is above, or is below. The term homogeneous was introduced by W. H. Young, who has investigated* the properties of sequences which oscillate uniformly and homogeneously. Any oscillating sequence $\{s_n(x)\}$ has the following property:

If all the functions of the set of lower functions of the sequence $\{s_n(x)\}$ are lower semi-continuous, then all the upper functions are also lower semi-continuous.

By changing the signs of all the functions, we see that *if all the upper functions are upper semi-continuous, then all the lower functions are also upper semi-continuous.*

To prove the theorem, let $u(x)$ be the upper function of a sub-sequence $\{s_{n_p}(x)\}$; we can then choose a sub-sequence of $\{s_{n_p}(x)\}$ which converges at the point ξ to the unique limit $u(\xi)$. The upper function $\bar{u}(x)$ of this last sub-sequence is $\leqq u(x)$, and $\bar{u}(\xi) = u(\xi)$. Denoting its lower function by $\bar{l}(x)$, we have, since $\bar{l}(x)$ is lower semi-continuous,

$$u(\xi) = \bar{u}(\xi) = \bar{l}(\xi) \leqq \underline{\lim}_{x\sim\xi} \bar{l}(x) \leqq \underline{\lim}_{x\sim\xi} \bar{u}(x) \leqq \underline{\lim}_{x\sim\xi} u(x);$$

and it follows that $u(x)$ is lower semi-continuous at ξ. Since ξ is an arbitrary point, $u(\xi)$ is a lower semi-continuous function.

It is easily seen that the semi-continuity assumed may be on one side only, and the semi-continuity proved will be on that same side.

* See *Cambridge Phil. Trans.* vol. XXI (1909), p. 241; also *Proc. Lond. Math. Soc.* (2), vol. VIII (1910), p. 353.

122. We proceed to establish the following theorem:

If either all the upper functions, or all the lower functions, of a sequence of functions, are continuous functions, then, in any sub-sequence, a sub-sequence can be determined which converges to a continuous function.

The theorem holds good if the continuity presupposed is on one side only, the same side for all the functions, and the limiting function whose existence is asserted will then be continuous on the same side.

Let us assume that all the upper functions are continuous. Let an enumerable set of points $\xi_1, \xi_2, \ldots \xi_n, \ldots$ be so defined as to be everywhere dense in the domain of the functions of the sequence.

A sub-sequence $\{s_{1n}(x)\}$ can be so chosen as to converge at the point ξ_1 to a unique limit. Omitting the function $s_{11}(x)$, a sub-sequence $\{s_{2n}(x)\}$ belonging to the sequence $s_{12}(x), s_{13}(x), \ldots$ can be defined, which converges, at ξ_2, to a unique limit. Proceeding indefinitely in a similar manner, we obtain a set of sub-sequences $\{s_{1n}(x)\}, \{s_{2n}(x)\}, \{s_{3n}(x)\} \ldots$. The sub-sequence $s_{11}(x), s_{22}(x), s_{33}(x), \ldots s_{nn}(x) \ldots$ has unique limits at $\xi_1, \xi_2, \ldots \xi_n \ldots$; for it belongs to $\{s_{1n}(x)\}$ and therefore has a unique limit at ξ_1, it also belongs to $\{s_{2n}(x)\}$, and therefore it has a unique limit at ξ_2; and so on. Since the set of points ξ is everywhere-dense, the values of the unique limits determine at most one continuous function having those values at the points ξ. Since the upper functions of the sequence $\{s_{nn}(x)\}$ are continuous, and therefore upper semi-continuous, it follows from the last theorem that its lower function is upper semi-continuous. Denoting by $u(x), l(x)$ the upper and lower functions of the sequence $\{s_{nn}(x)\}$, we have, since $l(x)$ is upper semi-continuous, $l(x) \geqq \overline{\lim\limits_{\xi \sim x}}\, l(\xi) = \lim\limits_{\xi \sim x} u(\xi) \geqq u(x)$, since $u(x)$ is continuous. But $l(x) \leqq u(x)$, and therefore $l(x) = u(x)$, or the sequence $\{s_{nn}(x)\}$ is convergent at x. Thus $\{s_{nn}(x)\}$ has everywhere a unique limiting function, and this is continuous, since all the upper functions are continuous.

123. The following theorem is an extension of, and includes, Arzelà's theorem given in § 120:

If a sequence of continuous functions $\{s_n(x)\}$ oscillates continuously and homogeneously, then all the upper functions and all the lower functions of the sequence are continuous, and in every sub-sequence there is contained a sequence of functions which converges to a unique limiting function which is continuous.

The oscillation may be taken to be continuous and homogeneous on one side only, then the continuity is on that side only.

Since the sequence oscillates continuously, from a theorem proved in § 117 it follows that $\bar{s}(x)$ is upper semi-continuous, and that $\underline{s}(x)$ is lower semi-continuous. From the theorem in § 120, it follows that $\bar{s}(x)$ is lower

semi-continuous, hence it must be continuous. Similarly it can be shewn that $\underline{s}(x)$ is continuous. As this may be applied to any sub-sequence it follows that all the upper functions and all the lower functions are continuous. The second part of the theorem then follows from § 122.

In order to exhibit the connection of this theorem with that of Arzelà, let it be supposed that the sequence is such that, for a fixed point ξ, and for each arbitrarily chosen positive number ϵ, a neighbourhood Δ, of ξ, exists such that, for each function $s_n(x)$, of the sequence, the upper boundary of $s_n(x)$ in Δ exceeds $s_n(\xi)$ by less than ϵ. It then follows that $p(\xi) - \bar{s}(\xi) \leq \epsilon$; and since ϵ is arbitrary, $p(\xi) = \bar{s}(\xi)$, or the sequence is continuously oscillatory above at ξ. If the neighbourhood of ξ, can be so determined that the lower boundary of $s_n(x)$ in Δ is, for every value of n, less than $\underline{s}(\xi)$ by less than ϵ, it follows that $c(\xi) = \underline{s}(\xi)$. When both these conditions are satisfied the sequence oscillates continuously. As the same argument can be applied to any sub-sequence, the continuous oscillation is homogeneous.

When the field of the functions is a continuous interval or cell, all the points of that field are, by the Heine-Borel theorem, interior to a finite set of the neighbourhoods Δ corresponding to all the points of the field. Accordingly, every interval, or cell, of length or span less than a number d, dependent only on ϵ, may be taken to be the neighbourhood Δ of the point which is its centre. When the conditions are satisfied, the fluctuation of each of the functions $s_n(x)$ in every such cell or interval is $< 2\epsilon$. Hence, the theorem reduces to Arzelà's theorem.

CHAPTER III

POWER-SERIES

124. A series of which the $(n+1)$th term is $a_n x^n$ is said to be a power-series for the variable x, when the coefficients a_n are assigned in accordance with some norm. It will be shewn that:

If $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ *be a prescribed power-series, then either* (1), *there exists a positive number* R *such that the series converges absolutely for every value of* x *for which* $|x| < R$, *and does not converge for any value of* x *such that* $|x| > R$, *or* (2), *the series converges for each value of* x, *positive or negative, or* (3), *the series does not converge for any value of* x, *except zero.*

The cases (2) and (3) may be regarded as arising from (1) when $R = +\infty$, or $R = 0$, respectively.

It has been shewn in § 13 that, in accordance with Cauchy's test, the series $|a_0| + |a_1 x| + \ldots + |a_n x^n| + \ldots$ is convergent if $\overline{\lim\limits_{n\sim\infty}} |a_n x^n|^{\frac{1}{n}} < 1$, and is divergent if $\overline{\lim\limits_{n\sim\infty}} |a_n x^n|^{\frac{1}{n}} > 1$. Writing $\underline{\lim\limits_{n\sim\infty}} |a_n|^{-\frac{1}{n}} = R$, and assuming first that $0 < R < \infty$, the series $a_0 + a_1 x + \ldots + a_n x^n + \ldots$ is seen to be absolutely convergent if $|x| < R$. If $|x| > R$, $|a_n x^n| = \left(\frac{|x|}{R}\right)^n |a_n R^n|$ and therefore $\overline{\lim\limits_{n\sim\infty}} |a_n x^n| = \infty$, and thus the series

$$a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

cannot be convergent when $|x| > R$, but must be divergent or oscillatory. In case $\underline{\lim\limits_{n\sim\infty}} |a_n|^{-\frac{1}{n}} = \infty$, the series $a_0 + a_1 x + \ldots + a_n x^n + \ldots$ is convergent for every value of x. In case $\underline{\lim\limits_{n\sim\infty}} |a_n|^{-\frac{1}{n}} = 0$, the series cannot converge when $|x| > 0$.

By writing $\frac{x}{R}$ instead of x, the series may always be changed, provided $0 < R < \infty$, into a series which converges when $|x| < 1$, and is non-convergent when $|x| > 1$.

The interval $(-R, R)$ is said to be the interval of convergence of the series $a_0 + a_1 x + \ldots + a_n x^n + \ldots$. This interval must be in general regarded as an open interval, since the question of the convergence of the series when $x = R$, and $x = -R$ remains undecided, until further investigation, in any particular case.

It will be shewn that:

If $a_n = O(n^k)$, where k is a fixed number, and the series $\sum\limits_{n=0} a_n$ be divergent or oscillating, the series $\sum\limits_{n=0} a_n x^n$ has the interval $(-1, 1)$ for its interval of convergence.

If x have any fixed value numerically less than 1, the series $\sum\limits_{n=0} n^k \mid x \mid^n$ is convergent, since $\lim\limits_{n\sim\infty} \left(\frac{n+1}{n}\right)^k \mid x \mid < 1$. Since each term of the series $\sum\limits_{n=0} a_n x^n$ is less, in absolute magnitude, than a fixed multiple of the corresponding term of the convergent series $\sum\limits_{n=0} n^k \mid x \mid^n$, from the last theorem in § 24 it follows that $\sum\limits_{n=0} a_n x^n$ is absolutely convergent. Since Σa_n is not convergent, the interval of convergence is not greater than $(-1, 1)$; hence it must be $(-1, 1)$.

125. It will now be shewn that, in the case of a power series with a finite, or infinite, interval of convergence:

The series converges uniformly in any interval interior to the interval of convergence.

It is sufficient to consider the cases in which $(-1, 1)$ or $(-\infty, \infty)$ are the intervals of convergence, since any finite interval of convergence may be reduced to $(-1, 1)$ by a change of the variable x.

If (α, β) be any interval interior to the interval of convergence, a point ρ, exterior to (α, β) may be chosen, so that ρ is greater than $\mid \alpha \mid$ and $\mid \beta \mid$, and is itself interior to the interval of convergence.

If x be any number in the interval (α, β), we have $\mid x/\rho \mid < 1$; and the partial remainder

$$\begin{aligned} R_{n,m}(x) &= a_n x^n + a_{n+1}x^{n+1} + \ldots + a_{n+m-1}x^{n+m-1} \\ &= R_{n,1}(\rho)\left(\frac{x}{\rho}\right)^n + \sum_{p=2}^{p=m}\{R_{n,p}(\rho) - R_{n,p-1}(\rho)\}\left(\frac{x}{\rho}\right)^{n+p-1} \\ &= \left(1 - \frac{x}{\rho}\right)\sum_{p=1}^{p=m-1} R_{n,p}(\rho)\left(\frac{x}{\rho}\right)^{n+p-1} + R_{n,m}(\rho)\left(\frac{x}{\rho}\right)^{n+m-1} \end{aligned}$$

If n be chosen so large that $\mid R_{n,p}(\rho) \mid < \epsilon$, for $p = 1, 2, 3, \ldots$, we have

$$\mid R_{n,m}(x) \mid < \epsilon\left(1 - \frac{x}{\rho}\right)\frac{\left|\frac{x}{\rho}\right|^n}{1 - \left|\frac{x}{\rho}\right|} + \epsilon \leq \epsilon \,.\, \frac{2 - \frac{x}{\rho} - \left|\frac{x}{\rho}\right|}{1 - \left|\frac{x}{\rho}\right|} < \frac{3\epsilon}{1 - \left|\frac{x}{\rho}\right|}.$$

Since $\left|\frac{x}{\rho}\right|$ is less than some fixed positive number less than 1, for all values of x in the interval (α, β), it follows that $\mid R_{n,m}(x) \mid < A\epsilon$, for all

values of x in (α, β), and for all values of m, the value of n having been chosen when ϵ has been assigned; the number A is fixed, being dependent only on α, β, and ρ. Since $A\epsilon$ is arbitrarily small, the condition for uniform convergence of the series in the interval (α, β) is satisfied.

Let $(-1, 1)$ be the interval of convergence for the series

$$a_0 + a_1 x + a_2 x^2 + \dots,$$

and let it be assumed that the series is convergent at the point $x = 1$. In the transformation used in the proof of the last theorem we may then take $\rho = 1$, and thus

$$|R_{n,m}(x)| < \epsilon(1 - x)\frac{x^n}{1 - |x|} + \epsilon,$$

for $|x| < 1$, if n is so chosen that $|R_{n,m}(1)| < \epsilon$, for $m = 1, 2, 3, \dots$. Let x be any point of the closed interval $(-\alpha, 1)$, where α is positive and < 1. We have $\dfrac{1 - x}{1 - |x|} = 1$, if x is such that $0 \leq x < 1$, and if x is in the interval $(-\alpha, 1)$ we have $\dfrac{1 - x}{1 - |x|} \leq \dfrac{1 + \alpha}{1 - \alpha}$; therefore in the closed interval $(-\alpha, 1)$, we have $|R_{n,m}(x)| < \dfrac{2\epsilon}{1 - \alpha}$, for $m = 1, 2, 3, \dots$, provided n has a sufficiently large value. It follows that the series converges uniformly in the interval $(-\alpha, 1)$.

In case the series converges when $x = -1$, the series

$$a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \dots$$

converges when $x = 1$. Applying the result obtained to this series, we see that, in case the series $a_0 + a_1 x + a_2 x^2 + \dots$ is convergent when $x = -1$, it converges uniformly in the closed interval $(-1, \beta)$, where $\beta < 1$.

If the series is convergent both for $x = 1$, and for $x = -1$, it is uniformly convergent in both the intervals $(-\alpha, 1)$ and $(-1, \beta)$, where α and β are positive and less than 1; since these intervals overlap one another, it follows that the convergence of the series is uniform in the closed interval $(-1, 1)$. It has now been established that:

If $(-1, 1)$ be the interval of convergence of a power series, then, if the series converge when $x = 1$, the convergence of the series is uniform in any interval $(\alpha, 1)$, where $\alpha > -1$; if the series converge when $x = -1$, the series converges uniformly in any interval $(-1, \beta)$, where $\beta < 1$; and if the series converges both when $x = 1$ and when $x = -1$, the convergence is uniform in the interval $(-1, 1)$.

126. Since a series, all the terms of which are continuous, has a sum which is continuous in any interval of uniform convergence of the series, it is seen that:

A power series $a_0 + a_1 x + a_2 x^2 + \ldots$ has a sum $s(x)$ which is continuous at any point interior to its interval of convergence.

Moreover, if $(-1, 1)$ be the interval of convergence, and the series be convergent for $x = 1$, the sum-function $s(x)$ is continuous in any interval $(a, 1)$, where $a > -1$; the continuity being reckoned only on one side at the point 1. A proof of Abel's theorem* has thus been obtained, that:

If $(-1, 1)$ be the interval of convergence of the power-series

$$a_0 + a_1 x + a_2 x^2 + \ldots;$$

$s(x)$ denoting the sum-function, and if $a_0 + a_1 + a_2 + \ldots$ is convergent, having $s(1)$ for its sum, then $\lim\limits_{x \sim 1} s(x) = s(1)$, when x converges to 1 through values < 1.

Abel's theorem may also be deduced from the theorem in § 80. For we have $x^n \geqq x^{n+1}$, for $0 \leqq x \leqq R$; hence, since the series $\sum\limits_{n=0} a_n$ is, by hypothesis, convergent, it follows that the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent in the interval $(0, R)$; and thus the sum-function is continuous in that closed interval.

It should be observed that Abel's theorem has been established only for a series in which the powers of the variable are ascending, and that it is not necessarily true in any other case. For example, the series

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

is convergent within the interval $(-1, 1)$; and as the series is absolutely convergent, for such values of x, the series

$$x + \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{4}x^4 + \ldots$$

has the same sum-function $\log_e(1 + x)$, as the original series within the interval. At $x = 1$, the series $1 - \frac{1}{2} + \frac{1}{3} - \ldots$ is convergent. and in accordance with the theorem, its sum is $\log_e 2$; but the series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \ldots,$$

although it is convergent, has the sum $\frac{3}{2}\log_e 2$ (see Ex. 1, § 26); which is not continuous with the sum of the series $x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \ldots$ for $x < 1$.

127. With a view to the extension of Abel's theorem, the following lemma will be established:

If $\sum\limits_{n=0} a_n x^n$, $\sum\limits_{n=0} \beta_n x^n$ both converge within the interval $(-1, 1)$, a_n being positive and such that $\sum\limits_{n=0}^{\infty} a_n$ is divergent, and if $\dfrac{\beta_n}{a_n}$ oscillates between the

* *Crelle's Journal*, vol. I (1826), p. 314, also *Œuvres*, vol. I, p. 223. See also Pringsheim, *Münch. Sitzungsber.* vol. XXVII (1897), p. 344.

limits U and L, then the upper and lower limits of $\dfrac{\sum\limits_{n=0}^{\infty}\beta_n x^n}{\sum\limits_{n=0}^{\infty}\alpha_n x^n}$, *as x converges to 0, are in the interval (L, U). In particular, if* $\dfrac{\beta_n}{\alpha_n}$ *converges to a definite limit, as $n \sim \infty$,* $\dfrac{\sum\limits_{n=0}^{\infty}\beta_n x^n}{\sum\limits_{n=0}^{\infty}\alpha_n x^n}$ *converges to the same limit, as x converges to* 1.

If $\dfrac{\beta_n}{\alpha_n}$ *diverges to ∞, so also does* $\dfrac{\sum\limits_{n=0}^{\infty}\beta_n x^n}{\sum\limits_{n=0}^{\infty}\alpha_n x^n}$, *as $x \sim 1$.*

Let u_m, l_m denote the greatest and least (algebraically) of the numbers $\dfrac{\beta_0}{\alpha_0}, \dfrac{\beta_1}{\alpha_1}, \ldots \dfrac{\beta_{m-1}}{\alpha_{m-1}}$; and let u_{mn}, l_{mn} denote the greatest and least (algebraically) of the numbers $\dfrac{\beta_m}{\alpha_m}, \dfrac{\beta_{m+1}}{\alpha_{m+1}}, \ldots \dfrac{\beta_n}{\alpha_n}$; where $m < n$.

We have then

$$\beta_0 + \beta_1 x + \ldots + \beta_n x^n < u_m(\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1}) + u_{mn}(\alpha_m x^m + \ldots + \alpha_n x^n);$$

therefore

$$\frac{\sum\limits_{n=0}^{n}\beta_n x^n}{\sum\limits_{n=0}^{n}\alpha_n x^n} < u_{mn} + (u_m - u_{mn})\frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1}}{\sum\limits_{n=0}^{n}\alpha_n x^n};$$

and similarly, we have

$$\frac{\sum\limits_{n=0}^{n}\beta_n x^n}{\sum\limits_{n=0}^{n}\alpha_n x^n} > l_{mn} - (l_{mn} - l_m)\frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1}}{\sum\limits_{n=0}^{n}\alpha_n x^n}.$$

If ϵ be a prescribed positive number, m may be so fixed that $u_{mn} < U + \epsilon$, $l_{mn} > L - \epsilon$, for all values of n. Moreover $u_m - u_{mn}$, $l_{mn} - l_m$ are numerically less than fixed positive numbers.

Keeping m fixed, let $n \sim \infty$, we have then, if A and B denote certain fixed numbers,

$$U + \epsilon + A\frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1}}{\sum\limits_{n=\infty}^{\infty}\alpha_n x^n}$$
$$> \frac{\sum\limits_{0}^{\infty}\beta_n x^n}{\sum\limits_{0}^{\infty}\alpha_n x^n} > L - \epsilon + B\frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1}}{\sum\limits_{n=0}^{\infty}\alpha_n x^n},$$

for all values of x such that $0 < x < 1$. Now let x converge to the value 1; since $\sum_{n=0}^{n} \alpha_n x^n$ is monotone increasing both with respect to n and with respect to x, the repeated limits (see § 30)

$$\lim_{x \sim 1} \lim_{n \sim \infty} \sum_{n=0}^{n} \alpha_n x^n, \quad \lim_{n \sim \infty} \lim_{x \sim 1} \sum_{n=0}^{n} \alpha_n x^n$$

have the same value ∞; thus $\lim_{x \sim 1} \sum_0^\infty \alpha_n x^n = \infty$. Accordingly, we have

$$U + \epsilon > \overline{\lim_{x \sim 1}} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} \geqq \underline{\lim_{x \sim 1}} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} > L - \epsilon;$$

and, since ϵ is arbitrary, we have

$$U \geqq \overline{\lim_{x \sim 1}} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} \geqq \underline{\lim_{x \sim 1}} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} \geqq L.$$

It follows that if $U = L$

$$\lim_{x \sim 1} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} = \lim_{n \sim \infty} \frac{\beta_n}{\alpha_n}.$$

In case $\frac{\beta_n}{\alpha_n}$ diverges to ∞, as $n \sim \infty$, n can be so chosen that $l_{mn} > N$, where N is a prescribed positive number, for every value of n. We have then, for $x < 1$,

$$\frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} > N \left[1 - \frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1} x^{m-1}}{\sum_0^\infty \alpha_n x^n} \right] + l \frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1} x^{m-1}}{\sum_0^\infty \alpha_n x^n},$$

where l is the lower boundary of all the numbers $\frac{\beta_0}{\alpha_0}, \frac{\beta_1}{\alpha_1}, \ldots$.

We have now $\underline{\lim_{x \sim 1}} \frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n} \geqq N$; and since N is arbitrarily large, it follows that $\frac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty \alpha_n x^n}$ diverges to ∞, as $x \sim 1$.

limits U and L, then the upper and lower limits of $\dfrac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n}$, *as x converges to* 0, *are in the interval* (L, U). *In particular, if* $\dfrac{\beta_n}{\alpha_n}$ *converges to a definite limit, as* $n \sim \infty$, $\dfrac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n}$ *converges to the same limit, as x converges to* 1.

If $\dfrac{\beta_n}{\alpha_n}$ *diverges to* ∞, *so also does* $\dfrac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n}$, *as* $x \sim 1$.

Let u_m, l_m denote the greatest and least (algebraically) of the numbers $\dfrac{\beta_0}{\alpha_0}, \dfrac{\beta_1}{\alpha_1}, \dots \dfrac{\beta_{m-1}}{\alpha_{m-1}}$; and let u_{mn}, l_{mn} denote the greatest and least (algebraically) of the numbers $\dfrac{\beta_m}{\alpha_m}, \dfrac{\beta_{m+1}}{\alpha_{m+1}}, \dots \dfrac{\beta_n}{\alpha_n}$; where $m < n$.

We have then

$$\beta_0 + \beta_1 x + \dots + \beta_n x^n < u_m (\alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1}) + u_{mn} (\alpha_m x^m + \dots + \alpha_n x^n);$$

therefore

$$\frac{\sum_{n=0}^{n} \beta_n x^n}{\sum_{n=0}^{n} \alpha_n x^n} < u_{mn} + (u_m - u_{mn}) \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1}}{\sum_{n=0}^{n} \alpha_n x^n};$$

and similarly, we have

$$\frac{\sum_{n=0}^{n} \beta_n x^n}{\sum_{n=0}^{n} \alpha_n x^n} > l_{mn} - (l_{mn} - l_m) \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1}}{\sum_{n=0}^{n} \alpha_n x^n}.$$

If ϵ be a prescribed positive number, m may be so fixed that $u_{mn} < U + \epsilon$, $l_{mn} > L - \epsilon$, for all values of n. Moreover $u_m - u_{mn}$, $l_{mn} - l_m$ are numerically less than fixed positive numbers.

Keeping m fixed, let $n \sim \infty$, we have then, if A and B denote certain fixed numbers,

$$U + \epsilon + A \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1}}{\sum_{n=\infty}^{\infty} \alpha_n x^n} > \frac{\sum_0^{\infty} \beta_n x^n}{\sum_0^{\infty} \alpha_n x^n} > L - \epsilon + B \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1}}{\sum_{n=0}^{\infty} \alpha_n x^n},$$

for all values of x such that $0 < x < 1$. Now let x converge to the value 1; since $\sum_{n=0}^{n} a_n x^n$ is monotone increasing both with respect to n and with respect to x, the repeated limits (see § 30)

$$\lim_{x\sim 1} \lim_{n\sim\infty} \sum_{n=0}^{n} a_n x^n, \quad \lim_{n\sim\infty} \lim_{x\sim 1} \sum_{n=0}^{n} a_n x^n$$

have the same value ∞; thus $\lim\limits_{x\sim 1} \sum\limits_0^\infty a_n x^n = \infty$. Accordingly, we have

$$U + \epsilon > \overline{\lim_{x\sim 1}} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} \geqq \underline{\lim_{x\sim 1}} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} > L - \epsilon;$$

and, since ϵ is arbitrary, we have

$$U \geqq \overline{\lim_{x\sim 1}} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} \geqq \underline{\lim_{x\sim 1}} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} \geqq L.$$

It follows that if $U = L$

$$\lim_{x\sim 1} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} = \lim_{n\sim\infty} \frac{\beta_n}{a_n}.$$

In case $\frac{\beta_n}{a_n}$ diverges to ∞, as $n \sim \infty$, n can be so chosen that $l_{mn} > N$, where N is a prescribed positive number, for every value of n. We have then, for $x < 1$,

$$\frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} > N\left[1 - \frac{a_0 + a_1 x + \dots + a_{m-1} x^{m-1}}{\sum\limits_0^\infty a_n x^n}\right] + l\,\frac{a_0 + a_1 x + \dots + a_{m-1} x^{m-1}}{\sum\limits_0^\infty a_n x^n},$$

where l is the lower boundary of all the numbers $\frac{\beta_0}{a_0}, \frac{\beta_1}{a_1}, \dots$.

We have now $\underline{\lim_{x\sim 1}} \frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} \geqq N$; and since N is arbitrarily large, it follows that $\frac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n}$ diverges to ∞, as $x \sim 1$.

128. In order to apply the above lemma, let $a_0 + a_1 x + \dots$ be a power series of which the interval of convergence is $(-1, 1)$.

(1) Let $\alpha_n = 1$, $\beta_0 = a_0$, $\beta_1 = a_0 + a_1 = s_1$, ... and in general,

$$\beta_n = a_0 + a_1 + \dots + a_n = s_n.$$

We have then the limit, as $x \sim 1$, of $\dfrac{\sum\limits_0^\infty s_n x^n}{(1-x)^{-1}}$, or of $\sum\limits_0^\infty a_n x^n$.

It follows that, if s_n is bounded for all values of n, with U and L as upper and lower boundaries,

$$U \geqq \overline{\lim_{x\sim 1}}\, s(x) \geqq \underline{\lim_{x\sim 1}}\, s(x) \geqq L.$$

Moreover, if s_n diverges to ∞, or to $-\infty$, so also does $\lim\limits_{x\sim 1} s(x)$.

Thus, *if the series $a_0 + a_1 + \dots + a_n + \dots$ oscillates between finite limits of indeterminancy, the upper and lower limits of $s(x)$, as $x \sim 1$, are in the interval formed by the limits of $\sum\limits_0 a_n$; and if the series diverges to $+\infty$, or to $-\infty$, then $\lim\limits_{n\sim\infty} s(x) = +\infty$, or $-\infty$.*

This is a generalization of Abel's theorem, which includes that theorem as a particular case.

For the case in which $a_n > 0$, for $n \geqq 0$, the last part of this theorem was proved by Abel*.

(2) Let $S_n^{(1)}$ denote, as in § 47, the sum

$$s_n + s_{n-1} + \dots + s_0, \text{ or } a_n + 2a_{n-1} + 3a_{n-2} + \dots + (n+1)\, a_0,$$

and let $\beta_n = S_n^{(1)}$, $\alpha_n = n + 1$; we have then to consider the limit, as $x \sim 1$, of $\dfrac{S_0^{(1)} + S_1^{(1)} x + S_2^{(1)} x^2 + \dots}{1 + 2x + 3x^2 + \dots}$, or of $(1-x)^2 \{S_0^{(1)} + S_1^{(1)} x + S_2^{(2)} x^2 + \dots\}$, which is equal to $a_0 + a_1 x + a_2 x^2 + \dots$.

The Cesàro sum of the series is defined as the limit of $\dfrac{S_n^{(1)}}{n+1}$, when that limit exists. In accordance with the lemma we obtain the following theorem due to Frobenius†:

If the series $a_0 + a_1 + a_2 + \dots$ is summable $(C, 1)$, then the sum-function $s(x)$ converges, as $x \sim 1$, to the Cesàro sum of $\sum\limits_{n=0}^\infty a_n$.

Moreover we obtain the following extension of this theorem:

If the series $\sum\limits_{n=0}^\infty a_n$ is bounded $(C, 1)$, then the sum-function $s(x)$, as $x \sim 1$, has its upper and lower limits in the interval bounded by the upper and lower Cesàro sums of $\sum\limits_{n=0} a_n$.

* See *Crelle's Journal*, vol. LXXXIX (1880), p. 262.

† *Œuvres*, vol. II, p. 203.

(3) Let $S_n^{(r)}$ denote, as in § 47,

$$s_n + \binom{r}{1} s_{n-1} + \binom{r+1}{2} s_{n-2} + \dots + \binom{r+n-1}{n} s_0;$$

then $C_n^{(r)}$ denotes $S_n^{(r)} \Big/ \binom{r+n}{n}$. It will here be assumed that r has some value, not necessarily integral, that is $\geqq 0$.

Let $\beta_n = S_n^{(r)}$, $\alpha_n = \binom{r+n}{n}$; then the limit in the lemma is that of

$$\frac{\sum\limits_0^\infty S_n^{(r)} x^n}{\sum\limits_0^\infty \binom{r+n}{n} x^n}, \text{ or of } (1-x)^{r+1} \sum_0^\infty S_n^{(r)} x^n,$$

which is equal to $a_0 + a_1 x + \dots + a_n x^n + \dots$.

We thus obtain the following theorems which include as particular cases those given above in (1) and (2):

If the series $a_0 + a_1 + a_2 + \dots$ is summable (C, r), for some value of r $(\geqq 0)$, the sum-function $s(x)$ converges, as $x \sim 1$, to the sum (C, r) of the series $\sum\limits_0 a_n$.

If the series $a_0 + a_1 + a_2 + \dots$ is bounded (C, r), for some value of r $(\geqq 0)$, then $\lim\limits_{x \sim 1} s(x)$ has finite limits of indeterminancy, in the interval bounded by the upper and lower sums (C, r) of $\sum\limits_{n=0} a_n$.

It should be observed that $\lim\limits_{x \sim 1} s(x)$ may exist in a case in which $\sum\limits_{n=0} a_n$ is neither convergent nor summable (C, r) for any value of r. An example of this has been given by Landau*.

Let $f_m(x) = (1+x)^{-m-1} = \sum\limits_{n=0}^\infty (-1)^n \binom{n+m}{m} x^n$, so that

$$\lim_{x \sim 1} f_m(x) = \frac{1}{2^{m+1}}, \text{ and } |f_m(x)| < 1, \text{ for } 0 < x < 1.$$

Consider the function $f(x)$, or $\frac{1}{1+x} e^{\frac{1}{1+x}}$, defined by $\sum\limits_{m=0}^\infty \frac{1}{m!} f_m(x)$; it is easily seen that, for $|x| < 1$, $f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$ where

$$(-1)^n a_n = \sum_{m=0}^\infty \frac{1}{m!} \binom{n+m}{m}.$$

It is seen that $$\lim_{x \sim 1} f(x) = \tfrac{1}{2} e^{\frac{1}{2}};$$

* *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1916,

but the series $a_0 + a_1 + a_2 + \dots$ is not summable (C, r) for any value of r; for if it were so summable, $a_n = o(n^r)$, for some integral value of r ($\geqq 0$). It appears, however, that $(-1)^n a_n > \frac{1}{(r+1)!}\binom{n+r+1}{r+1} > \frac{n^{r+1}}{\{(r+1)!\}^2}$, which is inconsistent with $a_n = o(n^r)$. It should be observed that the series for $f_m(1)$ is summable $(C, m+1)$, but is not summable (C, m); and thus $f(1)$ appears as the sum of terms which are summable $(C, 1)$, $(C, 2)$, ..., respectively; and thus the series for $f(m)$ is not summable with any order. This principle of construction may be employed to construct other series which have the required property; for example,

$$f(x) = e^{\sqrt{1}}x - e^{\sqrt{2}}x^2 + e^{\sqrt{3}}x^3 - \dots.$$

(4) Let $\beta_n = a_n$, $\alpha_n = \frac{p(p+1)\dots(p+n-1)}{n!}$,

then $$\lim_{x \sim 1}[(1-x)^p s(x)] = \lim_{n \sim \infty} \frac{a_n n!}{p(p+1)\dots(p+n-1)};$$

provided the limit on the right-hand side exists. It is easily seen by means of Stirling's theorem that

$$\lim_{n \sim \infty} \frac{a_n \Gamma(p)\Gamma(n+1)}{\Gamma(n+p)} = \Gamma(p) \lim_{n \sim \infty} \frac{a_n}{n^{p-1}}.$$

Thus we have the following theorem*:

If $p > 0$, *and* $\lim_{n \sim \infty} \frac{a_n}{n^{p-1}} = c$, *then* $\lim_{x \sim 1}[(1-x)^p s(x)] = c\Gamma(p)$.

In a similar manner, by taking $\beta_n = s_n$, $\alpha_n = \frac{(p+1)(p+2)\dots(p+n)}{n!}$, it can be shewn that:

If $p > 0$, *and* $\lim_{n \sim \infty} \frac{s_n}{n^p} = c$, *then* $\lim_{x \sim 1}[(1-x)^p s(x)] = c\Gamma(p+1)$.

These theorems express the mode of divergence of $s(x)$, as $x \sim 1$, in the cases in which the coefficients satisfy the prescribed conditions.

The following very general theorem has been established by Pringsheim†:

Denoting $(\log u)^{\alpha_1}(\log\log u)^{\alpha_2}(\log\log\log u)^{\alpha_3}\dots$ *by* $L(u)$; *if*

$$\lim_{n \sim \infty} \frac{s_n}{n^{\alpha} L(n)} = c$$

where $\alpha \geqq 0$, *the indices* $\alpha_1, \alpha_2, \alpha_3, \dots$ *being such that* $n^{\alpha} L(u)$ *tends to infinity, then* $$\lim_{x \sim 1}\left[\frac{(1-x)^{\alpha}}{L\left(\frac{1}{1-x}\right)} s(x)\right] = c\Gamma(\alpha+1).$$

* See Appell, *Comptes Rendus*, vol. LXXXVII (1878), p. 689; see also Pringsheim, *Acta Math.* vol. XXVIII (1904), p. 11.

† *Loc. cit.*, p. 29.

Appell's theorem is the case which arises when $a_1, a_2, \ldots$ are all zero. A theorem closely related to the general theorem was given by Lasker*.

Various theorems of a similar character will be found in Hardy's tract†, *Orders of Infinity*, p. 56. Reference may also be made to a memoir by Bromwich‡ in which various extensions of Abel's lemma are utilized.

The converse of this general theorem has been proved by Hardy and Littlewood§, in the case in which all the coefficients are positive. Thus, for example, if $\lim\limits_{x\sim 1}[(1-x)^a s(x)] = k$, $a_n \geqq 0$; $a > 0$ then

$$\lim_{n\sim\infty}\frac{s_n}{n^a} = \frac{k}{\Gamma(1+a)}.$$

The condition $a_n \geqq 0$ is essential, as may be seen, for example, by taking $s(x) = (1-x)^{-\frac{1}{2}} + (1+x)^{-1}$, $a = \frac{1}{2}$.

129. It may be observed that the lemma of § 127, which has been applied in these cases, can be made wider in its scope. Instead of

$$\frac{\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots}{a_0 + a_1 x + a_2 x^2 + \ldots},$$

we may take the equivalent form $\dfrac{\sum\limits_{n=0}^{\infty} S_n^{(r)} x^n}{\sum\limits_{n=0}^{\infty} S_n'^{(r)} x^n}$, which is obtained by multiplying the numerator and the denominator by $(1-x)^{-r-1}$. Here $S_n^{(r)}$, $S_n'^{(r)}$ denote, as in § 47, the expressions

$$\beta_n + (r+1)\beta_{n-1} + \frac{(r+1)(r+2)}{2}\beta_{n-2}, \ldots,$$

$$a_n + (r+1)a_{n-1} + \frac{(r+1)(r+2)}{2}a_{n-2} + \ldots.$$

Since $a_0, a_1, a_2, \ldots$ are all positive, so also are $S_0'^{(r)}, S_1'^{(r)}, S_2'^{(r)} \ldots$, where $r > 0$. By applying the lemma, in the case in which $\lim\limits_{n\sim\infty}\dfrac{S_n^{(r)}}{S_n'^{(r)}}$ exists, we have the theorem that:

If $\Sigma a_n x^n$, $\Sigma \beta_n x^n$ both converge within the interval $(-1, 1)$, a_n being positive and such that $\sum\limits_0^\infty a_n$ diverges, then, if $\dfrac{S_n^{(r)}}{S_n'^{(r)}}$ $(r \geqq 0)$ has a limit, finite or infinite, as $n \sim \infty$, $\lim\limits_{x\sim 1}\dfrac{\sum\limits_0^\infty \beta_n x^n}{\sum\limits_0^\infty a_n x^n} = \lim\limits_{n\sim\infty}\dfrac{S_n^{(r)}}{S_n'^{(r)}}$.

* *Phil. Trans.* (A), vol. CXCVI (1901), p. 444.

† Cambridge tracts in Mathematics and Mathematical Physics, No. 12 (1910).

‡ *Proc. Lond. Math. Soc.* (2), vol VI (1908), p. 58.

§ *Ibid.* (2), vol. XIII (1914), p. 174.

In particular, if $r = 0$, $\lim_{x\sim 1} \dfrac{\sum_0^\infty \beta_n x^n}{\sum_0^\infty a_n x^n} = \lim_{n\sim\infty} \dfrac{\beta_0 + \beta_1 + \dots + \beta_n}{a_0 + a_1 + \dots + a_n}$, *provided the limit on the right hand side exists.*

The particular case of the theorem was given by Cesàro.

130. Abel's theorem suggests the general question as to conditions under which the two repeated limits

$$\lim_{n\sim\infty} \lim_{x\sim 1} (a_0 + a_1 x + \dots + a_n x^n), \quad \lim_{x\sim 1} \lim_{n\sim\infty} (a_0 + a_1 x + \dots + a_n x^n)$$

have one and the same value, where $(-1, 1)$ is the interval of convergence of the infinite series $\sum_{n=0}^{\infty} a_n x^n$. The first of these repeated limits is the single limit $\lim_{n\sim\infty} (a_0 + a_1 + \dots + a_n)$, or $\lim_{n\sim\infty} s_n(1)$; and the second is $\lim_{x\sim 1} s(x)$, where $s(x)$ denotes the sum-function of the infinite series. Abel's theorem itself asserts that, when $\lim_{n\sim\infty} s_n(1)$ exists, as a finite number, or is infinite, then $\lim_{x\sim 1} s(x)$ exists, and has the same value as $\lim_{n\sim\infty} s_n(1)$. The converse question arises, whether it is possible, with suitable restrictions to infer the existence of $\lim_{n\sim\infty} s_n(1)$ from hypotheses concerning $s(x)$? In connection with this question, a series of investigations have been made, leading to theorems which are converse to that of Abel, or to its extensions; and in view of the fact that the earliest and simplest of them was given by Tauber, they are designated Tauberian theorems.

If all the numbers $a_0, a_1, a_2, \dots$ are positive, the function

$$a_0 + a_1 x + \dots + a_n x^n$$

is monotone increasing, both with respect to x and with respect to n. In accordance with a theorem of § 30, the existence of either of the repeated limits involves that of the other, and the equality of the two. We have accordingly the following theorem:

If $a_0 + a_1 x + \dots + a_n x^n + \dots$ *has* $(-1, 1)$ *for its interval of convergence, and all the numbers* $a_0, a_1, a_2, \dots$ *be* $\geqq 0$, *then* $\lim_{n\sim\infty} s_n(1) = \lim_{x\sim 1} s(x)$, *provided either of these limits exists as a finite number, or is infinite.*

It is easily seen that the theorem remains correct if, for a finite set of values of n, the condition $a_n \geqq 0$ is not satisfied.

The theorem first established by Tauber* is the following:

If $a_0 + a_1 x + \dots + a_n x^n + \dots$ *have* $(-1, 1)$ *for its interval of convergence, and if* $n a_n = o(1)$, *then if* $\lim_{x\sim 1} s(x)$ *has a definite value* $s(1)$, *the series* $a_0 + a_1 + \dots + a_n + \dots$ *converges to* $s(1)$.

* See *Monatshefte f. Math. u. Phys.* vol. VIII (1897), p. 273.

That the converse of Abel's theorem does not hold without restriction is exemplified by the series $1 - x + x^2 - x^3 + \dots$, for which $s(x) = \frac{1}{1+x}$, and $\lim_{x\sim 1} s(x) = \frac{1}{2}$, whereas $\lim_{x\sim 1} s_n(x)$ does not exist, but oscillates between 0 and 1. In this case the theorem of Frobenius is however applicable, since the sum $C(1)$ of the series $1 - 1 + 1 - 1 + \dots$ is $\frac{1}{2}$, which is equal to $\lim_{x\sim 1} s(x)$. Historically this example is of interest, as it was supposed by Leibnitz and subsequent writers that the sum of the series

$$1 - 1 + 1 - 1 + \dots$$

might in some sense be regarded as $\frac{1}{2}$.

To prove Tauber's theorem, let m be so chosen that $n \mid a_n \mid < \epsilon$, for $n \geqq m$, then $\left|\sum_m^\infty a_n x^n\right| < \sum_m^\infty \frac{\epsilon}{n} x^n < \frac{\epsilon}{m}\frac{1}{1-x} < \epsilon$, if x be chosen to be equal to $1 - \frac{1}{m}$. Next we have

$$\left|\sum_0^{m-1} a_n - \sum_0^\infty a_n x^n\right| < \epsilon + (1-x)\{|a_1| + 2|a_2| + \dots + (m-1)|a_{m-1}|\}$$

since $|a_r(1-x^r)| < (1-x) r |a_r|$.

Using the theorem of § 6, Ex. 2, we see that, as $n|a_n| = o(1)$, we have $\sum_1^{m-1} n|a_n| = o(m)$; and therefore, when $x = 1 - \frac{1}{m}$,

$$\left|\sum_0^{m-1} a_n - s\left(1 - \frac{1}{m}\right)\right| < \epsilon + \frac{1}{m}\sum_1^{m-1} n|a_n| < 2\epsilon,$$

provided m be chosen sufficiently large. Letting m increase indefinitely, we have $\left|\sum_0^\infty a_n - \lim_{x\sim 1} s(x)\right| \leqq 2\epsilon$; and since ϵ is arbitrary, $\sum_0^\infty a_n = \lim_{x\sim 1} s(x)$, which establishes the theorem.

This proof also suffices to shew that if $\overline{\lim_{x\sim 1}}\, s(x)$, $\underline{\lim_{x\sim 1}}\, s(x)$ are finite, and different from one another, $\sum_{n=0} a_n$ oscillates, and has these two numbers for its upper and lower limits.

It has been proved that, when $na_n = o(1)$, then

$$\lim_{m\sim\infty}\left|\sum_{n=0}^{m-1} a_n - \sum_{n=0}^{\infty} a_n\left(1 - \frac{1}{m}\right)^n\right| = 0.$$

This suffices to prove the more general theorem that:

If $na_n = o(1)$, and if either of the limits $\sum_{n=0}^\infty a_n$, $\lim_{x\sim 1}\sum_{n=0}^\infty a_n x^n$ exists, then the other exists, and the two have the same value. Moreover if either $\sum_{n=0} a_n$, $\lim_{x\sim 1}\sum_{n=0}^\infty a_n x^n$ oscillates between finite limits, then the other oscillates between the same limits.

131. The following more general theorem is also due to Tauber:

If $a_1 + 2a_2 + 3a_3 + \dots + na_n = o(n)$, *and* $\lim_{x\sim 1} s(x)$ *exists, then* $\sum_{n=0} a_n$ *converges to* $\lim_{x\sim 1} s(x)$. *The two conditions are both sufficient and necessary for the convergence of* $\sum_{n=0} a_n$.

Let $u_n = a_1 + 2a_2 + 3a_3 + \dots + na_n$; then $a_n = \frac{1}{n}(u_n - u_{n-1})$, where $n > 0$, $u_0 = 0$. We have then

$$\begin{aligned}\sum_{r=1}^{r=n} a_r x^r &= \sum_{r=1}^{r=n} \frac{u_r}{r} x^r - \sum_{r=1}^{r=n-1} \frac{u_r}{r+1} x^{r+1} \\ &= \frac{1}{n} u_n x^n + \sum_{r=1}^{n-1} \frac{u_r}{r(r+1)} x^r + (1-x) \sum_{r=1}^{n-1} \frac{u_r}{r+1} x^r.\end{aligned}$$

The series $\sum_{n=1} u_n x^n$ is convergent, since $u_n = o(n)$ (§ 124), and thus $\lim_{n\sim\infty} u_n x^n = 0$. It is thus seen that

$$\sum_{r=1}^{\infty} a_r x^r = \sum_{r=1}^{\infty} \frac{u_r}{r(r+1)} x^r + (1-x) \sum_{r=1}^{\infty} \frac{u_r}{r+1} x^r.$$

Employing the theorem of § 128 (4), since $\lim_{n\sim\infty} \frac{u_n}{n} = 0$, we have

$$\lim_{x\sim 1} \left[(1-x) \sum_{r=1}^{\infty} \frac{u_r}{r+1} x^r\right] = 0;$$

and thus it appears that

$$\lim_{x\sim 1} \sum_{r=1}^{\infty} \frac{u_r}{r(r+1)} x^r = \lim_{x\sim 1} s(x) - a_0.$$

The coefficient in the series on the left-hand side is $o\left(\frac{1}{r}\right)$, and therefore $\sum_{r=1} \frac{u_r}{r(r+1)}$ converges to $\lim_{x\sim 1} s(x) - a_0$.

We have also
$$\begin{aligned} s_n &= a_0 + \frac{1}{n} u_n + \sum_{r=1}^{n-1} \frac{u_r}{r(r+1)} \\ &= a_0 + \sum_{r=1}^{r=n} \frac{u_r}{r(r+1)} + \frac{u_n}{n+1},\end{aligned}$$

and therefore $\lim_{n\sim\infty} s_n = \lim_{x\sim 1} s(x)$, which is the result to be proved. That the conditions are necessary has been shewn in § 6, Ex. 1.

132. The theorem of Tauber, established in § 130, that when $\lim_{x\sim 1} s(x)$ exists, and $na_n = o(1)$, then $\sum_{n=0} a_n$ converges to $\lim_{x\sim 1} s(x)$, was extended by Littlewood*, who obtained the remarkable result that the condition $na_n = o(1)$, can be replaced by $na_n = O(1)$. This result has been shewn by Littlewood to be final, in the sense that, in $na_n = O(1)$, $O(1)$ cannot be

* *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 434.

replaced by $O\{\phi(n)\}$, where $\phi(n)$ is any function which diverges to $+\infty$ as $n \sim \infty$. Ingham* has proved that, if $\phi(n)$ be any such function, a function $s(x) = \sum_{n=0} a_n x^n$ can be so defined within the interval $(-1, 1)$ that (1) $a_n = O\left(\frac{\phi(n)}{n}\right)$, (2) $\lim_{x\sim 1} s(x)$ has an arbitrarily assigned value, and (3) $\sum_{n=0} a_n$ is either not summable or has an assigned index of summability. Littlewood had surmised the correctness of (3).

The still more general theorem was obtained by Hardy and Littlewood† that, in Littlewood's theorem, the condition that na_n should be bounded on one side may replace the condition $na_n = O(1)$, in Tauber's theorem. This result includes that of Littlewood as the special case when na_n is bounded on both sides. It will accordingly be sufficient to prove the general theorem of Hardy and Littlewood. This will be done in § 133.

In the first instance the following theorem due to Hardy and Littlewood will be established‡:

If the series $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $0 \leq x < 1$, *and the coefficients* a_n *are all non-negative, then if* $\lim_{x\sim 1}\{(1-x)s(x)\} = 1$, *the relation* $\lim_{n\sim\infty} \frac{s_n}{n} = 1$ *is satisfied, where* $s_n = a_0 + a_1 + \ldots + a_n$.

This theorem is a particular case of the general converse theorem referred to in § 128.

It is convenient to prove three Lemmas which can be employed in the proof of the theorem:

Lemma I. *If* $f(x)$ *be a function which has a differential coefficient at each point of the open interval* $(0, 1)$, *and if* $\lim_{x\sim 1}[(1-x)^a f(x)] = 1$, *where* $a > 0$; *and if further* $f'(x)$ *continually increases with* x, *then*

$$\lim_{x\sim 1}[(1-x)^{a+1} f'(x)] = a.$$

Let x, x_1 be two points such that $0 < x < x_1 < 1$; then, from I, § 262,

$$f'(x) \leq \frac{f(x_1) - f(x)}{x_1 - x} \leq f'(x_1).$$

If $x_1 - x = \lambda(1-x)$, we may suppose x and x_1 to increase together in such a manner that λ has a constant value; then

$$\lim_{x\sim 1}\left[(1-x)^{a+1}\frac{f(x_1) - f(x)}{x_1 - x}\right] = \frac{(1-\lambda)^{-a} - 1}{\lambda}.$$

* *Proc. Lond. Math. Soc.* (2), vol. XXIII (1924), p. 326.

† *Ibid.* (2), vol. XIII (1913), p. 188.

‡ *Proc. Lond. Math. Soc.* (2), vol. XIII (1913), p. 174. The form of the proof in the text is founded upon that given by Landau in his work, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, pp. 45–55.

It follows that

$$\overline{\lim_{x\sim 1}}\,[(1-x)^{\alpha+1}f'(x)] \leqq \frac{(1-\lambda)^{-\alpha}-1}{\lambda} \leqq \underline{\lim_{x_1\sim 1}}\,[(1-x)^{\alpha+1}f'(x_1)];$$

for every value of λ in the open interval $(0, 1)$. Since $\lim\limits_{\lambda\sim 0}\frac{(1-\lambda)^{-\alpha}-1}{\lambda}=\alpha$, we have

$$\overline{\lim_{x\sim 1}}\,[(1-x)^{\alpha+1}f'(x)] \leqq \alpha,$$

and $$\underline{\lim_{x_1\sim 1}}\,[(1-x_1)^{\alpha+1}f'(x_1)] \geqq \lim_{\lambda\sim 0}\frac{(1-\lambda)^{-\alpha}-1}{\lambda(1-\lambda)^{\alpha+1}} \geqq \alpha.$$

Therefore $$\lim_{x\sim 1}\,[(1-x)^{\alpha+1}f'(x)] = \alpha.$$

Lemma II. *If $s(x)$ denote the sum-function of the series $\sum\limits_{n=0} a_n x^n$, convergent in the open interval $(0, 1)$, and all the coefficients a_n are non-negative; and if $\lim\limits_{x\sim 1}[(1-x)^\beta s(x)] = 1$, where $\beta > 0$, then*

$$\lim_{x\sim 1}[(1-x)^{\beta+r}\sum_{n=0}^{\infty} a_n n^r x^n] = \beta(\beta+1)\ldots(\beta+r-1),$$

when r may have any positive integral value.

If, in Lemma I, we write β for α, and $s(x)$ for $f(x)$, we have

$$\lim_{x\sim 1}[(1-x)^{\beta+1}\sum_{n=1}^{\infty} n a_n x^n] = \beta;$$

and thus the theorem holds good for $r = 1$. Assuming that it is true for $r-1$, or that

$$\lim_{x\sim 1}[(1-x)^{\beta+r-1}\sum_{n=1}^{\infty} n^{r-1} a_n x^n] = \beta(\beta+1)\ldots(\beta+r-2),$$

by writing in Lemma I, $\alpha = \beta + r - 1$, $f(x) = \sum\limits_{x=1}^{\infty} n^{r-1} a_n x^n$, we have

$$\lim_{x\sim 1}[(1-x)^{\beta+r}\sum_{n=1}^{\infty} n^r a_n x^n] = \beta(\beta+1)\ldots(\beta+r-1).$$

If we introduce, instead of x, the variable t, where $x = e^{-t}$, then t is capable of having all positive values; and the lemma may be stated in the form

$$\lim_{t\sim 0}[t^{\beta+r}\sum_{n=1}^{\infty} n^r a_n e^{-nt}] = \beta(\beta+1)\ldots(\beta+r),$$

provided $$\lim_{t\sim 0}[t^\beta\sum_{n=0}^{\infty} a_n e^{-nt}] = 1, \quad a_n \geqq 0.$$

Lemma III. *If $\epsilon\ (>0)$ be a prescribed number, $\phi_1(m)$, $\phi_2(m)$ can be determined as functions of m, which is restricted to be a positive integer, so that $\phi_1(m) < m < \phi_2(m)$, $m - \phi_1(m) = o(m)$, $\phi_2(m) - m = o(m)$, and that*

$$\sum_{n\leqq\phi_1(m)/t} n^m e^{-nt} < \frac{\epsilon m!}{t^{m+1}},$$

$$\sum_{n>\phi_2(m)/t} n^m e^{-nt} < \frac{\epsilon m!}{t^{m+1}},$$

for all values of m, not less than an integer m_ϵ dependent on ϵ, and for $0 < t < 1$.

The first summation is from $n = 1$, to the largest integral value of n not greater than $\phi_1(m)/t$, and the second is taken for all integral values of n greater than $\phi_2(m)$.

We have, since $x^m e^{-xt}$ has, for $0 < t$, its maximum for $x = \frac{m}{t}$,

$$\sum_{n \leqq \phi_1(m)/t} n^m e^{-nt} < \frac{m}{t}\left\{\frac{\phi_1(m)}{t}\right\}^m e^{-\phi_1(m)} < \frac{1}{t^{m+1}} e^{\log m + m \log \phi_1(m) - \phi_1(m)},$$

where it is assumed that $\phi_1(m) < m$.

Let it be assumed that $\phi_1(m) = m - m^k$, where $0 < k < 1$; we then have

$$\log m + m \log \phi_1(m) - \phi_1(m) = \log m + m\left[\log m - m^{k-1} - \tfrac{1}{2}m^{2k-2} - O(m^{3k-3})\right] - m + m^k,$$

and this is equal to $\log m + m \log m - m - \frac{1}{2}m^{2k-1} - O(m^{3k-2})$, or to $(m+1)\log m - m - \frac{1}{2}m^{2k-1} + O(m^{3k-2})$.

From Stirling's theorem we have

$$\log m! = m \log m - m + O(\log m),$$

or

$$m! = e^{m \log m - m + O(\log m)};$$

thus we have $\displaystyle\sum_{n \leqq \phi_1(m)/t} n^m e^{-mt} < \frac{m!}{t^{m+1}} e^{-\frac{1}{2}m^{2k-1} + O(\log m) + O(m^{3k-2})}$.

If we take $\frac{1}{2} < k < 1$, the exponential factor converges to 0, as $m \sim \infty$, since $\log m = o(m^{2k-1})$, and $O(m^{3k-2}) = o(m^{2k-1})$. An integer m_ϵ can be so chosen that the factor is $< \epsilon$, provided $m \geqq m_\epsilon$. The first part of the theorem has thus been established, the value of $\phi(m)$ being $m - m^k$, where k is any number such that $\frac{1}{2} < k < 1$.

To prove the second part of the theorem, we have

$$\sum_{n > \phi_2(m)/t} n^m e^{-nt} < \left(\frac{\phi_2(m)}{t}\right)^m e^{-\phi_2(m)}\{1 + \lambda + \lambda^2 + \ldots\},$$

where λ is not less than $\left(1 + \frac{t}{\phi_2(m)}\right)^m e^{-t}$: we have then

$$\sum_{n > \phi_2(m)/t} n^m e^{-nt} < \left(\frac{\phi_2(m)}{t}\right)^m \frac{e^{-\phi_2(m)}}{1 - \lambda}.$$

Since $$\left\{1 + \frac{t}{\phi_2(m)}\right\}^m e^{-t} < e^{\frac{mt}{\phi_2(m)} - t} < e^{-t\left\{\frac{\phi_2(m) - m}{\phi_2(m)}\right\}},$$

let us take $\phi_2(m) = m + m^k - 2$, where $\frac{1}{2} < k < 1$; it is then sufficient to take $\lambda = e^{-t\frac{m^k - 2}{m + m^k - 2}}$, and thus $\frac{1}{1-\lambda} = \frac{e^\mu}{e^\mu - 1} < \frac{e}{\mu}$, where μ denotes $\frac{m^k - 2}{m + m^k - 2}t$, which is positive when $m > 3$, and is < 1.

The sum of the series is less than

$$\frac{e}{t^{m+1}} \frac{(m + m^k)^{m+1}}{m^k - 2} e^{-m-m^k+2},$$

and this is equal to $\dfrac{1}{t^{m+1}} e^{1+(m+1)\log(m+m^k)-m-m^k+2-\log(m^k-2)}$,

which is $\qquad \dfrac{1}{t^{m+1}} e^{O(\log m)+m\log m-\frac{1}{2}m^{2k-1}+O(m^{2k-2})-m-+O(m^{3k-2})}$.

Employing again the expression $m! = e^{m\log m - m + O(\log m)}$, we see that

$$\sum_{n>\phi_2(m)/t} n^m e^{-nt} < \frac{m!}{t^{m+1}} e^{-\frac{1}{2}m^{2k-1}+O(\log m)+O(m^{3k-2})} < \frac{\epsilon m!}{t^{m+1}},$$

provided m is sufficiently large, say $\geqq m_\epsilon$, where m_ϵ may be taken to be the same number in both parts of the theorem; the function $\phi_2(m)$ being taken to be $m + m^k - 2$, where k is between $\frac{1}{2}$ and 1.

It is clear that $(m+1) - (m+1)^k > m - m^k$, for all values of m that are sufficiently large; also we have $(m+1) + (m+1)^k - 2 < m + m^k$. In the theorem, we may change m into $m+1$, then, employing these facts, we have the following corollary:

$$\sum_{n\leqq(m-m^k)/t} n^{m+1}e^{-mt} < \frac{\epsilon\,(m+1)!}{t^{m+2}},$$

$$\sum_{n>(m+m^k)/t} n^{m+1}e^{-mt} < \frac{\epsilon\,(m+1)!}{t^{m+2}},$$

where ϵ is a prescribed positive number, provided m is not less than some integer, dependent on ϵ, and k is a fixed number between $\frac{1}{2}$ and 1.

We are now in a position to prove the theorem of Hardy and Littlewood stated above.

We have $s_n = \sum\limits_{r=0}^{r=n} a_r \leq \sum\limits_{r=0}^{r=n} a_r e^{\frac{n-r}{n}}$; therefore $s_n < e\sum\limits_{r=0}^{r=n} a_r e^{-\frac{r}{n}} < e\sum\limits_{r=0}^{\infty} a_r e^{-\frac{r}{n}}$; hence $s_n < es\left(e^{-\frac{1}{n}}\right)$, or $s_n = O(n)$. We may suppose that $s_n < cn$, where c is some positive number. Since

$$\sum_{n=0}^{\infty} s_n x^n = \frac{1}{1-x} s(x),$$

where $0 < x < 1$, we have

$$\lim_{x\sim 1}\left[(1-x)^2 \sum_{n=0}^{\infty} s_n x^n\right] = 1.$$

Employing Lemma II, we have, for every integer m (> 0),

$$\lim_{x\sim 1}\left[(1-x)^{m+2} \sum_{n=0}^{\infty} s_n n^m x^n\right] = (m+1)!$$

or

$$\lim_{t\sim 0}\left[t^{m+2} \sum_{n=0}^{\infty} s_n n^m e^{-nt}\right] = (m+1)!.$$

The particular case when $a_n = 1$, for every value of n, gives

$$\lim_{t \sim 0} \left[t^{m+1} \sum_{n=0}^{\infty} n^m e^{-nt} \right] = m!.$$

If I denote the interval $(m - m^k)/t \leqq n < (m + m^k)/t$, we have, by means of Lemma III, and the corollary,

$$\left| \sum_{n=0}^{\infty} n^m e^{-nt} - \sum_I n^m e^{-nt} \right| < 2\epsilon m!\, t^{-m-1};$$

and also $$\left| \sum_{n=0}^{\infty} s_n n^m e^{-nt} - \sum_I s_n n^m e^{-nt} \right| < 2c\epsilon\, (m+1)!\, t^{-m-2},$$

since $s_n < cn$. These results hold for all values of m not less than an integer m_ϵ.

For each value of m ($\geqq m_\epsilon$), there exists a number t_m, such that, for $0 < t < t_m$,

$$(1 - 3\epsilon)\, m!\, t^{-m-1} < \sum_I n^m e^{-nt} < (1 + 3\epsilon)\, m!\, t^{-m-1},$$

and $(1 - 3c\epsilon)\,(m+1)!\, t^{-m-2} < \sum_I s_n n^m e^{-nt} < (1 + 3c\epsilon)\,(m+1)!\, t^{-m-2}$.

Since $$s_{(m-m^k)/t} \sum_I n^m e^{-nt} \leqq \sum_I s_n n^m e^{-nt} \leqq s_{(m+m^k)/t} \sum_I n^m e^{-nt},$$

we have $$s_{(m-m^k)/t} < \frac{1 + 3c\epsilon}{1 - 3\epsilon} \frac{m+1}{t},$$

and $$s_{(m+m^k)/t} > \frac{1 - 3c\epsilon}{1 + 3\epsilon} \cdot \frac{m+1}{t}.$$

Since $(m - m^k)/t$, as t converges continuously to zero, takes successively all sufficiently large integral values, we have, for all values of n greater than some integer depending on ϵ and m,

$$\frac{1 - 3c\epsilon}{1 + 3\epsilon} \frac{m+1}{m + m^k} < \frac{s_n}{n} < \frac{1 + 3c\epsilon}{1 - 3\epsilon} \frac{m+1}{m - m^k}.$$

If η be an arbitrarily chosen positive number, ϵ can be so chosen that

$$1 - \eta < \frac{1 - 3c\epsilon}{1 + 3\epsilon}, \quad 1 + \eta > \frac{1 + 3c\epsilon}{1 - 3\epsilon};$$

then m can be chosen so that

$$1 - \eta < \frac{1 - 3c\epsilon}{1 + 3\epsilon} \cdot \frac{m+1}{m + m^k}, \quad 1 + \eta > \frac{1 + 3c\epsilon}{1 - 3\epsilon} \cdot \frac{m+1}{m - m^k}.$$

It is now seen that, for all sufficiently large values of n, the inequalities $1 - \eta < \frac{s_n}{n} < 1 + \eta$ are satisfied. Since η is arbitrary, it follows that $\lim\limits_{n \sim \infty} \frac{s_n}{n} = 1$.

133. The important theorem of Hardy and Littlewood, already referred to in § 132, will now be established; the theorem may be stated as follows:

If the series $\sum_{n=0}^{\infty} a_n x^n$, *of which the sum-function is* $s(x)$, *converge for* $0 \leqq x < 1$, *and if* $\lim_{x \sim 1} s(x)$ *has a definite value, then if* $a_n < \frac{K}{n}$, *where* K *is a fixed number, for all values of* n, *the series* $\sum_{n=0} a_n$ *converges to the value of* $\lim_{x \sim 1} s(x)$.

The condition $a_n < \frac{K}{n}$ may be replaced by $a_n > \frac{K}{n}$, for we have only to change the sign of all the terms of the series to reduce the latter condition to the former one. There is no loss of generality if we suppose $\lim_{x \sim 1} f(x)$ to have the value zero, as this only involves an alteration in the value of a_0.

Besides the theorem of § 132, two further lemmas will be required.

Lemma I. *If* $w_m = \sum_{n=1}^{n=m} na_n = o(m)$, *and* $s(x) = \sum_{n=0}^{\infty} a_n x^n$, *where*

$$\lim_{x \sim 1} s(x) = l, \text{ then } \sum_{n=0}^{\infty} a_n = l.$$

This has already been proved in § 130.

Lemma II. *If* $f(x)$ *be a function defined for* $0 < x < 1$, *such that* $\lim_{x \sim 1} f(x) = 0$; *and if further* $f''(x)$ *exists everywhere in the open interval, and is such that* $(1-x)^2 f''(x) < K$, *where* K *is a fixed positive number, then* $\lim_{x \sim 1} [(1-x) f'(x)] = 0$.

Let $x < x_1 < 1$, then $f(x_1) - f(x) = (x_1 - x) f'(x) + \frac{1}{2}(x_1 - x)^2 f''(\xi)$, where ξ is in the interval (x, x_1). Let $x_1 - x = \lambda(1-x)$, where λ is kept fixed as x, x_1 vary continuously; we have then

$$(1-x) f'(x) = \frac{f(x_1) - f(x)}{\lambda} - \frac{\lambda}{2}(1-x)^2 f''(\xi)$$

$$> \frac{f(x_1) - f(x)}{\lambda} - \frac{\lambda K}{2}\left(\frac{1-x}{1-x_1}\right)^2 > \frac{f(x_1) - f(x)}{\lambda} - \frac{\lambda K}{2(1-\lambda)^2}.$$

If x converges to 1, we have, λ being constant,

$$\underline{\lim} [(1-x) f'(x)] \geqq -\frac{\lambda K}{2(1-\lambda)^2},$$

and therefore since λ is arbitrary within the interval $(0, 1)$, we have $\underline{\lim} [(1-x) f'(x)] \geqq 0$.

Using the equation $f(x_1) - f(x) = (x_1 - x) f'(x_1) - \frac{1}{2}(x_1 - x)^2 f''(\xi')$, where ξ' is in the interval (x, x_1), we find that

$$(1 - x_1) f'(x_1) < \frac{1-\lambda}{\lambda}\{f(x_1) - f(x)\} + \frac{\lambda(1-\lambda)}{2}\left(\frac{1-x}{1-x_1}\right)^2 K;$$

and hence $\overline{\lim_{x_1 \sim 1}}\, [(1 - x_1) f'(x_1) \leq \frac{\lambda K}{2(1-\lambda)}$. As before, we have

$$\overline{\lim_{x_1 \sim 1}}\, [(1 - x_1) f'(x_1)] \leq 0;$$

and it is now seen that $\lim_{x \sim 1} [(1 - x) f'(x)] = 0$.

In order to prove the main theorem, if $s(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n < \frac{K}{n}$, and $\lim_{x \sim 1} s(x) = l$, we have

$$s''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} < K \sum_{n=2}^{\infty} (n-1) x^{n-2} < K \sum_{n=1}^{\infty} n x^{n-1} < \frac{K}{(1-x)^2}.$$

Since $(1 - x)^2 s''(x) < K$, we have, by Lemma II, $\lim_{x \sim 1} [(1 - x) s'(x)] = 0$; and therefore $\lim_{x \sim 1} [x(1 - x) s'(x)] = 0$, or $\lim_{x \sim 1} \left[(1 - x) \sum_{n=1}^{\infty} \frac{n a_n}{K} x^n\right] = 0$. This is equivalent to $\lim_{x \sim 1} \left[(1 - x) \sum_{x=1}^{\infty} \left(1 - \frac{n a_n}{K}\right) x^n\right] = 1$; and since $1 - \frac{n a_n}{K} > 0$, we have, by applying the theorem of § 132, $\lim_{m \sim \infty} \left\{\frac{1}{m} \sum_{n=1}^{n=m} \left(1 - \frac{n a_n}{K}\right)\right\} = 1$, or $\lim_{m \sim \infty} \left\{\frac{1}{m} \sum_{n=1}^{m} n a_n\right\} = 0$, which may be written in the form $\sum_{n=1}^{n=m} n a_n = o(m)$. Employing Lemma I, we have $\sum_{n=0}^{\infty} a_n = l$. Therefore $\sum_{n=0}^{\infty} a_n$ converges to $\lim_{x \sim 1} \sum_{n=1}^{\infty} a_n x^n$, which is the theorem to be established.

It will be observed that the differentiation of a large number of terms of the series $\Sigma a_n e^{-nt}$ is the essential means by which this striking theorem is established. This process is exhibited in Lemma III of § 132; the reason for adopting it may be explained in general terms. The behaviour of a function $\sum_{n=0}^{\infty} c_n x^n$, in the general theory of functions, when it converges for all values of x, is more or less dominated by that of the maximum term. In case the interval of convergence is finite, this phenomenon of the maximum term does not naturally occur. By introducing a factor n^m by means of differentiation a large number of times, we may obtain a series in which it does occur. For $n^m x^n$ has a steep peak, when m is very large, which naturally accentuates the importance of the far away terms.

PROPERTIES OF POWER-SERIES

134. *A power-series with an interval of convergence, finite or infinite, cannot be such that every interval* $(-\delta, \delta)$, *interior to the interval of convergence, contains a point* x, *distinct from zero, for which* $s(x)$ *has the value zero.*

The function $s(x)$ is continuous in any interval $(-\delta, \delta)$, interior to the interval of convergence; the number δ may be so chosen that

$$|s(x) - a_0| < \epsilon,$$

if x is in the interval $(-\delta, \delta)$, where ϵ is an arbitrarily fixed positive number. If there be a value of x in $(-\delta, \delta)$ for which $s(x) = 0$, we have $|a_0| < \epsilon$; and, since ϵ is arbitrary, it follows that $a_0 = 0$.

The series $a_1 + a_2 x + a_3 x^2 + \ldots$ converges in the same interval as the original series, and its sum-function vanishes for some value of x which is not zero, and is contained in an interval $(-\delta', \delta')$. Hence, by the same argument as before, it is seen that $a_1 = 0$; proceeding in the same manner, it can be shewn that $a_2, a_3, \ldots$, all vanish. Thus no power-series with an interval of convergence exists which satisfies the prescribed condition.

The following theorem is an immediate corollary from the foregoing:

There cannot be two distinct power-series, each of which converges within some interval, such that, in every interval $(-\delta, \delta)$ *interior to the intervals of convergence, there is a point, distinct from zero, at which the sum-functions have an identical value.*

135. Let us suppose that the series

$$\begin{array}{l}
a_{11} + a_{12}x + a_{13}x^2 + \ldots + a_{1r}x^{r-1} + \ldots \\
a_{21} + a_{22}x + a_{23}x^2 + \ldots + a_{2r}x^{r-1} + \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{n1} + a_{n2}x + a_{n3}x^2 + \ldots + a_{nr}x^{r-1} + \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

are all absolutely convergent at the point $x = 1$, where n may have all the values in the sequence of positive integers. Each series is then absolutely and uniformly convergent in the interval $(-1, 1)$.

Let $u_n(x)$ denote the sum of the nth series in the interval $(-1, 1)$, and let U_n denote the sum of the series

$$|a_{n1}| + |a_{n2}| + \ldots + |a_{nr}| + \ldots.$$

If the series $U_1 + U_2 + \ldots + U_n + \ldots$ be convergent, the series

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

is, in accordance with § 77, uniformly and absolutely convergent in $(-1, 1)$; it follows that $s(x)$ the sum of the series is continuous in the interval.

The series $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ can be arranged as a series of type ω^2, by substituting the various power-series for

$$u_1(x),\ u_2(x), \dots,$$

and the series so obtained is absolutely convergent; for the terms of the series $|a_{11}| + |a_{12}x| + \dots + |a_{1r}x^{r-1}| + \dots + |a_{21}| + |a_{22}x| + \dots$ are each less than the corresponding terms of the series obtained by putting 1 for x; and the latter series is $U_1 + U_2 + \dots$, which is convergent.

Since the series $u_1(x) + u_2(x) + \dots$ is absolutely convergent when the power-series are substituted for $u_1(x)$, $u_2(x), \dots$, it remains (see § 29) absolutely convergent when it is arranged in the form

$$b_1 + b_2x + b_3x^2 + \dots,$$

where

$$\begin{aligned} b_1 &= a_{11} + a_{21} + a_{31} + \dots + a_{n1} + \dots \\ b_2 &= a_{12} + a_{22} + a_{32} + \dots + a_{n2} + \dots \\ &\dots\dots \\ b_r &= a_{1r} + a_{2r} + \dots + a_{nr} + \dots; \end{aligned}$$

and its sum is unaltered. It has thus been shewn that the continuous function $s(x)$ can be represented in the interval $(-1, 1)$ by the power-series $b_1 + b_2x + b_3x^2 + \dots$.

The following theorem has now been established:

If $u_1(x)$, $u_2(x)$, ... $u_n(x)$, ..., be functions which can be represented by power-series that are all absolutely convergent at the point R, and therefore in the interval $(-R, R)$, and if the series $v_1(R) + v_2(R) + \dots + v_n(R) + \dots$, where $v_n(R)$ denotes the sum of the series obtained from that for $u_n(R)$ by replacing each coefficient by its absolute value, is convergent, then the series $u_1(x) + u_2(x) + \dots$ converges in the interval $(-R, R)$ to a sum-function $s(x)$ which is the sum of the power-series obtained by substituting the various power-series for the terms $u_1(x)$, $u_2(x)$, ..., and rearranging the resulting series as a single power-series.

It should be observed that the absolute convergence of the series $u_1(x) + u_2(x) + \dots$, at $x = R$, is not sufficient to ensure the convergence of the power-series obtained by substitution and rearrangement of the power-series for $u_1(x)$, $u_2(x)$, ..., to the sum of the series $u_1(x) + u_2(x) + \dots$.

For example, the series

$$\begin{aligned} 1 - x + x^2 - x^3 + \dots &= \frac{1}{1+x} \\ -1 + 2x - 3x^2 + \dots &= -\frac{1}{(1+x)^2} \\ 1 - 3x + 6x^2 - \dots &= \frac{1}{(1+x)^3} \\ &\dots\dots \end{aligned}$$

are all absolutely convergent when $x = \frac{1}{2}$, and the series

$$\frac{1}{1+x} - \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} - \dots$$

is absolutely convergent when $x = \frac{1}{2}$, but the coefficients in the rearranged series are not definite, and thus the series cannot be rearranged as a single power-series. The condition required by the above theorem is that $\frac{1}{1-x} + \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3} + \dots$ should be convergent when $x = \frac{1}{2}$; and this condition is not satisfied.

THE MULTIPLICATION OF POWER-SERIES

136. If the two power-series $a_0 + a_1x + a_2x^2 + \dots$, $b_0 + b_1x + b_2x^2 + \dots$ both converge within the interval $(-1, 1)$, since their convergence is absolute, in accordance with the theorem of § 38, the Cauchy-product series

$$c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots$$

where $c_n = a_0b_n + a_1b_{n-1} + \dots + a_nb_0$, is absolutely convergent when $|x| < 1$, and $s_1(x)\, s_2(x) = s(x)$, where $s_1(x)$, $s_2(x)$, $s(x)$ denote the sum-functions of the three series.

If all three series converge when $x = 1$, their sums, in accordance with Abel's theorem (§ 127) are $\lim_{x\sim1} s_1(x)$, $\lim_{x\sim1} s_2(x)$, $\lim_{x\sim1} s(x)$; consequently we obtain the theorem, already established in § 37, that if the series $\sum_{n=0} a_n$, $\sum_{n=0} b_n$, $\sum_{n=0} c_n$ are all convergent, then the product of the sums of the first two of these series is the sum of the third.

Employing the extension of Abel's theorem given in § 128 (3), that, if $\sum_{n=0} a_n$ is summable (C, r), for any positive value of r, then $\lim_{x\sim1} s_1(x)$ exists and is equal to this Cesàro sum, we obtain the following theorem:

If the two series $\sum_{n=0} a_n$, $\sum_{n=0} b_n$, are summable (C, r), (C, s) respectively, then the product of the Cesàro sums of the first two series is the Cesàro sum $(C, r+s+1)$ of the third series.

That the sum $(C, r+s+1)$ of the third series exists has been proved in § 51.

137. A special case of the multiplication of a power-series

$$a_0 + a_1x + a_2x^2 + \dots$$

which is convergent within a finite or infinite interval $(-\lambda, \lambda)$ arises when the series is multiplied by itself. If y denote the sum of the series when it is convergent, the series $a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)\,x^2 + \dots$ formed as before, has the same interval of convergence as the original series, and

its sum within that interval is y^2. Proceeding in a similar manner, series may be formed which represent y^3, y^4, ... within their intervals of convergence which are the same as that of the original series.

It is sometimes of importance to possess sufficient conditions for the convergence of the series obtained by substitution of the series for y, y^2, y^3, ... in the terms of a series $b_0 + b_1 y + b_2 y^2 + \ldots$, of which the interval of convergence is $(-k, k)$, when the resulting series is rearranged as a series in powers of x.

In accordance with the theorem given in § 135, it is sufficient for the convergence of the resulting power-series obtained by substitution of the series for y, y^2, ... in $b_0 + b_1 y + b_2 y^2 + \ldots$, and rearrangement of the result, that the series $|b_0| + |b_1|\eta + |b_2|\eta^2 + \ldots$ should be convergent; where η denotes the sum of the series $|a_0| + |a_1 x| + |a_2 x^2| + \ldots$ which certainly converges when $|x| < \lambda$. It is in fact clear that the series for η, η^2, ... are obtained from the series for y, y^2, ... by replacing all the coefficients a_0, a_1, a_2, ... by their absolute values. If $\eta < k$, the series

$$|b_0| + |b_1|\eta + |b_2|\eta^2 + \ldots$$

is convergent; and thus $|x|$ must be such that $|x| < \lambda$, and

$$|a_0| + |a_1 x| + |a_2 x^2| + \ldots < k.$$

Choosing a positive number ρ $(< \lambda)$, $|a_n|\rho^n$ converges to zero as $n \sim \infty$; let then M be the maximum value of $|a_n|\rho^n$ for all values of n.

The sum of the series $|a_0| + |a_1 x| + |a_2 x^2| + \ldots$ is then less than $|a_0| + M\left\{\frac{|x|}{\rho} + \frac{|x|^2}{\rho^2} + \ldots\right\}$, or than $|a_0| + \frac{M|x|}{\rho - |x|}$. It is then sufficient that $|x| < \lambda$, and that $|a_0| + \frac{M|x|}{\rho - |x|} < k$, or $|x| < \frac{(k - |a_0|)\rho}{M + k - |a_0|}$. If $|a_0| < k$, it follows that $|x| < \rho < \lambda$; thus it is sufficient that

$$|a_0| < k, \quad |x| < \frac{(k - |a_0|)\rho}{M + k - |a_0|}.$$

In case $a_0 = 0$, it is sufficient that $|x| < \frac{k\rho}{M + k}$. In case k is infinite, the condition $|x| < \lambda$ is sufficient.

The following theorem has been established:

If the series $a_0 + a_1 x + a_2 x^2 + \ldots$ of which the interval of convergence is $(-\lambda, \lambda)$, and of which the sum-function is y, be substituted in the series $b_0 + b_1 y + b_2 y^2 + \ldots$, of which $(-k, k)$ is the interval of convergence, and the resulting expression be rearranged as a power-series in x, the resulting series converges to the sum of the series in powers of y, provided $|a_0| < k$ and $|x| < \frac{(k - |a_0|)\rho}{M + k - |a_0|}$, where ρ is some number $< \lambda$ and $|a_n|\rho^n \leqq M$,

for $n = 1, 2, 3, \ldots$. *In case* $a_0 = 0$, *it is sufficient that* $|x| < \dfrac{k\rho}{M+k}$. *If the series* $b_0 + b_1 y + b_2 y^2 + \ldots$ *converges for all values of* y, *it is sufficient that* x *should be within the interval of convergence of the series*

$$a_0 + a_1 x + a_2 x^2 + \ldots.$$

As an example of the last case of the theorem, it will be seen that, since the power-series for e^y converges for all values of y, the value of $e^{a_0+a_1x+a_2x^2+\ldots}$ may be expressed by rearranging as a power-series the terms of the series

$$1 + (\sum_{n=0} a_n x^n) + \frac{1}{2!}(\sum_{n=0} a_n x^n)^2 + \ldots;$$

for all values of x for which the series $\sum\limits_{n=0} a_n x^n$ is convergent.

138. In order to obtain a power-series for $\dfrac{1}{a_0 + a_1 x + a_2 x^2 + \ldots}$ when x is within the interval of convergence of the power-series in the denominator, let y denote the sum-function of the series $a_1 x + a_2 x^2 + \ldots$. Now $\dfrac{1}{a_0 + y}$ can be represented by $\dfrac{1}{a_0}\left\{1 - \dfrac{y}{a_0} + \dfrac{y^2}{a_0{}^2} - \ldots\right\}$, provided $|y| < |a_0|$. It now follows from the foregoing theorem that a power-series for

$$\frac{1}{a_0 + a_1 x + a_2 x^2 + \ldots}$$

may be obtained by substituting in $\dfrac{1}{a_0}\left\{1 - \dfrac{y}{a_0} + \dfrac{y^2}{a_0{}^2} - \ldots\right\}$, and rearrangement, if $|x| < \dfrac{|a_0|\rho}{M + |a_0|}$, where ρ is a number less than λ, $(-\lambda, \lambda)$ being the interval of convergence of $a_0 + a_1 x + a_2 x^2 + \ldots$, and $|a_n|\rho^n \leq M$, for $n = 1, 2, 3, \ldots$. The precise range of values of x for which the resulting power-series is convergent can be obtained by employing the theory of complex variables.

TERM BY TERM DIFFERENTIATION AND INTEGRATION OF POWER-SERIES

139. Let $s(x)$ denote the sum of the power-series $a_0 + a_1 x + a_2 x^2 + \ldots$ which converges at all points within the interval $(-R, R)$ of convergence. Provided $|x + k|$ is less than R, the series

$$a_0 + a_1(x + h) + a_2(x + h)^2 + \ldots$$

converges to $s(x + h)$. Assuming that x also is within $(-R, R)$ it will be shewn that, if $|x| + |h| < R$, the series may be rearranged in a power-series according to powers of h, without altering the sum; that series is

$$(a_0 + a_1 x + a_2 x^2 + \ldots) + (a_1 + 2a_2 x + \ldots + na_n x^{n-1} + \ldots)h$$
$$+ \left(a_2 + 3a_3 x + \ldots + \frac{n(n-1)}{2} a_n x^{n-2} + \ldots\right)h^2$$
$$+ \ldots.$$

That this may be the case it is sufficient that

$$|a_0| + |a_1| (|x| + |h|) + |a_2| (|x| + |h|)^2 + \dots$$

should be convergent, which is the case, since $|x| + |h| < R$. The above power-series in h is convergent within the interval $(-R + |x|, R - |x|)$ of h, and its sum is $s(x+h)$. The coefficients of h are all absolutely convergent when $|x| < R$.

We have then

$$\frac{s(x+h) - s(x)}{h} = a_1 + 2a_2 x + \dots + na_n x^{n-1} + \dots + v_2(x) h + v_3(x) h^2 + \dots,$$

where $v_2(x)$, $v_3(x)$, ..., are all continuous functions of x, provided

$$|x| < R, \ |x| + |h| < R:$$

For a fixed value of x the series $v_2(x) h + v_3(x) h^2 + \dots$ has for its sum a continuous function of h, which converges to zero as $h \sim 0$. It follows that

$$\lim_{h \sim 0} \frac{s(x+h) - s(x)}{h} = a_1 + 2a_2 x + \dots + na_n x^{n-1} + \dots,$$

and thus $s(x)$ has a differential coefficient $s'(x)$ which is the sum-function of the series obtained by differentiating the terms of the series

$$a_0 + a_1 x + a_2 x^2 + \dots.$$

It has thus been shewn that:

If $s(x)$ be the sum-function of a power-series which converges within an interval, the function $s(x)$ has a differential coefficient $s'(x)$ at each point within the interval of convergence; moreover the power-series obtained by term by term differentiation of the given series converges at such a point to $s'(x)$.

By successive employment of this theorem it is clear that:

If $s(x)$ be the sum-function of a power-series, then $s(x)$ has differential coefficients of all orders at any point interior to the interval of convergence of the given power-series; moreover, if term by term differentiation of any order be applied to the given series, a power-series is obtained which converges at all points within the interval of convergence to the value of the differential coefficient, of the corresponding order, of $s(x)$.

140. If $r < R$, we have $s(x) = a_0 + a_1 x + \dots + a_n x^n + p(x)$, where, for all sufficiently large values of n, $|p(x)| < \epsilon$, for all points x in the interval $(-r, r)$ interior to the interval of convergence $(-R, R)$ of the power-series. This is equivalent to the statement that the series converges uniformly in $(-r, r)$ (§ 66). We have now

$$\left| \int_0^x \{s(x) - (a_0 + a_1 x + \dots + a_n x^n)\} dx \right| \leqq \int_0^x |p(x)| dx < \epsilon R,$$

provided $|x| \leq r$. Since ϵ is arbitrary it follows that the integrated series

$$a_0 x + a_1 \frac{x^2}{2} + \ldots + a_n \frac{x^{n+1}}{n+1} + \ldots$$

converges to $\int_0^x s(x)\,dx$ uniformly in the interval $(-r, r)$. Thus we have the result that:

Term by term integration of a power-series produces a new power-series which converges to $\int_0^x s(x)\,dx$, *for all points interior to the interval* $(-R, R)$ *of convergence of the given series.*

In case the series $a_0 R + a_1 \frac{R^2}{2} + a_2 \frac{R^3}{3} + \ldots$ is convergent, its sum, in accordance with Abel's theorem, is $\lim_{x \sim R} \int_0^x s(x)\,dx$, or $\int_0^R s(x)\,dx$. This may be the case when the series $a_0 + a_1 R + a_2 R^2 + \ldots$ is not convergent.

In case the series $a_0 R + a_1 \frac{R^2}{2} + \ldots$ is summable (C, r), $(r > 0)$, its sum (C, r) is $\int_0^R s(x)\,dx$.

TAYLOR'S SERIES

141. It has been shewn in § 139 that, if a function $f(x)$ be such that, within the interval $(-R, R)$, it is the sum of the convergent series

$$a_0 + a_1 x + a_2 x^2 + \ldots,$$

the differential coefficients $f^{(r)}(x)$ exist, for all values of the integer r, and that $f^{(r)}(x)$ is the sum-function of the series

$$1.2.3 \ldots r a_r + 2.3 \ldots (r+1)\, a_{r+1} x + 3.4 \ldots (r+2)\, a_{r+2} x^2 + \ldots$$

obtained by differentiating the terms of the given power-series r times, within the interval $(-R, R)$. It has further been shewn that, if $|x| + |h|$ also lies within the interval $(-R, R)$, the series obtained by arranging the series $a_0 + a_1(x+h) + a_2(x+h)^2 + \ldots$ as a series in powers of h converges to $f(x+h)$. The coefficient of h^r in this series is

$$a_r + (r+1)\, a_{r+1} x + \frac{(r+2)(r+1)}{2!} a_{r+2} x^2 + \ldots$$

which converges to $\frac{f^{(r)}(x)}{r!}$.

It has thus been shewn that, if $|x|$ and $|x| + |h|$ are both less than R, the series

$$f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^r}{r!} f^{(r)}(x) + \ldots$$

converges to the value $f(x+h)$.

This theorem is a particular case of Taylor's theorem for the expansion of a function $f(x+h)$ in powers of h, and has here been established for the particular case of a function $f(x)$ which represents, in some interval, the sum of a convergent power-series. It has, moreover, been shewn that such a function possesses differential coefficients of all orders, within the interval of convergence of the power-series.

We proceed to investigate the necessary and sufficient conditions that a corresponding theorem may hold for a function defined in a more general manner.

142. The following theorem will be established:

If a function $f(x)$, defined in the closed interval $(a, a+h)$, be such that, (1), the functions $f(x)$, $f'(x)$, $f''(x)$, $\dots f^{(n-1)}(x)$ are all continuous in the closed interval $(a, a+h)$, and (2), $f^{(n)}(x)$ exists at every point of the open interval $(a, a+h)$, being either finite, or infinite with fixed sign, at each point, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^\nu}{(n-\nu)(n-1)!}f^n(a+\theta h),$$

for some value of θ such that $0 < \theta < 1$; provided the number ν, not necessarily integral or positive, is such that $n - \nu > 0$. At the points a, $a+h$, the differential coefficients are interpreted to mean the successive derivatives on the right and left respectively.

It may be observed that the conditions (1) and (2) are not, as stated, reduced to the minimum number.

Let $$F(x) \equiv f(a+h) - f(x) - (a+h-x)f'(x) - \frac{(a+h-x)^2}{2!}f''(x) - \dots$$
$$- \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - (a+h-x)^{n-\nu}K,$$

where the number K is defined by

$$f(a+h) - f(a) - hf'(a) - \frac{h^2}{2!}f''(a) - \dots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) = h^{n-\nu}K.$$

In the closed interval $(a, a+h)$, $F(x)$ is continuous, since $n > \nu$; also $F'(x)$ exists everywhere in the open interval $(a, a+h)$. Moreover $F(a) = 0$, $F(a+h) = 0$, and therefore, in accordance with the mean value theorem (I, § 262), $F'(x)$ has the value zero, for some value of x interior to the interval $(a, a+h)$. Let this value be $a+\theta h$, where $0 < \theta < 1$. We find on differentiation,

$$F'(x) = -\frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) + (n-\nu)(a+h-x)^{n-\nu-1}K,$$

and thus $K = \dfrac{h^\nu (1-\theta)^\nu}{(n-\nu)(n-1)!} f^{(n)}(a+\theta h)$. Equating this value of K to the value by which it was defined, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n (1-\theta)^\nu}{(n-\nu)(n-1)!} f^n(a+\theta h).$$

In case the value of K, as defined, is zero, the proof remains valid; in this case we have $f^{(n)}(a+\theta h) = 0$.

It is clear that a corresponding result holds for an interval on the left of a, provided corresponding conditions are satisfied as regards differential coefficients, the derivatives at a being in this case on the left.

If the conditions of the theorem are satisfied for the interval $(a, a+h)$, they are clearly satisfied for any interval $(a, a+h')$, where $0 < h' < h$.

In case $f(x)$ be defined for the interval $(a-h_2, a+h_1)$, the conditions of existence and continuity of $f(x), f'(x), \dots f^{(n-1)}(x)$ in a closed interval being satisfied in the closed interval $(a-h_2, a+h_1)$, and $f^{(n)}(x)$ being assumed to exist at every point of the open interval, the theorem holds for every value of h in the closed interval $(-h_2, h_1)$, the value of θ depending upon the value of h.

The theorem which has been established above is frequently spoken of as Taylor's theorem, although that name was originally, and is still usually, applied to the case in which it is possible to suppose n to be indefinitely increased, so that the series becomes an infinite convergent one.

The expression $R_n = \dfrac{h^n (1-\theta)^\nu}{(n-\nu)(n-1)!} f^n(a+\theta h)$, where $n > \nu$, is spoken of as "the remainder in Taylor's theorem." In this general form it was obtained by Schlömilch* and by Roche†. The particular case in which ν is taken to be zero, $R_n = \dfrac{h^n}{n!} f^{(n)}(a+\theta h)$ is known as Lagrange's‡ form of the remainder in Taylor's theorem; another particular case, due originally to Cauchy§, of the general form given by Schlömilch, is that in which ν is taken to be $n-1$, or $R_n = \dfrac{h^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$.

143. It was first shewn by Stolz|| that the theorem of § 140 can be extended to the case in which the functions $f'(x), f''(x), \dots f^{(n-1)}(x)$ are

* *Handbuch der Differential- und Integralrechnung*, 1847.

† *Mém. de l'Acad. de Montpellier*, 1858. See also *Liouville's Journal* (2), vol. III (1858), pp. 271 and 384.

‡ *Théorie des fonctions*, vol. I, p. 40.

§ *Calcul Diff.* p. 77.

|| See *Grundzüge*, vol. I, p. 97. It should be observed that Stolz omits to state the restriction, necessary to his proof, that ν is not to be a positive integer less than n.

assumed to exist, and to be continuous, only in the half-closed interval $a \leqq x < a + h$, so that their existence at the point $a + h$ is unnecessary. The following theorem will be here established:

If a function $f(x)$, continuous in the closed interval $(a, a + h)$, be such that, (1), *the functions $f'(x)$, $f''(x)$, ... $f^{(n-1)}(x)$ are all continuous in the domain $a \leqq x < a + h$, and* (2), *the function $f^{(n)}(x)$ exists, as a finite number, or as infinite with fixed sign, at each point of the open interval $(a, a + h)$, then*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n (1 - \theta)^\nu}{(n - \nu)(n - 1)!} f^n(a + \theta h),$$

for some value of θ such that $0 < \theta < 1$; where ν may have any value $< n$, positive, negative, or zero, and not necessarily integral, except that it may not be a positive integer.

We cannot in the present theorem take ν to have the value $n - 1$, so that Cauchy's form of the remainder is not here included in the result.

To prove the theorem, let

$$F(x) = f(x) - f(a) - (x - a) f'(a) - \frac{(x - a)^2}{2!} f''(a) - \dots$$
$$- \frac{(x - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - K \Big\{ \phi(x) - \phi(a) - (x - a)\phi'(a)$$
$$- \frac{(x - a)^2}{2!} \phi''(a) - \dots - \frac{(x - a)^{n-1}}{(n-1)!} \phi^n(a) \Big\},$$

where $\phi(x)$ denotes a function which possesses finite differential coefficients of the first n orders in the closed interval $(a, a + h)$, and such that $\phi^{(n)}(x)$ is nowhere zero in the interior of the interval $(a, a + h)$. Let K have the value

$$\frac{f(a + h) - f(a) - hf'(a) - \dfrac{h^2}{2!} f''(a) - \dots - \dfrac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)}{\phi(a + h) - \phi(a) - h\phi'(a) - \dfrac{h^2}{2!} \phi''(a) - \dots - \dfrac{h^{n-1}}{(n-1)!} \phi^{(n-1)}(a)};$$

it being assumed that the denominator is not zero. We have $F(a) = 0$, $F(a + h) = 0$; hence, since $F'(x)$ exists at every interior point of the interval $(a, a + h)$, $F'(x_1) = 0$, for some point x_1 such that $a < x_1 < a + h$.

Since $F'(a) = 0$, $F'(x_1) = 0$, and $F''(x)$ exists at every point of the interval (a, x_1), it follows that at some point x_2, such that $a < x_2 < x_1$, $F''(x_2) = 0$. Proceeding in this manner we see that $F^n(x_n) = 0$, at some point $x_n = a + \theta h$, interior to the interval $(a, a + h)$.

Thus we have $f^{(n)}(a+\theta h) - K\phi^{(n)}(a+\theta h) = 0$. It has now been shewn that

$$\frac{f(a+h) - f(a) - hf'(a) - \frac{h^2}{2!}f''(a) - \dots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)}{\phi(a+h) - \phi(a) - h\phi'(a) - \frac{h^2}{2!}\phi''(a) - \dots - \frac{h^{n-1}}{(n-1)!}\phi^{(n-1)}(a)} = \frac{f^{(n)}(a+\theta h)}{\phi^{(n)}(a+\theta h)}.$$

In case $K = 0$, we have $f^{(n)}(a+\theta h) = 0$, and since $\phi^{(n)}(a+\theta h) \neq 0$, the result holds good.

Now let $\phi(x) = (a+h-x)^{n-\nu}$, where ν may have any value whatever ($< n$), except the values $1, 2, 3, \dots n-1$; then

$$\phi^{(n)}(a+\theta h) = (-1)^n (n-\nu)(n-\nu-1)(n-\nu-2)\dots (-\nu+1) h^{-\nu}(1+\theta)^{-\nu}.$$

Since $n > \nu$, the value of

$$\phi(a+h) - \phi(a) - h\phi'(a) - \frac{h^2}{2!}\phi''(a) - \dots - \frac{h^{n-1}}{(n-1)}\phi^{(n-1)}(a)$$

is
$$-h^{n-\nu}\left\{1 - (n-\nu) + \frac{(n-\nu)(n-\nu-1)}{2!} - \dots + (-1)^{n-1}\frac{(n-\nu)(n-\nu-1)\dots(\nu+2)}{(n-1)!}\right\};$$

and it can easily be shewn that this is equal to

$$(-1)^n h^{n-\nu}\frac{(n-\nu-1)(n-\nu-2)(n-\nu-3)\dots(-\nu+1)}{(n-1)!}.$$

We have thus shewn that, subject to the conditions stated in the theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^\nu}{(n-\nu)(n-1)!}f^{(n)}(a+\theta h).$$

144. Let it now be supposed that $f(x)$ is defined only in the open interval $(a, a+h)$, and that all the functional limits

$$f(a+0),\ f'(a+0),\ f''(a+0),\ \dots f^{(n-1)}(a+0)$$

exist as definite numbers, and that also $f^{(n)}(x)$ exists everywhere in the open interval, as a finite number or as infinite with fixed sign. The proof of the last theorem may be employed to prove that

$$f(a+h-0) = f(a+0) + (x-a)f'(a+0) + \frac{(x-a)^2}{2!}f''(a+0) + \dots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a+0) + \frac{h^n(1-\theta)^\nu}{(n-\nu)(n-1)!}f^n(a+\theta h);$$

where $f(a+h-0)$ is any one of the limits of $f(x)$ at the point $a+h$. In case $f(a+h-0)$ is not unique, it is known that $f'(x)$ takes every finite value within $(a, a+h)$ (see I, § 266).

Let us next suppose that $f^{(n-1)}(x)$ has no definite limit at a, on the right, so that it has a discontinuity of the second kind at that point, and that $f^{(n)}(x)$ is everywhere finite. Let $F(x)$ be defined by

$$F(x) = f(x) - c_0 - c_1(x-a) - c_2(x-a)^2 - \dots \\ - c_{n-1}(x-a)^{n-1} - K(x-a)^n,$$

where $\quad K.h^n = f(a+h) - c_0 - c_1 h - c_2 h^2 - \dots - c_{n-1} h^{n-1};$

the numbers $c_0, c_1, c_2, \dots c_{n-1}$ being arbitrarily chosen, and $f(a+h)$ being a definite number.

We have $F^{(n)}(x) = f^{(n)}(x) - Kn!$, where $a < x < a+h$. Now since $F^{(n-1)}(x)$ has a discontinuity of the second kind at a, and $F^{(n)}(x)$ exists in the open interval $(a, a+h)$ and is finite, it follows, by a theorem established in I, § 266, that $F^{(n)}(x)$ has the value zero at interior points of the interval. Hence θ can be so chosen that

$$F^n(a+\theta h) \equiv f^{(n)}(a+\theta h) - Kn! = 0;$$

and thence we have

$$f(a+h) = c_0 + c_1 h + c_2 h^2 + \dots + c_{n-1} h^{n-1} + \frac{h^n}{n!} f^n(a+\theta h).$$

The following theorem has now been proved:

If $f(x)$ be defined in the interval $a < x \leq a+h$, and $f^{(n-1)}(x)$ exists in the open interval $a < x < a+h$, but has no definite limit on the right at a, and if $f^{(n)}(x)$ exists in the open interval, being everywhere finite, then

$$f(a+h) = c_0 + c_1 h + c_2 h^2 + \dots + c_{n-1} h^{n-1} + \frac{h^n}{n!} f^{(n)}(a+\theta h),$$

where $c_0, c_1, c_2, \dots c_{n-1}$ are arbitrarily chosen numbers, and θ is a number such that $0 < \theta < 1$.

Theorems of a similar character have been given by W. H. Young*, based upon the lemma that, if $f(x)$ is continuous in the open interval $a < x < b$, and that $f(x)$ either has a differential coefficient, or else that there is no distinction between right and left with regard to its derivatives in the open interval, then there is a point x of the open interval such that

$$f(b-0) - f(a+0) = (b-a) f'(x),$$

where $f(a+0), f(b-0)$ denote any two of the limits of $f(x)$ on the right at a and on the left at b. It should, however, be observed that, unless both the limits $f(b-0)$, $f(a+0)$ are unique, $f'(x)$ takes every finite value

* See *Quart. Journ.* vol. XL (1909), p. 146.

at points in the open interval (see I, § 266), and thus the lemma in that case is unduly restricted, unless either $f(b-0)$ or $f(a+0)$ is capable of having all finite values.

145. If $f(x)$ possess differential coefficients of all orders within a prescribed interval $(a-\lambda', a+\lambda)$, of x, then, provided R_n have the limit zero, when n is indefinitely increased, for each value of h, the series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a) + \ldots,$$

where $-\lambda' < h < \lambda$, is convergent, and has $f(a+h)$ for its limiting sum. This is Taylor's theorem in the original meaning of that theorem.

It will be observed that the existence of the differential coefficients at the extreme points $a-\lambda'$, $a+\lambda$ has, in § 143, not been presupposed, but only their existence everywhere in the open interval $(a-\lambda', a+\lambda)$. If the condition $\lim\limits_{n\sim\infty} R_n = 0$ be satisfied for each value of h within the interval $(-\lambda', \lambda)$, and if the series converge also for $h=\lambda$, then since it is a power-series, it follows from the theorem of § 124, that, at $h=\lambda$, it converges to $f(a+\lambda)$, provided $f(x)$ is continuous on the left at the point $a+\lambda$.

The value of θ, in any of the forms of the expression for R_n, is in general dependent upon n; and consequently it is not a sufficient condition of convergence of the Taylor's series throughout the half-open interval $0 \leqq h < \lambda$ that R_n have the limit zero whilst θ remains fixed, even if this be the case for each fixed value of θ in the open interval $(0, 1)$. In connection with the theory of non-uniform convergence of series, we have already seen, in § 84, that a function $R_n(x)$ may have the limit zero, as $n \sim \infty$, for each value of x in the interval, and yet that $\lim\limits_{n\sim\infty} R_n(x)$ may not be zero when x varies with n. For example, if $R_n = \dfrac{n\theta h}{(1+\theta h)^n}$, then R_n has the limit zero for each fixed value of θ; but if $\theta = 1/n$, R_n has the limit he^{-h}.

A sufficient condition for the convergence of the series, within a given interval of h, is that R_n, for each fixed value of h, within the given interval, should converge to zero, as $n \sim \infty$, uniformly for all values of θ in the interval $(0, 1)$. Thus, for each value of h, and each value of an arbitrarily chosen positive number ϵ, a value n_ϵ, of n would exist, such that

$$\left| \frac{h^n (1-\theta)^\nu}{(n-\nu)(n-1)!} f^{(n)}(a+\theta h) \right| < \epsilon,$$

provided $n \geqq n_\epsilon$, for every value of θ in the closed interval $(0, 1)$.

This condition, though sufficient to ensure the convergence of the series, has not been shewn to be necessary. An investigation, due in its

original form to Pringsheim*, will now be given of necessary and sufficient conditions for the convergence of Taylor's series.

146. The following lemma will be first established:

If $f^{(n)}(x)$ be defined for every value of n, where x is in the semi-closed interval $a \leqq x < a + R$, and if, for some fixed value of p which is a positive or negative integer, or which may be zero, the condition is satisfied that

$$\frac{1}{(n+p)!} f^{(n)}(a+h)\, k^{n+p}$$

converges to zero, as $n \sim \infty$, uniformly for all values of h and k such that $0 \leqq h \leqq h + k \leqq r$, for each value of r that is $< R$, then the same condition is satisfied when p has any other value which is a positive or negative integer, or zero, and such that $p + n > 0$.

Denoting $\frac{1}{(n+p)!} f^{(n)}(a+h)\, k^{n+p}$ by $F_{p,n}(h, k)$, we have

$$F_{p+1,n}(h, k) = \frac{k}{n+p+1} F_{p,n}(h, k);$$

and hence, since $k \leqq r$, we have

$$|F_{p+1,n}(h, k)| < |F_{p,n}(h, k)|,$$

provided $n + p + 1 > r$. It follows that $F_{p+1,n}(h, k)$, and more generally $F_{p+q,n}(h, k)$, for $q > 0$, converges uniformly if $F_{p,n}(h, k)$ does so.

Again, we have

$$F_{p-1,n}(h, k) = \left(\frac{k}{k+\delta}\right)^{n+p-1} \frac{n+p}{k+\delta} F_{p,n}(h, k+\delta);$$

if $r\ (< R)$ be fixed, δ can be so chosen as to be positive and such that $r + \delta < R$. Hence, if $0 \leqq k \leqq r$,

$$|F_{p-1,n}(h, k)| < \left(\frac{r}{r+\delta}\right)^{n+p-1} \frac{n+p}{\delta} |F_{p,n}(h, k+\delta)|.$$

If n be so chosen that $\left(\frac{r}{r+\delta}\right)^{n+p-1} \frac{n+p}{\delta} \leqq 1$, and if $|F_{p,n}(h, k)| < \epsilon$, for $0 \leqq h \leqq h + k \leqq r + \delta$, and $n \geqq n_\epsilon$, we have then $|F_{p-1,n}(h, k)| < \epsilon$, for $0 \leqq h \leqq h + k \leqq r$, provided n is not less than some integer n_ϵ'. This can at once be extended to shew that $|F_{p-q,n}(h, k)| < \epsilon$, if $q > 0$, provided n is not less than some integer dependent on ϵ. The lemma has now been established.

For the above lemma given by Pringsheim another lemma, which does not involve the notion of uniform convergence, has been substituted by W. H. Young†:

* *Math. Annalen*, vol. XLIV (1894), p. 57. See also *Münch. Sitzungsber.* (1912), p. 137.

† *Quart. Journ.* vol. XL (1908), p. 157, also his Tract, *The Fundamental Theorems of the Differential Calculus*, Cambridge (1910), pp. 57–8.

If, for some given value of p which may be a positive or negative integer, or zero, the condition is satisfied that

$$\left|\frac{1}{(n+p)!} f^{(n)}(a+h)\, k^{n+p}\right|$$

for each value of r ($< R$) *is less than some fixed positive number dependent only on r, for all values of h, k, n such that* $0 \leq h \leq h+k \leq r$, $n+p>0$, *the same condition is satisfied when p has any other value, which is a positive or negative integer, or zero.*

The foregoing proof can easily be adapted to prove the second lemma. As before, if $F_{p,n}(h,k)$ satisfies the condition, so does $F_{p+1,n}(h,k)$; and if $F_{p,n}(h,k)$ satisfies the condition when $r+\delta$ is taken instead of r, $F_{p-1,n}(h,k)$ also satisfies the condition.

147. We proceed to investigate the necessary and sufficient conditions of convergence, which may be stated as follows:

Necessary conditions that the series $\sum\limits_0^\infty c_n h^n$ *shall converge for every positive value of h that is* $< R$, *are that, if* $f(x)$ *denote the sum of the series* $\sum\limits_0^\infty c_n (x-a)^n$, *where a is a fixed number, and* $0 \leq x-a < R$, (1), $f(x)$ *possesses, for every value of x such that* $a \leq x < a+R$, *a definite finite value,* (2), *that, for every value of x such that* $a < x < a+R$, $f(x)$ *possesses finite differential coefficients of every order, and at a, definite derivatives on the right, of every order; and* (3), *that, for each fixed value of p* ($> -n$) *which may be a positive or negative integer, or zero,* $\dfrac{1}{(n+p)!} f^{(n)}(a+h)\, k^{n+p}$ *converges uniformly for all values of h, k such that* $0 \leq h \leq h+k < r$ *to zero, as* $n \sim \infty$, *for each value of r* ($< R$). *Moreover if the condition* (2) *is satisfied, and if* (3) *is satisfied for any one value of p, this is sufficient to ensure the convergence of the Taylor's series corresponding to* $f(x)$ *for the interval* $a \leq x < R$.

Instead of the condition (3), the following condition may be substituted:

(3)′, *that, for some value of p, a positive or negative integer, or zero, and for each value of r* ($< R$), $\left|\dfrac{1}{(n+p)!} f^{(n)}(a+h)\, k^{n+p}\right|$ *is less than some fixed number, dependent only on r, for all values of n (for which* $n+p>0$) *and for all values of h and k such that* $0 \leq h \leq h+k < r$.

A similar statement holds as regards an interval on the left of a; and it is clear that the theorem can be so stated as to apply to the more general case of a neighbourhood which contains a in its interior.

Assuming that $\sum\limits_0^\infty c_n (x-a)^n$ converges for $a \leq x < R$, it follows from § 139 that its sum-function $f(x)$ is differentiable in that interval, and that

$f'(x)$ is represented in the interval $a \leq x < a + R$ by the series obtained by term by term differentiation of the series $\Sigma c_n (x-a)^n$. The same remark applies to the function $f'(x)$, and to the series which represents it, and then successively to the higher differential coefficients of $f(x)$. We have therefore $f^s(x) = \sum\limits_{n=s}^{\infty} n(n-1)\dots(n-s+1)c_n(x-a)^{n-s}$, for all values of s; hence $f(a) = c_0$, and $f^{(s)}(a) = n!\,c_n$, and thus

$$f(a+h) = \sum_0^\infty \frac{1}{n!} f^{(n)}(a)\,h^n, \quad f^{(p)}(a+h) = \sum_p^\infty \frac{1}{(n-p)!} f^{(n)}(a)\,h^{n-p},$$

where $0 \leq h < R$.

Since a power-series converges absolutely at all points within the interval of convergence, we see that the function $\phi(x)$ defined, for the interval $a \leq x < a + R$, by $\phi(x) = \sum\limits_0^\infty |c_n|(x-a)^n$, has properties similar to those of $f(x)$; and thus that

$$\phi(a+h) = \sum_0^\infty |c_n|\,h^n = \sum_0^\infty \frac{1}{n!}\phi^{(n)}(a)\,h^n,$$

and
$$\phi^{(p)}(a+h) = \sum_p^\infty \frac{1}{(n-p)!}\phi^{(n)}(a)\,h^{n-p}, \text{ for } 0 \leqq h < R,$$
$$= \sum_p^\infty \frac{n!}{(n-p)!}|c_n|\,h^{n-p}.$$

The functions $\phi(a+h)$, $\phi^{(p)}(a+h)$ are continuous functions of h in the interval $0 \leq h < R$; and for each value of p,

$$|f^{(p)}(a+h)| \leq |\phi^{(p)}(a+h)|.$$

In order to prove that the condition (3) is satisfied it will be sufficient to prove that the corresponding condition is satisfied by the function $\phi^{(p)}(a+h)$.

If $0 \leqq h \leq h + k < R$, we have

$$\phi(a+h+k) = \sum_0^\infty \frac{1}{n!}\phi^{(n)}(a)\,(h+k)^n = \sum_0^\infty \frac{1}{n!}\phi^{(n)}(a+h)\,k^n;$$

and it will now be shewn that the series $\Sigma \frac{1}{n!}\phi^{(n)}(a+h)\,k^n$ converges uniformly for all values of h and k which are such that $0 \leq h \leq h + k \leqq r$, where r is any positive number $< R$.

Since the terms of the series are all positive, and the sum-function is a continuous function of (h, k), it follows from the theorem of § 78 that the series converges uniformly in the closed domain $0 \leq h \leqq h + k \leqq r$, where $r < R$; it is a necessary consequence of the uniform convergence that $\frac{1}{n!}\phi^{(n)}(a+h)\,k^n$ should converge to zero uniformly in the domain $0 \leqq h \leqq h + k \leqq r$, as n is indefinitely increased; r being any assigned

positive number $< R$. It follows that $\frac{1}{n!} f^{(n)}(a+h) k^n$ has the same property, and by using the lemma of § 146, it follows that

$$\frac{1}{(n+p)!} f^{(n)}(a+h) k^{n+p}$$

also converges uniformly to zero, where p is any integer ($> -n$), positive or negative. Thus the condition (3) has been shewn to be a necessary condition. Moreover, when (3) is satisfied, the condition (3)′ is also satisfied; thus (3)′ is a necessary condition.

In order to shew that, if (2) is satisfied, and if, for some particular value of p, (3) is satisfied, then the series converges to $f(x)$, for all values of x such that $a \leq x < a + R$, we observe that, in accordance with the lemma, the condition (3) must be satisfied for every value of p ($> -n$). Thus, if p be a positive integer $< n$, $\frac{1}{(n-p)!} f^{(n)}(a+h) k^{n-p}$ converges uniformly to zero in the domain $0 \leq h \leq h + k < r$. Writing θh for h, and $(1-\theta) h$ for k, we see that $\frac{1}{(n-p)!} f^n(a+\theta h)(1-\theta)^{n-p} h^{n-p}$ converges uniformly to zero in the domain $0 < \theta < 1$, $0 \leq h \leq r$. Referring to the expression $\frac{h^n (1-\theta)^\nu}{(n-\nu)(n-1)!} f^{(n)}(a+\theta h)$, in § 142, we may take $\nu = n - p$, and the expression becomes $\frac{1}{p(n-1)!} h^n (1-\theta)^{n-p} f^n(a+\theta h)$ which can be written in the form

$$\frac{h^p}{p(n-1)(n-2)\ldots(n-p+1)} \cdot \left\{\frac{1}{(n-p)!} f^{(n)}(a+\theta h)(1-\theta)^{n-p} h^{n-p}\right\}.$$

For any fixed value of p this converges to zero as $n \sim \infty$, for every value of θ, and for $0 \leq h \leq r$. Consequently the remainder in Taylor's theorem converges to zero for every value of h such that $0 \leq h < R$.

It is clearly sufficient for the convergence of the remainder that

$$\left|\frac{1}{(n-p)!} f^{(n)}(a+\theta h)(1-\theta)^{n-p} h^{n-p}\right|$$

should be less than some fixed positive number independent of n; thus the condition (3)′ is sufficient, when (2) is satisfied.

148. The necessary and sufficient conditions that Taylor's theorem should hold for the function $f(a+h)$, where $0 \leq h < R$, can be most simply expressed when Cauchy's form of the remainder is used, and they may be obtained as follows:

The condition as to the existence of differential coefficients of all orders being assumed to be satisfied in the interval $0 \leq h < R$, it has been shewn in § 147 to be necessary and sufficient for the validity of Taylor's

series in the half-closed interval that $\frac{1}{(n-1)!} f^{(n)}(a+h)\,k^{n-1}$ should converge to zero, as $n \sim \infty$, uniformly for all values of h and k such that $0 \leqq h \leqq h + k < r$, for each value of r $(<R)$. Writing θh for h, and $(1-\theta)\,h$ for k, the condition takes the form that

$$\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$$

should converge to zero uniformly in the domain $0 < \theta < 1$, $0 \leqq h \leqq r$; and this is Cauchy's form of the remainder in Taylor's series obtained in § 142. It is then necessary that $\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ should converge to zero for each value of r such that $0 < r < R$, uniformly for all values of θ such that $0 < \theta < 1$; moreover, this condition has been shewn in § 145 to be sufficient. The following theorem has now been established:

In order that the function $f(a+h)$, defined for all values of h such that $0 \leqq h < R$, may be represented for all the values of h by the series $\sum_0^\infty \frac{1}{n!} f^{(n)}(a)\,h^n$, it is necessary and sufficient, (1) *that $f(x)$ have differential coefficients of all orders, for $a < x < a + R$, and definite derivatives on the right at a, of all orders, and* (2) *that Cauchy's remainder*

$$\frac{h^n}{(n-1)!} f^{(n)}(a+\theta h)\,(1-\theta)^{n-1},$$

for each value of h such that $0 \leqq h < R$, converge to zero, as $n \sim \infty$, uniformly for all values of θ in the closed interval $(0, 1)$.

149. In case Lagrange's form of the remainder in Taylor's theorem is employed, instead of that due to Cauchy, the necessary and sufficient conditions cannot be expressed in so simple a form. The following theorem has reference to this form of the remainder:

In order that the function $f(a+h)$, defined for all values of h such that $0 \leqq h < R$, may be represented, for all the values of h, by the series $\sum_0^\infty \frac{h^n}{n!} f^{(n)}(a)$, it is necessary, besides the condition of unrestricted differentiability previously stated, that $\frac{h^n}{n!} f^{(n)}(a+\theta h)$ should converge, for each value of h such that $0 \leqq h < \frac{1}{2}R$, to the limit zero, as $n \sim \infty$, uniformly for all values of θ in the closed interval $(0, 1)$. *It is sufficient, but not necessary, that the expression should converge to zero, for each value of h such that $0 \leqq h < R$, uniformly for all values of θ in the closed interval* $(0, 1)$.

In accordance with the theorem of § 147, it is necessary that $\frac{k^n}{n!} f^{(n)}(a+h)$ should converge to zero, as $n \sim \infty$, uniformly for all values of (h, k) such

that $0 \leqq h \leqq h + k \leqq r$, where r is an arbitrarily chosen positive number $< R$. Writing h for k, and θh for h, we see that this condition includes the condition that $\frac{h^n}{n!} f^{(n)}(a + \theta h)$ should converge to zero, for each value of h such that $0 \leqq h < \frac{1}{2}R$, uniformly for all values of θ in the closed interval $(0, 1)$.

To shew that the convergence for each value of h such that $0 \leqq h < R$ is not necessary, let us consider the function $f(x) = (1 - x)^{-1}$, defined in the interval $0 \leqq x < 1$. The Lagrangian form of the remainder is $\frac{x^n}{(1 - \theta x)^{n+1}}$; this converges to zero only when $x < 1 - \theta x$, hence, if $x > \frac{1}{2}$ it does not everywhere converge in the interval $(0, 1)$ of θ, but if $x < \frac{1}{2}$ it converges uniformly with respect to θ in the interval $(0, 1)$. This shews that the condition that the remainder shall converge for every value of h that is $< R$ is not always satisfied when the Taylor's series converges in the interval $0 \leqq h < R$.

150. It was remarked by Cauchy* that the series $\Sigma \frac{h^n}{n!} f^{(n)}(a)$ may be convergent in an interval, and yet that its sum need not be $f(a + h)$. This happens whenever the remainder R_n, in Taylor's theorem, which is defined as the difference between $f(a + h)$ and the sum of the first n terms of the series, converges, for each value of x, to a limit which is different from zero, as n is indefinitely increased.

Let the function $f(x)$ be defined by $f(x) = e^{-\frac{1}{x^2}}$, for $x^2 > 0$, and $f(0) = 0$; it can easily be shewn that this function and all its differential coefficients exist, and are zero at the point $x = 0$; and that for $x^2 > 0$, the remainder in the Taylor's series has for its limit $e^{-\frac{1}{x^2}}$. If now $\phi(x) = e^x + e^{-\frac{1}{x^2}}$, $(x^2 > 0)$, $\phi(0) = 1$, and the series $\Sigma \frac{h^n}{n!} \phi^{(n)}(0)$ in the neighbourhood of the point $x = 0$ be formed, then the series converges, not to the value $\phi(h)$, but to the value e^h.

EXAMPLES

(1) Let $f(x) = (1 + x)^p$; then, in a neighbourhood of the point $x = 0$, we have

$$f(x) = 1 + px + \frac{p(p-1)}{2!} x^2 + \ldots + \frac{p(p-1)\ldots(p-n+2)}{(n-1)!} x^{n-1} + R_n,$$

where R_n can be expressed in Lagrange's form by

$$\frac{p(p-1)\ldots(p-n+1)}{n!} \frac{x^n}{(1+\theta x)^{n-p}},$$

or in Cauchy's form by $\frac{p(p-1)\ldots(p-n+1)}{(n-1)!} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-p}} x^n$.

* *Calcul Diff.* p. 103; see also P. Du Bois-Reymond, *Math. Annalen*, vol. XXI (1883), p. 114.

Using Cauchy's form, we see that

$$| R_n | \leq \left| \frac{p(p-1)\dots(p-n+1)}{(n-1)!} x^n \right|,$$

provided $n > p$. If $| x | < 1$, the expression

$$\left| \frac{p(p-1)\dots(p-n+1)}{(n-1)!} x^n \right|$$

continually diminishes as n is increased: for, denoting it by u_n, we find

$$\frac{u_{n+1}}{u_n} = \left| \frac{p-n}{n} x \right| < 1 - \epsilon,$$

where ϵ is a fixed positive number $< 1 - | x |$, provided n be sufficiently great, and it follows that the limit of u_n is zero; and thus $\lim R_n = 0$. The series therefore converges for all values of x such that $| x | < 1$.

To find the limit of $\left| \frac{p(p-1)\dots(p-n+1)}{n!} \right|$ when n is indefinitely increased, suppose first that $p+1$ is negative, say $-k$. We may write the expression in the form

$$\left(1 + \frac{k}{1}\right)\left(1 + \frac{k}{2}\right)\dots\left(1 + \frac{k}{n}\right),$$

and this is $> 1 + k\left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}\right)$; thus the limit is indefinitely great. Next suppose that $p+1$ is positive. Then the expression may be written in the form

$$\frac{p(p-1)\dots(p-\lambda+2)}{(\lambda-1)!}\left(1 - \frac{p+1}{\lambda}\right)\left(1 - \frac{p+1}{\lambda+1}\right)\left(1 - \frac{p+1}{\lambda+2}\right)\dots\left(1 - \frac{p+1}{n}\right)$$

where λ is the integer next greater than $p+1$; this is less than

$$\frac{p(p-1)\dots(p-\lambda+2)}{(\lambda-1)!} \frac{1}{\left(1 + \frac{p+1}{\lambda}\right)\left(1 + \frac{p+1}{\lambda+1}\right)\dots\left(1 + \frac{p+1}{n}\right)},$$

or than
$$\frac{p(p-1)\dots(p-\lambda+2)}{(\lambda-1)!} \frac{1}{1 + (p+1)\left(\frac{1}{\lambda} + \frac{1}{\lambda+1} + \dots \frac{1}{n}\right)},$$

hence the limit, when n is indefinitely increased, is zero. If $p = -1$, the limit is unity.

If $x = 1$, Lagrange's form of the remainder shews that the series converges if $p > -1$. The series diverges if $p < -1$, because the general term of the series increases indefinitely with n. The series oscillates if $p = -1$.

If $x = -1$, Cauchy's form of the remainder shews that if $p - 1 > -1$, or $p > 0$, the series is convergent. It is divergent if $p < 0$, for the sum of n terms of the series is

$$(-1)^{n-1} \frac{(p-1)(p-2)\dots(p-n+1)}{(n-1)!}.$$

(2) Let* $f(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{a^r}{a^{-2r} + x^2}$, where $a > 1$. For this function

$$f(0) = e^{-a}, \quad f^{(2k-1)}(0) = 0, \quad f^{(2k)}(0) = (-1)^k (2k)!\, e^{-a^{2k+1}};$$

thus the series for $f(x)$ is

$$\sum_0^{\infty} (-1)^k e^{-a^{2k+1}} x^{2k},$$

which is everywhere convergent.

The sum of the series, for $x = 0$, is $f(0)$, but in every neighbourhood of $x = 0$, the sum of the series and the value of $f(x)$ are different except at most at a finite number of points.

* Pringsheim, *Münchener Sitzungsber.*, 1892, p. 222.

(3) Let $f(x) = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{a^{-r}}{a^{-2r} + x^2}$, where $a > 1$. For this function, the Maclaurin's series is $\Sigma (-1)^k e^{a^{2k+1}} x^{2k}$, which diverges for every value of x except $x = 0$.

(4) Let* $f(x) = \sum_0^{\infty} \frac{(-1)^n}{n!} \frac{1}{1 + a^n x}$, where $a > 1$. This function is continuous on the right of the point $x = 0$, and has derivatives on the right of all orders at that point; the Maclaurin's series $\Sigma (-1)^n \left(\frac{1}{e}\right)^{a^n} . x^n$ thus obtained, converges for all positive values of x, but does not represent the function $f(x)$.

(5) Let $f(x) = \sum_0^{\infty} \frac{1}{n!} \frac{1}{1 + a^n x}$, where $a > 1$. For this function the Maclaurin's series does not converge in any neighbourhood of the point $x = 0$.

MAXIMA AND MINIMA OF A FUNCTION OF ONE VARIABLE

151. It has been shewn, in I, § 268, to be a necessary condition that a function $f(x)$ may have an extreme at the point $x = 0$, that the differential coefficient at that point should be zero, provided the function be such that a differential coefficient at $x = 0$ exists. Let us assume the function to be such that the first n differential coefficients $f'(x)$, $f''(x)$, ... $f^{(n)}(x)$ all exist and are continuous, at every point x such that $-\delta < x < \delta$. Let us further assume that $f'(0)$, $f''(0)$, ... $f^{(n-1)}(0)$ are all zero, and thus that $f^{(n)}(0)$ is that differential coefficient of lowest order which does not vanish at $x = 0$.

We have then $f(x) - f(0) = \frac{x^n}{n!} f^{(n)}(\theta x)$; where $0 < \theta < 1$, and x is such that $-\delta < x < \delta$. Since $f^{(n)}(x)$ is continuous at $x = 0$, a neighbourhood $(-\delta', \delta')$ of that point, interior to $(-\delta, \delta)$, can be so determined that $f^{(n)}(\theta x)$ has the same sign as $f^{(n)}(0)$, provided $-\delta' \leq x \leq \delta'$. If n be odd, the sign of $f(x) - f(0)$, in the interval $(-\delta', \delta')$, depends upon that of x; and therefore $f(x)$ has neither a maximum nor a minimum at the point $x = 0$. If n be even, the sign of $f(x) - f(0)$ is the same as that of $f^{(n)}(0)$, in the whole interval $(-\delta', \delta')$, and therefore $f(x)$ has a maximum or minimum at $x = 0$, according as $f^{(n)}(0)$ is negative or positive. The following theorem for determining whether a maximum, or a minimum, exists at a point at which the differential coefficient of a function $f(x)$ vanishes has therefore been established:

If the first n differential coefficients of a function $f(x)$ all exist, and are continuous at all interior points of the interval $(-\delta, \delta)$; and if $f^{(n)}(x)$ be the differential coefficient of lowest order which does not vanish at the point $x = 0$, then (1), *if n be odd, there is neither a maximum nor a minimum of the function $f(x)$ at the point $x = 0$; and* (2), *if n be even, the point $x = 0$ is a maximum or a minimum of $f(x)$, according as $f^{(n)}(0)$ is negative or positive.*

* Pringsheim, *Math. Annalen*, vol. XLII (1893), p. 161, and vol. XLIV (1894), p. 54.

It is unnecessary for the application of the criterion given in this theorem that $f(x)$ be capable of representation in a neighbourhood of the point $x = 0$ by a convergent power-series. There exist functions with differential coefficients of all orders, which all vanish at the point $x = 0$.

EXAMPLES

(1)* Let $f(x) = x^2 - e^{-\frac{1}{x^2}}$, and $f(0) = 0$. In this case $f'(0) = 0$, $f''(0) = 2$; and $f'(x)$, $f''(x)$ are continuous in any neighbourhood of $x = 0$. The theorem establishes that $f(x)$ has a minimum at $x = 0$, although $f(x)$ cannot be represented by a power-series in any neighbourhood of the point.

(2)* The function defined by $f(x) = e^{-\frac{1}{x^2}}$, $f(0) = 0$, has a minimum at $x = 0$; and yet the theorem is not applicable, because the differential coefficients of all orders vanish at $x = 0$.

(3)* The function defined by $f(x) = xe^{-\frac{1}{x^2}}$, $f(0) = 0$, has neither a maximum nor a minimum at $x = 0$. As in (2), the above theorem is in this case inapplicable.

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

152. Let us assume a function $f(x, y)$ to be defined for all values of x, y in the domain defined by $a - \delta \leq x \leq a + \delta$, $b - \delta' \leq y \leq b + \delta'$. Under proper conditions as to the existence and continuity of the partial differential coefficients of $f(x, y)$, of a finite number n of orders, it is possible to obtain an expression for $f(a + h, b + k) - f(a, b)$ consisting of terms involving the first n powers of h and k, together with a remainder analogous to the remainder in Taylor's theorem, such expression being valid for values of h, k, such that $|h| < \delta$, $|k| < \delta'$. It is however, for the present purpose, unnecessary to consider the least stringent set of conditions relating to the partial differential coefficients of the various orders, which are sufficient to allow the extension of Taylor's theorem to the case of a function of two variables. It will here be assumed that, for all values of x and y such that $a - \delta < x < a + \delta$, $b - \delta' < y < b + \delta'$, the partial differential coefficients of $f(x, y)$ of the first n orders all exist, and are finite; and further, that they are all continuous, for this range of values of x and y, with respect to (x, y). In accordance with the theorem of I, § 314, the order of differentiation, in each of the mixed partial differential coefficients, is in this case immaterial.

Taking values of h and k which are numerically less than δ, δ' respectively, let $f(a + th, b + tk)$ be denoted by $F(t)$, the variable t having the domain $(-1, +1)$. The conditions contained in the last theorem of I, § 309 being in this case satisfied, the differential coefficient $F'(t)$ of $F(t)$

* These examples are given by Scheeffer, *Math. Annalen*, vol. XXXV (1890), p. 542.

exists, and is equal to $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x, y)$, where $x=a+th$, $y=b+tk$. Similarly, it is seen that all the differential coefficients

$$F''(t),\ F'''(t),\ \ldots\ F^{(n)}(t)$$

exist, and are continuous; and that

$$F^{(r)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^r f(x, y).$$

In accordance with the theorem of § 142, we have

$$F(t)=F(0)+tF'(0)+\frac{t^2}{2!}F''(0)+\ldots+\frac{t^{n-1}}{(n-1)!}F^{(n-1)}(0)+\frac{t^n}{n!}F^{(n)}(\theta t),$$

where θ is a number such that $0<\theta<1$.

Since this holds for $t=1$, we see that

$$f(a+h, b+k)=f(a, b)+\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)f(a, b)+\frac{1}{2!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^2 f(a, b)+\ldots$$
$$\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^{n-1}f(a, b)+\frac{1}{n!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^n f(a+\theta h, b+\theta k).$$

This is an extension of Taylor's theorem to the case of a function of two variables. It has been established for all values of h, k such that $|h|<\delta$, $|k|<\delta'$, on the hypothesis that $f(x, y)$ and all its partial differential coefficients exist for all values of x and y such that

$$a-\delta<x<a+\delta,\ b-\delta'<y<b+\delta',$$

and that they are all continuous with respect to the two-dimensional continuum (x, y).

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

153. Necessary and sufficient conditions have been stated, in the theorem of I, § 321, that the point $(0, 0)$ may be a point at which a function $f(x, y)$ has a maximum, or a minimum. The general theory of maxima and minima of functions of two variables has been discussed by Scheeffer*, Dantscher†, and Stolz‡, the last of whom has extended Scheeffer's method to the case of functions of any number of variables. The account which will here be given of the general theory is based upon the investigations of Scheeffer, as modified by Stolz.

Let the function $f(x, y)$ be such that either $f(x, y)-f(0, 0)$ is representable in a neighbourhood of the point $(0, 0)$ by a convergent series

* *Math. Annalen*, vol. XXXV (1890), p. 541. † *Ibid.* vol. XLII (1893), p. 89.

‡ *Sitzungsber.* of the Vienna Academy, vols. XCIX (IIa), C(IIa); also *Grundzüge*, vol. I, p. 211. An account of the various theories is given in Hancock's treatise, *Theory of Maxima and Minima*, Boston.

consisting of powers of x and y, or else that it is such that the theorem of § 152 is applicable, so that

$$f(x, y) - f(0, 0) = G_n(x, y) + R_{n+1}(x, y),$$

where $G_n(x, y)$ consists of terms of dimensions not higher than n, in x and y; and $R_{n+1}(x, y)$ is either a convergent series of which the terms of lowest dimension are of the order $n + 1$, or has the form of the remainder given in § 152, consisting of terms of dimension $n + 1$ in θx, θy, where $0 < \theta < 1$; and in the latter case it will be assumed that the differential coefficients in that remainder are bounded in the whole domain. It will be shewn that, under a certain condition, the problem of determining whether the point $(0, 0)$ is a point at which $f(x, y)$ has a maximum or a minimum is reducible to the solution of the corresponding problem relating to the rational integral function $G_n(x, y)$. The following general theorem will be established:

The function $f(x, y)$ having in the neighbourhood of $(0, 0)$ the character above described, if an index n, and two positive numbers c, δ can be so determined that (1), for all values of x such that $0 < |x| < \delta$, the upper and lower boundaries of $G_n(x, y)$, for a constant value of x, and for all values of y in the interval $(-x, x)$, are in absolute value not less than $c|x|^n$; and (2), that, if $0 < |y| < \delta$, the upper and lower boundaries of $G_n(x, y)$, for a constant value of y, and for all values of x in the interval $(-y, y)$, are in absolute value not less than $c|y|^n$; then the two functions $f(x, y)$, $G_n(x, y)$ have both either a proper maximum, or both a proper minimum, or both neither a maximum nor a minimum, at the point $(0, 0)$.

To prove this theorem, we first observe that $R_{n+1}(x, y)$ can be regarded as a homogeneous function of x and y of degree $n + 1$, in which the coefficients depend upon x and y. By giving each of the coefficients its greatest possible value, for $|x| < \delta$, $|y| < \delta$, we see that

$$|R_{n+1}(x, y)| < A_0|x|^{n+1} + A_1|x|^n|y| + \ldots + A_{n+1}|y|^{n+1};$$

where $A_0, A_1, \ldots A_{n+1}$ are positive numbers.

If now $|y| \leq |x|$, we have

$$|R_{n+1}(x, y)| < (A_0 + A_1 + \ldots + A_{n+1})|x||x|^n;$$

hence we see that a number $\delta' < \delta$ can be so chosen that

$$|R_{n+1}(x, y)| < \epsilon|x|^n,$$

where ϵ is an arbitrarily chosen positive number, provided $|x| < \delta'$, $|y| \leq |x|$. In a similar manner we can shew that δ' can be so chosen that $|R_{n+1}(x, y)| < \epsilon|y|^n$, provided $|x| \leq |y|$, and $|y| < \delta'$.

Let now the upper and the lower boundaries of $G_n(x, y)$, for a constant value of x, and for all values of y such that $|y| \leq |x|$, be denoted by $G_n(x, \bar{\phi}(x))$, $G_n(x, \phi(x))$ respectively. Also let the upper and the lower

boundaries of $G_n(x, y)$, for a constant value of y, and for all values of x such that $|x| \leqq |y|$, be denoted by $G_n(\bar{\psi}(y), y)$, $G_n(\underline{\psi}(y), y)$ respectively. We have then, provided $|x| < \delta'$, and $|y| \leqq |x|$,

$$G_n(x, \underline{\phi}(x)) - \epsilon |x|^n < f(x, y) - f(0, 0) < G_n(x, \bar{\phi}(x)) + \epsilon |x|^n;$$

also, provided $|y| < \delta'$, $|x| \leqq |y|$, we have

$$G_n(\underline{\psi}(y), y) - \epsilon |y|^n < f(x, y) - f(0, 0) < G_n(\bar{\psi}(y), y) + \epsilon |y|^n.$$

First, let us assume that $G_n(0, 0)$ is a proper minimum of $G_n(x, y)$, and that the conditions of the theorem are satisfied. By the theorem of I, § 321, $G_n(x, \phi(x))$, $G_n(\underline{\psi}(y), y)$ are both positive, for sufficiently small values of x and y; we may suppose δ' to be so small that these conditions are satisfied, provided $|x| < \delta'$, $|y| < \delta'$.

We have then $G_n(x, \phi(x)) \geqq c|x|^n$, if $0 < |x| < \delta'$, $|y| \leqq |x|$; and $G_n(\underline{\psi}(y), y) \geqq c|y|^n$, if $0 < y < \delta'$, $|x| \leqq |y|$.

It now follows that

$$(c - \epsilon)|x|^n < f(x, y) - f(0, 0), \text{ for } 0 < |x| < \delta', \ |y| \leqq |x|,$$

and that

$$(c - \epsilon)|y|^n < f(x, y) - f(0, 0), \text{ for } 0 < |y| < \delta', \ |x| \leqq |y|.$$

Since ϵ can be chosen so as to be less than c, we see that $f(x, y) - f(0, 0)$ is positive for all values of x and y such that $0 < |x| < \delta'$, $0 < |y| < \delta'$, and therefore $f(0, 0)$ is a proper minimum of $f(x, y)$.

Next, let us assume that $G_n(0, 0)$ is a proper maximum of $G_n(x, y)$; then $G_n(x, \bar{\phi}(x))$, $G_n(\bar{\psi}(y), y)$ are both negative, for sufficiently small values of x and y. We therefore assume that

$$G_n(x, \bar{\phi}(x)) \leqq -c|x|^n, \text{ for } 0 < |x| < \delta', \text{ and } |y| \leqq |x|;$$

and that $G_n(\psi(y), y) \leqq -c|y|^n$, for $0 < |y| < \delta'$, $|x| \leqq |y|$.

We have then $f(x, y) - f(0, 0) < -(c - \epsilon)|x|^n$, for $0 < |x| < \delta'$, and $|y| \leqq |x|$; and also $f(x, y) - f(0, 0) < -(c - \epsilon)|y|^n$, for $0 < |y| < \delta'$, $|x| \leqq |y|$. Since ϵ may be taken to be $< c$, it follows that $f(0, 0)$ is a proper maximum of $f(x, y)$.

Lastly, let us assume that $G_n(0, 0)$ is neither a maximum nor a minimum of $G_n(x, y)$. In this case we may, for example, assume that

$$G_n(x, \bar{\phi}(x)) \geqq cx^n, \ G_n(x, \underline{\phi}(x)) \leqq -cx^n, \text{ for } 0 < x < \delta.$$

We have then, $\quad f(x, \bar{\phi}(x)) - f(0, 0) > (c - \epsilon)x^n,$

and $\quad f(x, \underline{\phi}(x)) - f(0, 0) < -(c - \epsilon)x^n,$

provided $0 < x < \delta'$. Since ϵ may be taken to be less than c, these two differences are of opposite signs; therefore $f(0, 0)$ is neither a maximum nor a minimum of $f(x, y)$.

It should be observed that this theorem does not always suffice to decide whether the point (0, 0) is a point at which $f(x, y)$ has an extreme value, or not. For it may happen that, for a given function $f(x, y)$ of the assumed type, no value of n can be determined, for which the conditions stated in the theorem hold, and therefore the theorem is inapplicable however great n may be taken.

If $f(x, y) = [u(x, y)]^2$, where $u(x, y)$ vanishes at points of a locus which passes through the point (0, 0), then the function $f(x, y)$ is one for which the theorem is inapplicable; the point (0, 0) is in this case a point at which $f(x, y)$ has an improper minimum.

In general the theorem is inapplicable in the case of any function which attains the value zero, at points other than (0, 0), in every neighbourhood of that point, but which has one and the same sign at all points at which it does not vanish.

154. The simplest case in which the theorem of § 153 can be applied is that in which the function $G_n(x, y)$ is a homogeneous function of degree n. For such a function $G_n(x, y)$, three cases arise.

(1) If $G_n(x, y)$ be a definite form, *i.e.* if $G_n(x, y)$ has one and the same sign for all values of (x, y) except (0, 0), then $G_n(0, 0)$ is a proper minimum, or a proper maximum, according as that sign is positive or negative.

(2) If $G_n(x, y)$ be an indefinite form, *i.e.* if there are points in every neighbourhood of (0, 0) at which $G_n(x, y)$ is positive, and others at which it is negative, there are other points besides (0, 0) at which the function vanishes, and there is no extreme of the function $G_n(x, y)$ at the point (0, 0).

(3) If $G_n(x, y)$ be semi-definite, *i.e.* if $G_n(x, y)$ vanishes at points other than (0, 0), but has a fixed sign at all points at which it does not vanish, then $G_n(0, 0)$ is an improper extreme of $G_n(x, y)$.

It should be observed that, if n be odd, $G_n(x, y)$ is necessarily an indefinite form.

It will be shewn that, when $G_n(x, y)$ is definite or indefinite, it satisfies the conditions stated in the theorem of § 153; accordingly $f(x, y)$ has a proper maximum or else a proper minimum, when $G_n(x, y)$ is a definite form; and $f(x, y)$ has no extreme when $G_n(x, y)$ is an indefinite form.

When $G_n(x, y)$ is a semi-definite form, no conclusion can be drawn as to the existence of an extreme of the function $f(x, y)$, as the conditions contained in the theorem of § 153 are not satisfied.

If $G_n(x, y)$ be definite, it is of the form

$$G_n(x, y) = A \prod_{r=1}^{r=k} \{(y - \gamma_r x)^2 + \delta_r^2 x^2\},$$

where $n = 2k$. Let us assume that A is positive; then

$$G_n(x, y) \geqq A \prod_{r=1}^{r=k} \delta_r^2 . x^n,$$

for all values of x and y; it follows that the first condition of the theorem is satisfied.

The case in which A is negative may be treated in a similar manner.

Again

$$(y - \gamma_r x)^2 + \delta_r^2 x^2 = \left(x\sqrt{\gamma_r^2 + \delta_r^2} - \frac{\gamma_r y}{\sqrt{\gamma_r^2 + \delta_r^2}}\right)^2 + \frac{\delta_r^2}{\gamma_r^2 + \delta_r^2} y^2 > \frac{\delta_r^2}{\gamma_r^2 + \delta_r^2} y^2,$$

hence $\quad | G_n(\bar{\psi}(y), y) | > | G_n(\psi(y), y) | \geqq | A | \, y^n \prod_1^k \frac{\delta_r^2}{\gamma_r^2 + \delta_r^2}$:

and therefore the second condition of the theorem is satisfied.

Next let $G_n(x, y)$ be an indefinite form; in which case $G_n(x, y)$ has neither a maximum nor a minimum at $(0, 0)$. Let (x', y') be a point at which $G_n(x', y') > 0$; and first suppose that $| y' | \leqq | x' |$, so that $| x' | > 0$.

Let x, y be such that $y/x = y'/x'$, and let x, x' have the same sign; we have $G_n(x, y) > 0$, and it follows that

$$G_n(x, \bar{\phi}(x)) \geqq \frac{G_n(x', y')}{| x' |^n} | x |^n > 0.$$

Next suppose that $| x' | \leqq | y' |$, so that $| y' | > 0$; we then shew in the same manner that

$$G_n(\bar{\psi}(y), y) \geqq \frac{G_n(x', y')}{| y' |^n} | y |^n > 0,$$

where y has the same sign as y'.

Since there are also values of (x', y') such that $G_n(x', y') < 0$, we can shew as before that

$$G_n(x, \phi(x)) \leqq \frac{G_n(x', y')}{| x' |^n} | x |^n < 0,$$

where x and x' have the same sign, and that

$$G_n(\psi(y), y) \leqq \frac{G_n(x', y')}{| y' |^n} | y |^n < 0,$$

where y has the same sign as y'. It has thus been established that, when $G_n(x, y)$ is an indefinite form, the conditions of the theorem of § 153 are satisfied.

The following general result has now been obtained:

If $f(x, y) - f(0, 0)$ be of the form $G_n(x, y) + R_{n+1}(x, y)$, where $G_n(x, y)$ is a homogeneous function of degree n, then, if n be odd, $f(0, 0)$ is not an

extreme of $f(x, y)$. If n be even, and $G_n(x, y)$ be an indefinite form, $f(0, 0)$ is not an extreme of $f(x, y)$. If $G_n(x, y)$ be a definite form, $f(0, 0)$ is a proper minimum, or a proper maximum, of $f(x, y)$, according as $G_n(x, y)$ is positive or negative. If $G_n(x, y)$ be a semi-definite form, no conclusion can be drawn from the consideration of $G_n(x, y)$ by itself, as to the existence or non-existence of an extreme of $f(x, y)$ at the point $(0, 0)$.

155. When those terms in the expansion of $f(x, y)$ in powers of x and y, which are of the lowest degree, give a semi-definite form, it is necessary to take a value of n greater than this lowest degree; we have therefore to consider the case in which $G_n(x, y)$ is not homogeneous. We have then, in order to apply the theorem of I, § 321, to $G_n(x, y)$, to determine the four functions $G_n(x, \bar{\phi}(x))$, $G_n(x, \underline{\phi}(x))$, $G_n(\bar{\psi}(y), y)$, $G_n(\underline{\psi}(y), y)$. The values $y = \bar{\phi}(x)$, $y = \underline{\phi}(x)$, may be either in the interior, or at the ends of the interval $(-x, x)$. In the former case they must be such as to satisfy the condition $\dfrac{dG_n(x, y)}{dy} = 0$; in the latter case they will in general not satisfy this condition, although they may do so. The method of procedure, by which $G_n(x, \bar{\phi}(x))$, $G_n(x, \underline{\phi}(x))$ may be obtained, is to determine the various solutions of the equation $\dfrac{dG_n(x, y)}{dy} = 0$, in which y is expressed as a series of fractional or integral powers of x; only such values of y need be considered as vanish for $x = 0$.

Let $y = P_1(x)$, $y = P_2(x)$, ... $y = P_r(x)$ denote these series; we then form the expressions $G_n(x, -x)$, $G_n(x, x)$, $G_n(x, P_1(x))$, ... $G_n(x, P_r(x))$.

It is certain that the two expressions $G_n(x, \bar{\phi}(x))$, $G_n(x, \underline{\phi}(x))$ must both occur amongst these $r + 2$ expressions, and a comparison of the leading terms of these expressions will enable us to identify the two expressions required. If the indices of the leading terms in $G_n(x, \bar{\phi}(x))$, $G_n(x, \underline{\phi}(x))$, are not greater than n, the first condition of the general theorem is satisfied. A similar method, in which the equation $\dfrac{dG_n(x, y)}{dx} = 0$ is employed, will lead to the determination of $G_n(\bar{\psi}(y), y)$, $G_n(\underline{\psi}(y), y)$.

The details of the investigation have been fully carried out by Scheeffer, who employs the somewhat more symmetrical, but practically less simple, method, in which x and y are expressed as series involving a single parameter.

When, for any value of n, the result of this process is that $G_n(x, y)$ is such that the conditions contained in the theorem of § 153 are not satisfied, a larger value of n in which more terms of $f(x, y)$ are included in $G_n(x, y)$ must be taken, and the process repeated until a definite result is obtained.

EXAMPLES

(1) Let $f(x, y) - f(0, 0) = ax^2 + 2hxy + by^2 + R_3(x, y)$. The form $ax^2 + 2hxy + by^2$ is definite if $ab - h^2$ is positive; in this case $f(0, 0)$ is a proper minimum or a proper maximum of $f(x, y)$, according as a is positive or negative. If $ab - h^2$ is negative, then $ax^2 + 2hxy + by^2$ is an indefinite form, and in that case $f(0, 0)$ is not an extreme of $f(x, y)$. If $ab - h^2 = 0$, the form $ax^2 + 2hxy + by^2$ is semi-definite, and no conclusion can be drawn as to the existence of an extreme of $f(x, y)$. It will be necessary in the last case to consider terms of order higher than 2 as included in $G_n(x, y)$. By taking $n = 3, 4, \ldots$ a function $G_n(x, y)$ may be determinable which satisfies the conditions of the theorem of § 153.

(2)* Let $f(x, y) = ay^2 + 2bx^2y + cx^4 + R_5(x, y)$, where a is positive; in this case we have

$$\frac{\partial G_4}{\partial y} = 2(ay + bx^2),$$

and this vanishes for $y = -\frac{b}{a}x^2$. We have

$$G_4(x, -x) = ax^2 - 2bx^3 + cx^4, \qquad G_4(x, x) = ax^2 + 2bx^3 + cx^4,$$

and
$$G_4\left(x, -\frac{b}{a}x^2\right) = \frac{ac - b^2}{a}x^4.$$

It follows that $G_4(x, -x)$ or $G_4(x, x)$ is the value of $G_4(x, \phi(x))$, and that $G_4\left(x, -\frac{b}{a}x^2\right)$ is that of $G_4(x, \phi(x))$. If $ac - b^2$ be negative, the two expressions $G_4(x, \bar{\phi}(x))$, $G_4(x, \phi(x))$ have opposite signs; therefore $f(0, 0)$ is not an extreme of $f(x, y)$. If $ac - b^2$ be positive, the two expressions are both positive, and the first condition of the general theorem is satisfied, since the indices of x in the leading terms are not greater than 4.

We find that $\frac{\partial G_4}{\partial x} = 0$ has for roots $x = \pm\sqrt{-\frac{by}{c}}$, and $x = 0$; we thus form the expressions
$$G_4(0, y) = ay^2, \quad G_4(\pm y, y) = ay^2 + 2by^3 + cy^4.$$

It is unnecessary to consider the roots $x = \pm\sqrt{-\frac{by}{c}}$, because, for sufficiently small values of y, $|x| > |y|$, and thus these roots could not give the extremes for $|x| \leq |y|$. Remembering that a and c are both positive, let $b \geqq 0$, then the value of $G_4(\bar{\psi}(y), y)$ is $ay^2 + 2by^3 + cy^4$, and that of $G_4(\psi(y), y)$ is ay^2; these values being both positive, we see that $G_4(0, 0)$ is a proper minimum of $G_4(x, y)$. The same conclusion may be made when $b \leqq 0$. Therefore, when $ac - b^2 > 0$, $a > 0$, since the conditions of the theorem of § 153 are satisfied, $f(x, y)$ has a proper minimum at $(0, 0)$. If $ac - b^2 > 0$, $a < 0$, there is a proper maximum. If $ac - b^2 = 0$, we have

$$f(x, y) = \frac{1}{a}(ay + bx^2)^2 + R_5(x, y);$$

hence $G_4(x, y)$ has an improper extreme at $(0, 0)$, and no conclusion can be drawn as regards $f(x, y)$.

(3)† Let $f(x, y) = y^2 + x^2y + R_4(x, y)$. We find $\frac{\partial G_3}{\partial y} = 2y + x^2$, and thence we have

$$G_3(x, -\tfrac{1}{2}x^2) = -\tfrac{1}{4}x^4;$$

also
$$G_3(x, x) = x^2 + x^3, \qquad G_3(x, -x) = x^2 - x^3.$$

It is clear that, in this case, $G_3(x, \phi(x))$, $G_3(x, \phi(x))$ have opposite signs, provided x be sufficiently small, therefore $G(x, y)$ has no extreme at the point $(0, 0)$. Since

$$G_3(x, \phi(x)) = -\tfrac{1}{4}x^4,$$

* See Stolz, *Grundzüge*, vol. I, p. 234.

† Scheeffer, *loc. cit.* p. 573.

it is not the case that $|G_3(x, \phi(x))| \geqq c|x|^3$, for any value of c, in a neighbourhood of $x = 0$; the theorem of § 153 is therefore not applicable. No information is obtained as to whether $f(x, y)$ has an extreme at $(0, 0)$, or not. It will in fact be shewn, in the next example, that $y^2 + x^2y + x^4$ has a minimum at $(0, 0)$.

(4) Let $f(x, y) = y^2 + x^2y + x^4 + R_5(x, y)$. We find $\frac{\partial G_4}{\partial y} = 2y + x^2$, hence

$$\frac{\partial G_4}{\partial y} = 0 \text{ gives } y = -\tfrac{1}{2}x^2;$$

hence $$G_4(x, -\tfrac{1}{2}x^2) = \tfrac{3}{4}x^4 + \dots;$$

also $$G_4(x, x) = x^2 + x^3 + x^4, \quad G_4(x, -x) = x^2 - x^3 + x^4.$$

In this case $G_4(x, \phi(x))$, $G_4(x, \phi(x))$ are both positive, and are greater than $c|x|^4$ for a fixed c. It can be shewn that the other condition is also satisfied. It follows that $f(x, y)$ has a minimum at $(0, 0)$.

(5) Let $f(x, y) = x^2y^4 - 3x^4y^3 + x^6y^2 - 3xy^7 + y^8 - 10x^{10}y + 5x^{12} + R_{13}(x, y)$.

In this case $\frac{\partial G_{12}}{\partial y} = 0$ has the three roots

$$y = 5x^4 + \dots, \quad y = -\tfrac{1}{4}x^2 - \tfrac{40}{7}x^4 + \dots, \quad y = 2x^2 + \tfrac{9}{7}x^4 + \dots.$$

On substituting these values in $G_{12}(x, y)$, and forming also $G_{12}(x, x)$, $G_{12}(x, -x)$, we find that $G_{12}(x, -\phi(x))$ is $G(x, -x)$ or $G(x, x)$, according as x is positive or negative; and the expression commences with the term x^6. We find for $G_{12}(x, \phi(x))$ an expression $-4x^{10} + \dots$. Since $G_{12}(x, \phi(x))$, $G_{12}(x, \phi(x))$ have opposite signs, it follows that $(0, 0)$ is not a point at which $G_{12}(x, y)$ has an extreme. Since the indices of the leading terms of $G_{12}(x, \phi(x))$, $G_{12}(x, \phi(x))$ are both less than 12, the condition of the theorem of § 153 is satisfied, and we can therefore infer that $f(x, y)$ has no extreme at $(0, 0)$.

THE LIMITS OF A SERIES INVOLVING A PARAMETER

156. A generalization of the theorems of Abel and Tauber relating to the convergence or oscillation of a power-series at the boundaries of its domain of convergence can be obtained by the consideration of series of the form

$$a_1\phi(t) + a_2\phi(2t) + \dots + a_n\phi(nt) + \dots,$$

where $\phi(t)$ converges to 1, as $t \sim 0$. The following theorem will be established:

If $a_1 + a_2 + \dots + a_n + \dots$ is a numerical series which oscillates between finite limits, and the series $a_1\phi(t) + a_2\phi(2t) + \dots + a_n\phi(nt) + \dots$ converges, for each positive value of t, to a sum $S(t)$, then the upper and lower limits of $S(t)$ as $t \sim 0$, are finite if $\phi(t)$ satisfies the conditions that, (1), $\phi(t)$ converges to 1, as $t \sim 0$, and to 0, as $t \sim \infty$, (2), $\phi'(t)$ exists for all positive values of t as a finite number, and is absolutely summable in the indefinite interval $(0, \infty)$. In case $a_1 + a_2 + \dots$ converges to a definite sum s, $\lim_{t \sim 0} \sum_{n=1}^{\infty} a_n\phi(nt) = s$. If $\phi(t)$ steadily diminishes as t increases from zero indefinitely, the condition (2) may be omitted.

The condition of convergence of $\sum\limits_{n=1} a_n \phi(nt)$, for $t > 0$, is satisfied in particular if $t^{1+k}\phi(t)$ is bounded for all values of t greater than some positive number c, where k is some number > 0. For, since $\sum\limits_{n=1} a_n$ is bounded, a_n is bounded, and $\sum\limits_{n=p}^{\infty} a_n\phi(nt)$ has each term $a_n\phi(nt)$ numerically less than a fixed multiple of $\frac{1}{t^{1+k}} \cdot \frac{1}{n^{1+k}}$, where, for a fixed value of t, $nt > c$; and thus, for each value of t, $\sum\limits_{n=1}^{\infty} a_n\phi(nt)$ is absolutely convergent.

The partial sum s_n of the series $\sum\limits_{n=1}^{\infty} a_n$ may be expressed by

$$\tfrac{1}{2}(\bar{s} + \underline{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})\,\theta_n + \epsilon_n,$$

where θ_n is in the interval $(-1, 1)$, $\bar{s}$ and $\underline{s}$ are the upper and lower sums of $\sum\limits_{n=1}^{\infty} a_n$, and ϵ_n is a number such that $|\epsilon_n| < \delta$, provided n is not less than an integer m, dependent on the arbitrarily chosen positive number δ.

Since $S(t) = \sum\limits_{n=1}^{\infty} (s_n - s_{n-1})\,\phi(nt)$, and $s_n\phi(nt)$ converges to 0, as $n \sim \infty$, since s_n is bounded, $S(t)$ can be expressed by

$$\sum_{n=1}^{\infty} s_n\{\phi(nt) - \phi(\overline{n+1}t)\}$$

or by $\quad \tfrac{1}{2}(\bar{s} + \underline{s})\,\phi(t) + \sum\limits_{n=1}^{\infty} [\tfrac{1}{2}(\bar{s} - \underline{s})\,\theta_n + \epsilon_n]\{\phi(nt) - \phi(\overline{n+1}t)\}$.

The sum $\sum\limits_{n=1}^{\infty}$ may be divided into two parts $\sum\limits_{n=1}^{m}$, $\sum\limits_{n=m+1}^{\infty}$, where $|\epsilon_n| < \delta$, for $n \geqq m$. The first part of the sum converges to 0, as $t \sim 0$, since each term converges to 0, and the number of terms is fixed. Thus $\left|\sum\limits_{n=1}^{m}\right| < \eta$, provided t is $<$ a number τ_η dependent on the arbitrarily chosen number η.

The sum $\sum\limits_{n=m+1}^{\infty}$ may be written as

$$-\sum_{n=m+1}^{\infty} [\tfrac{1}{2}(\bar{s} - \underline{s})\,\theta_n + \epsilon_n] \int_{nt}^{(n+1)t} \phi'(t)\,dt,$$

and this lies between the two numbers

$$\pm [\tfrac{1}{2}(\bar{s} - \underline{s}) + \delta] \int_0^{\infty} |\phi'(t)|\,dt$$

which, by condition (2), are definite. Since δ is arbitrary, it now follows that $S(t)$ has its upper and lower limits, for $t \sim 0$, in the interval defined by the two numbers

$$\tfrac{1}{2}(\bar{s} + \underline{s}) \pm \tfrac{1}{2}(\bar{s} - \underline{s}) \int_0^{\infty} |\phi'(t)|\,dt,$$

which establishes the theorem.

It will be observed that, in case $\phi(t)$ steadily diminishes as t increases from 0 indefinitely, the sum

$$\sum_{n=m+1}^{\infty} [\tfrac{1}{2}(\bar{s} - \underline{s})\,\theta_n + \epsilon_n]\{\phi(nt) - \phi(\overline{n+1}t)\}$$

lies between the two numbers $\pm[\frac{1}{2}(\bar{s} - \underline{s}) + \delta]\,\phi(\overline{m+1}t)$, which converge to the numbers $\pm\frac{1}{2}(\bar{s} - \underline{s})$, as $t \sim 0$, $\delta \sim 0$. In this case the limits of $S(t)$ lie in the interval $(\underline{s}, \bar{s})$.

An important example of the above theorem is the case in which $\phi(t) = e^{-t}$. This function diminishes steadily as t increases indefinitely from the value 0, and the series $\Sigma a_n e^{-t}$ is convergent when Σa_n is bounded. It follows that the limits of the sum of the series $a_1 e^{-t} + a_2 e^{-2t} + \ldots$, as $t \sim 0$, are in the interval bounded by the upper and lower sums of $a_1 + a_2 + \ldots$.

Now let $e^{-t} = r$, and we obtain the following extension of Abel's theorem, already proved in § 127:

If the series $a_0 + a_1 + a_2 + \ldots$ is bounded, then $a_0 + a_1 r + a_2 r^2 + \ldots$, which is convergent for $r < 1$, has a sum $s(r)$, such that the upper and lower limits of $s(r)$, as $r \sim 1$, are in the interval bounded by the lower and upper sums of Σa_n.

A precisely similar theorem is obtained in the case $\phi(t) = e^{-t^2}$, for the series $\sum\limits_{n=1} a_n e^{-n^2 t^2}$.

157. Another important example of the theorem of § 156 is the case in which

$$\phi(t) = \left(\frac{\sin t}{t}\right)^2, \quad \phi'(t) = \frac{\sin 2t}{t^2} - \frac{2\sin^2 t}{t^3}.$$

In this case $t^2\phi(t) \leq 1$, $|t^2\phi'(t)| < 3$, for all values of t; also $\phi'(t)$ converges to 0 as $t \sim 0$.

The theorem is immediately applicable to shew that, when the series $a_0 + a_1 + a_2 + \ldots$ oscillates between finite limits, the limit, as $t \sim 0$, of the sum $S(t)$ of the convergent series

$$a_0 + a_1\left(\frac{\sin t}{t}\right)^2 + a_2\left(\frac{\sin 2t}{2t}\right)^2 + \ldots + a_n\left(\frac{\sin nt}{nt}\right)^2 + \ldots$$

is such that $\overline{S}_{t\sim 0}(t)$, $\underline{S}_{t\sim 0}(t)$ are both finite.

In order to estimate the interval in which $\lim\limits_{t\sim 0} S(t)$ lies, in terms of the upper and lower sums of Σa_n, it is convenient to consider the series

$$\sum_{m+1}^{\infty} [\tfrac{1}{2}(\bar{s} - \underline{s})\,\theta_n + \epsilon_n]\left\{\left(\frac{\sin nt}{nt}\right)^2 - \left(\frac{\sin \overline{n+1}t}{\overline{n+1}t}\right)^2\right\}$$

independently of the investigation in the general theorem; m being such that $|\epsilon_n| < \delta$, for $n \geq m$.

Assuming that t is so small that $mt < \pi$, let s be the greatest integer such that $st < \pi$; and thus $st < \pi \leqq (s+1)t$. If we divide the above summation into two parts, $\sum\limits_{m+1}^{s-1}$ and $\sum\limits_{s}^{\infty}$, it is seen that in the first part all the expressions $\left(\frac{\sin nt}{nt}\right)^2 - \left(\frac{\sin \overline{n+1}\,t}{\overline{n+1}\,t}\right)^2$ are positive, since $\frac{\sin\theta}{\theta}$ diminishes steadily as θ increases from 0 to π. Therefore the sum

$$\sum_{m+1}^{s-1}\left[\tfrac{1}{2}(\bar{s}-\underline{s})\,\theta_n + \epsilon_n\right]\left\{\left(\frac{\sin nt}{nt}\right)^2 - \left(\frac{\sin(\overline{n+1}\,t)}{\overline{n+1}\,t}\right)^2\right\}$$

lies between $$\pm\left[\tfrac{1}{2}(\bar{s}-\underline{s})+\delta\right]\left\{\left(\frac{\sin mt}{mt}\right)^2 - \left(\frac{\sin st}{st}\right)^2\right\}.$$

As $t \sim 0$, these numbers converge to $\pm\left[\tfrac{1}{2}(\bar{s}-\underline{s})+\delta\right]$, since st converges to π, because $\pi - st < t$.

In the second part of the sum, $\left(\frac{\sin nt}{nt}\right)^2 - \left(\frac{\sin \overline{n+1}\,t}{\overline{n+1}\,t}\right)^2$ may be written in the form

$$-\frac{\sin t \sin(2n+1)t}{n^2t^2} + \frac{\sin^2(n+1)t}{t^2}\left\{\frac{1}{n^2} - \frac{1}{(n+1)^2}\right\}.$$

It now follows that

$$\sum_{s}^{\infty}\left[\tfrac{1}{2}(\bar{s}-\underline{s})\,\theta_n + \epsilon_n\right]\left\{\left(\frac{\sin nt}{nt}\right)^2 - \left(\frac{\sin \overline{n+1}\,t}{\overline{n+1}\,t}\right)^2\right\}$$

lies between

$$\pm\left[\tfrac{1}{2}(\bar{s}-\underline{s})+\delta\right]\left[\frac{1}{t}\sum_{s}^{\infty}\frac{1}{n^2} + \frac{1}{t^2}\sum_{s}^{\infty}\left\{\frac{1}{n^2} - \frac{1}{(n+1)^2}\right\}\right],$$

or between $$\pm\left[\tfrac{1}{2}(\bar{s}-\underline{s})+\delta\right]\left(\frac{1}{\overline{s-1}\,t} + \frac{1}{s^2t^2}\right);$$

and these numbers converge to

$$\pm\tfrac{1}{2}(\bar{s}-\underline{s})\left(\frac{1}{\pi}+\frac{1}{\pi^2}\right),$$

since st converges to π, as $t \sim 0$.

It has now been shewn that the sum of the series

$$a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{nt}\right)^2$$

has for its limits, as $t \sim 0$, numbers which are in the interval bounded by the two numbers $\tfrac{1}{2}(\bar{s}+\underline{s}) \pm \tfrac{1}{2}(\bar{s}-\underline{s})\left(1+\frac{1}{\pi}+\frac{1}{\pi^2}\right)$.

By Du Bois Reymond* this interval was given by the numbers

$$\tfrac{1}{2}(\bar{s}+\underline{s}) \pm \tfrac{1}{2}(\bar{s}-\underline{s})\left(\frac{3}{2}+\frac{1}{\pi}+\frac{1}{\pi^2}\right).$$

* *Abh. der bayerisch. Akad.* vol. XII (1876), p. 136.

A more exact estimate than that given above has led G. C. Young* to determine the numbers as $\frac{1}{2}(\bar{s}+\underline{s}) \pm \frac{1}{2}(\bar{s}-\underline{s})\left(1+\frac{1}{\pi^2}\right)$.

158. A more general theorem than that of § 157 may be obtained by considering the case in which the series $\sum\limits_{n=0} a_n$, when summed by Césaro's method (C, r), oscillates boundedly. The general theorem, which is similar to a theorem given† by Fejér for the case $r = 1$, may be stated as follows:

If $a_0 + a_1 + a_2 + \ldots$ be a series which, when summed (C, r), where r is a positive integer, oscillates boundedly, and $\phi(t)$ be a function converging to 1, as $t \sim 0$, and such that $t^r \phi(t)$ converges to 0, as $t \sim \infty$, and is also such that

$$a_0 + a_1\phi(t) + a_2\phi(2t) + \ldots$$

is convergent for every value of t, (> 0), having $S(t)$ for its sum, then the upper and lower limits of $S(t)$, as $t \sim 0$, are both finite if the conditions are satisfied that (1), $\phi^{(r+1)}(t)$ exists and is continuous for all values of $t > 0$, and (2), that $t^{r+k+1}\phi^{(r+1)}(t)$ is bounded for all values of t greater than 1, where k is some number > 0. In case Σa_n is summable (C, r), the limit of $S(t)$, as $t \sim 0$, exists, and is equal to the sum (C, r) of Σa_n.

Since Σa_n oscillates finitely when summed (C, r), $\frac{a_n}{n^r}$ is bounded (see § 52); and the condition of convergence of the series $\Sigma a_n \phi(nt)$, for $t > 0$, is satisfied in particular if $t^{r+\lambda+1}\phi(t)$ is bounded for all values of $t > C > 0$, where λ is some number > 0. Moreover, if a_n is bounded, it will be sufficient that $t^{k+1}\phi(t)$ is bounded for all values of $t > C > 0$. The partial sum $C(n, r)$ of the series $\sum\limits_{n=0}^{\infty} a_n$ is denoted by $C_n^{(r)}$, or by $S_n^{(r)} \Big/ \frac{(n+r)!}{n!\,r!}$ (see § 47).

Let $C_n^{(r)} = \frac{1}{2}(\bar{C}^{(r)} + \underline{C}^{(r)}) + \frac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)})\,\theta_n + \epsilon_n$, where $-1 \leqq \theta_n \leqq 1$, and $|\epsilon_n| < \delta$, provided $n \geqq m$; where $\bar{C}^{(r)}$, $\underline{C}^{(r)}$ denote the upper and lower limits of $C_n^{(r)}$, as $n \sim \infty$.

Employing the expression (4), of § 48, the series $\sum\limits_{n=0} a_n\phi(nt)$ may be written $\sum\limits_{n=0} \phi(nt)\left\{S_n^{(r)} - \binom{r+1}{1} S_{n-1}^{(r)} + \binom{r+1}{2} S_{n-2}^{(r)} - \ldots\right\}$, where the series in the bracket contains $r + 2$ terms, or $n + 1$ terms according as r is $< n - 1$, or $r \geqq n$.

The sum of $n + 1$ terms of the series $\sum\limits_{n=0} a_n\phi(nt)$ differs from the sum of $n + 1$ terms of the series

$$\sum_{n=0} S_n^{(r)}\left[\phi(nt) - \binom{r+1}{1}\phi(\overline{n+1}\,t)\right.$$
$$\left. + \binom{r+1}{2}\phi(\overline{n+2}\,t) - \ldots + (-1)^{r+1}\phi(\overline{n+r+1}\,t)\right] \ldots (1)$$

* *Mess. of Math.* vol. XLIX (1919–20), p. 73.

† *Math. Annalen*, vol. LVIII (1904), p. 62.

by an expression of the form $\sum\limits_{p=0,\,q=1}^{p=r+1,\,q=r+1} a_{p,q} S_{n-p}^{(r)} \phi(\overline{n+q}t)$, of which the number of terms depends only on r. Each of these terms is less than a fixed multiple of $(n-p)^r \phi(\overline{n+q}t)$, or of $(n+q)^r \phi(\overline{n+q}t)$, which, for each value of t, converges to zero, as $n \sim \infty$. It follows that the series $\sum\limits_{n=0} a_n \phi(nt)$ may be rearranged in the form (1), without affecting its convergence, or its sum. Moreover the coefficient of $S_n^{(r)}$ in the series (1) can be expressed by $t^{r+1}\phi^{(r+1)}(X_n)$, where X_n is such that $nt < X_n < (n+r+1)t$.

The effect of substituting $\frac{1}{2}(\bar{C}^{(r)} + \underline{C}^{(r)})$ for all the expressions $C_n^{(r)}$ in the series $\sum\limits_{n=0} a_n \phi(nt)$ is to reduce it to the form

$$\tfrac{1}{2}(\bar{C}^{(r)} + \underline{C}^{(r)}) \sum_{n=0}^{\infty} \phi(nt) \left[\frac{(n+r)!}{r!\,n!} - (r+1)\frac{(n+r-1)!}{r!\,(n-1)!} + \frac{(r+1)\,r}{2!}\frac{(n+r-2)!}{r!\,(n-2)!} - \dots\right],$$

the numerical coefficient of $\phi(nt)$ is the coefficient of x^r in

$$(1+x)^{n+r} - (r+1)(1+x)^{n+r-1} + \frac{(r+1)\,r}{2!}(1+x)^{n+r-2} \quad \dots,$$

or in $(1+x)^{n+r}\left(1 - \dfrac{1}{1+x}\right)^{r+1}$, or in $x^{r+1}(1+x)^{n-1}$, and this is equal to 0, except that, when $n = 0$, the coefficient is 1. Thus this part of the series reduces to $\frac{1}{2}(\bar{C}^{(r)} + \underline{C}^{(r)})$.

We next consider the part $\frac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)})\,\theta_n + \epsilon_n$, of $C_n^{(r)}$, and take the summation from $n = 0$ to $n = m - 1$, where $|\epsilon_m| < \delta$, for $n \geqq m$. The limit of this part, as $t \sim 0$, is zero, since each term converges to zero, and the number of terms is fixed.

Next, we consider the series in which the n is taken from the value m to p, where p is an integer such that $(p+r+1)\,t < 1 < (p+r+2)\,t$, it being assumed that t is so small that $(m+r+1)\,t < 1$. If μ is the maximum of the continuous function $|\phi^{(r+1)}(t)|$ within the interval (0, 1), this part of the sum lies between the two numbers

$$\pm \{\tfrac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)}) + \delta\}\,\mu t^{r+1} \sum_{n=m}^{n=p} \frac{1}{r!}\frac{(r+n)!}{n!},$$

or between the two numbers

$$\pm \frac{\mu t^{r+1}}{r!}\frac{(r+p)!}{p!}\,[\tfrac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)}) + \delta]\left\{1 + \frac{p}{p+r} + \frac{p\,(p-1)}{(p+r)(p+r-1)} + \dots\right\},$$

or between the two numbers

$$\pm\,(p+r)^{r+1} t^{r+1} \cdot \frac{\mu}{r!}\,[\tfrac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)}) + \delta],$$

that is between the numbers $\pm \dfrac{\mu}{r!}\,[\tfrac{1}{2}(\bar{C}^{(r)} - \underline{C}^{(r)}) + \delta]$.

Lastly we have to consider the series

$$\sum_{n=p+1}^{\infty}\left[\tfrac{1}{2}\left(\bar{C}^{(r)}-\underline{C}^{(r)}\right)\theta_n+\epsilon_n\right]\frac{(n+r)!}{n!\,r!}(-1)^{r+1}\phi^{(r+1}(X_n)\,t^{r+1}.$$

The sum of this series lies between the two numbers

$$\pm\left[\tfrac{1}{2}\left(\bar{C}^{(r)}-\underline{C}^{(r)}\right)+\delta\right]\sum_{n=p+1}^{\infty}\frac{(n+r)!}{n!\,r!}\,\frac{N}{(nt)^{r+k+1}}\,t^{r+1},$$

where $|\,t^{r+k+1}\,\phi^{(r+1)}(t)\,|<N$, for $t>1$.

Thus the sum lies between the numbers

$$\pm\left[\tfrac{1}{2}\left(\bar{C}^{(r)}-\underline{C}^{(r)}\right)+\delta\right]\frac{N}{t^k r!}\sum_{p+1}^{\infty}\frac{(r+n)!}{n!}\,\frac{1}{n^{r+k+1}}.$$

The sum $\sum_{p+1}^{\infty}\frac{(r+n)!}{n!}\frac{1}{n^{r+k+1}}$ is less than $\sum_{p+1}^{\infty}\frac{(2n)^r}{n^{r+k+1}}$, if $r<p$, which holds for all sufficiently small values of t; and this sum is less than $\frac{2^r}{kp^k}$. It now follows that this part of the limit of the sum of the series to be estimated lies between the two numbers

$$\pm\left[\tfrac{1}{2}\left(\bar{C}^{(r)}-\underline{C}^{(r)}\right)+\delta\right]\frac{N.\,2^r}{r!\,k}$$

since pt converges to the value 1, as $t\sim 0$.

It has now been shewn that the limits of the sum of the convergent series $a_0+a_1\phi(t)+a_2\phi(2t)+\dots$, as $t\sim 0$, are between two numbers

$$\tfrac{1}{2}\left(\bar{C}^{(r)}+\underline{C}^{(r)}\right)\pm\lambda.\tfrac{1}{2}\left(\bar{C}^{(r)}-\underline{C}^{(r)}\right),$$

where λ is a fixed number; since δ is arbitrarily small. In case the series Σa_n is summable (C, r), the sum of the series $a_0+a_1\phi(t)+\dots$ converges to $C^{(r)}$.

A special case obtained by taking $\phi(t)=e^{-t}$, and then writing $h=e^{-t}$ is the following, already obtained in § 128 (3).

If the series $a_0+a_1h+a_2h^2+\dots$ is convergent, and have $s(h)$ for sum, for $0\leq h<1$, and the series $a_0+a_1+a_2+\dots$ is summable (C, r), then the limit $\lim_{h\sim 1}s(h)$ exists and is equal to the Césaro sum of order r of the series Σa_n. If the sum (C, r) oscillate between finite limits, so also does $\lim_{h\sim 1}s(h)$.

More general theorems of a similar kind have been obtained by C. N. Moore*, by Bromwich†, and by Hardy‡.

* *Trans. Amer. Math. Soc.* vol. VIII (1907). p. 299.

† *Math. Annalen*, vol. LXV (1908), pp. 359, 362.

‡ *Proc. Lond. Math. Soc.* (2), vol. IV (1906), p. 247, and (2), vol. VI (1907), p. 255; also *Math. Annalen*, vol. LXIV (1907), p. 77.

CHAPTER IV

FUNCTIONS REPRESENTABLE BY SERIES OR SEQUENCES OF CONTINUOUS FUNCTIONS

WEIERSTRASS' THEOREM

159. The general question will be here considered what conditions a function of one variable, or of several variables, defined in a given domain, must satisfy, in order that it can be represented, in that domain, as the limit of a sequence of continuous functions, and therefore as the sum of an infinite series of which the terms are continuous functions.

Before proceeding to the general case we shall consider the special case of a continuous function of a single variable, the function being defined in a closed interval.

The following fundamental theorem is due to Weierstrass*:

If a function $f(x)$ be continuous in a given closed interval (a, b), and if δ be an arbitrarily chosen positive number, a finite polynomial $P(x)$ can be so determined that $|f(x) - P(x)| < \delta$, throughout the interval (a, b).

In order to prove the theorem, it is convenient first to consider certain special cases. Let a function y be defined, for the interval $(-\alpha, \alpha)$, by the specifications $y = mx$, for $0 \leqq x \leqq \alpha$, and $y = -mx$, for $-\alpha \leqq x \leqq 0$; thus y is the continuous function which is represented geometrically by portions of straight lines which meet at the origin and are equally inclined to the x-axis.

The function is represented in the whole interval $(-\alpha, \alpha)$, by

$$y = m\alpha\left[1 + \left(\frac{x^2}{\alpha^2} - 1\right)\right]^{\frac{1}{2}},$$

where the positive value of the radical is to be taken. This expression for y can be expanded by the Binomial Theorem in a series of powers of $\frac{x^2}{\alpha^2} - 1$; and this series converges uniformly in the whole interval $(-\alpha, \alpha)$ of x. In this manner, by taking two, three, four, etc., terms of the series, we obtain a sequence of polynomials which converges uniformly, in the interval, to the value of the function. Thus a particular case of the theorem has been established.

Next, let the function y be defined, for the interval (a, b), as follows:

Let $y = 0$, for $a \leqq x \leqq c$; and $y = m(x - c)$, for $c \leqq x \leqq b$; where c is a fixed number between a and b. This function is represented geometrically by the portion of the x-axis between the points a and c, and by the portion of the straight line $y = m(x - c)$, between c and b. The function

* See the *Sitzungsberichte* of the Berlin Academy (1885), pp. 633 and 789; also *Werke*, vol. III,

may be represented by $y = \frac{1}{2}m(x-c) + |\frac{1}{2}m(x-c)|$; and since, as has been shewn in the last case, $|\frac{1}{2}m(x-c)|$ is representable as the limit of a sequence of polynomials converging uniformly, the same is true of the function now considered.

Next, let (a, b) be divided into a finite set of intervals

$$(a, x_1), (x_1, x_2), (x_2, x_3), \dots (x_{n-1}, b);$$

and let ordinates to the x-axis be erected at the points $a, x_1, x_2, \dots b$, the extremities of these ordinates being denoted by

$$P, P_1, P_2, \dots P_{n-1}, Q.$$

Let the consecutive pairs of these points be joined by straight lines, an open polygon $P, P_1, P_2, \dots Q$ being thus formed. It will be shewn that the continuous polygonal function $\phi(x)$, defined by the ordinates of this polygon, is such that a polynomial $P(x)$ can be determined so that

$$|\phi(x) - P(x)| < \eta,$$

for every value of x, in (a, b); where η is an arbitrarily chosen positive number. It is clear that $\phi(x)$ can be expressed as the sum of n functions $\phi_1(x), \phi_2(x), \dots \phi_n(x)$, such that $\phi_1(x)$ is linear in the whole of (a, b), that $\phi_2(x)$ vanishes in the interval (a, x_1), and is linear in (x_1, b); that $\phi_3(x)$ vanishes in the interval (a, x_2), and is linear in (x_2, b); and generally such that $\phi_r(x)$ is zero in the interval (a, x_{r-1}), and is linear in the interval (x_{r-1}, b). Since polynomials $P^{(r)}(x)$ satisfying the condition

$$|\phi_r(x) - P^{(r)}(x)| < \frac{\eta}{n}$$

can be determined for each of the functions $\phi_1(x), \phi_2(x), \dots \phi_n(x)$, the theorem is established for the polygonal function $\phi(x)$.

In the general case in which $f(x)$ is any function that is continuous in (a, b), it follows from the known theorem (I, § 217) that $f(x)$ is uniformly continuous, that, if ϵ be a prescribed positive number, the interval (a, b) can be so divided into parts $(a, x_1), (x_1, x_2), \dots (x_{n-1}, b)$, that the fluctuation of $f(x)$ in each part is less than ϵ.

If $\phi(x)$ denotes the polygonal function considered above, which we take to be equal to $f(x)$ at each of the points $a, x_1, x_2, \dots x_{n-1}, b$, we see that $|f(x) - \phi(x)| < \epsilon$, in the whole interval (a, b). As it has been shewn that a polynomial $P(x)$ exists, such that $|\phi(x) - P(x)| < \eta$, it follows that $|f(x) - P(x)| < \epsilon + \eta$. Since ϵ, η are both arbitrary, Weierstrass' theorem has been established.

If $\delta_1, \delta_2, \dots \delta_n, \dots$ be a diminishing sequence of positive numbers converging to zero, a sequence of polynomials $P_1(x), P_2(x), \dots P_n(x), \dots$ can be so determined that $|f(x) - P_n(x)| < \delta_n$, for $n = 1, 2, 3, \dots$; and for all values of x in (a, b).

Since the sequence $\{P_n(x)\}$ converges uniformly to $f(x)$ as its limit, $f(x)$ may be regarded as the sum-function of the uniformly convergent series

$$P_1(x) + \{P_2(x) - P_1(x)\} + \dots + \{P_n(x) - P_{n-1}(x)\} + \dots.$$

Thus the following theorem has been established:

If $f(x)$ be continuous in the linear interval (a, b), the function is the limiting sum of a uniformly convergent series, of which the terms are finite polynomials.

The proof of Weierstrass' theorem, given above, is substantially due to Lebesgue*. Other proofs have been given by Runge†, by Picard‡, by Volterra§, by Mittag-Leffler||, and by Lerch¶. We proceed to consider the extension of the theorem to continuous functions of any number of variables. Another proof of the theorem for a single variable will be given in § 300.

The original proof given by Weierstrass depended upon the theorem

$$\lim_{k \sim 0} \frac{1}{(\pi k)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x') e^{-\frac{(x'-x)^2}{k}} dx' = f(x),$$

where $f(x)$ is continuous in the infinite interval $(-\infty, \infty)$. It was deduced that a sequence of polynomials exists which converges to $f(x)$ uniformly in any finite interval.

WEIERSTRASS' THEOREM FOR FUNCTIONS OF TWO OR MORE VARIABLES

160. In order to extend Weierstrass' theorem to the case of a continuous function of two or more variables, defined in a closed cell, a proof by induction will be given. Other proofs have been given by Weierstrass** and by Tonelli††.

We consider the case of a continuous function of any number of variables, defined in a closed cell $(a^{(1)}, a^{(2)}, \dots a^{(p)}; b^{(1)}, b^{(2)}, \dots b^{(p)})$. Let it be assumed that the theorem holds good for a continuous function of $p-1$ variables, defined in the cell $(a^{(1)}, a^{(2)}, \dots a^{(p-1)}; b^{(1)}, b^{(2)}, \dots b^{(p-1)})$, it will then be shewn to hold good for a continuous function

$$f(x^{(1)}, x^{(2)}, \dots x^{(p)}),$$

defined in the p-dimensional cell.

If δ be an arbitrarily prescribed positive number, a net with closed meshes may be fitted on to the p-dimensional cell, such that the fluctuation of f in each closed mesh is $< \delta$. Let $x_0^{(p)}, x_1^{(p)}, x_2^{(p)}, \dots x_m^{(p)}$ be the successive

* *Bulletin d. Sc. Mat.* (2), vol. XXII (1) (1898), p. 278.

† *Acta Mat.* vol. VII (1885), p. 387 and vol. VI (1885), p. 236.

‡ *Traité d'Analyse*, vol. I, p. 258.

§ *Rend. del. cir. mat. di Palermo*, vol. XI (1897), p. 83.

|| *Ibid.* vol. XIV (1900), p. 217.

¶ *Acta Mat.* vol. XXVII (1903), p. 339.

** *Werke*, vol. III (1903), p. 27, but not in the original memoir.

†† *Rend. del. cir. mat. di Palermo* (2), vol. XXIX (1910), p. 9.

values of $x^{(p)}$ on the boundaries of the meshes which are perpendicular to the $x^{(p)}$-axis; where $x_0^{(p)} = a^{(p)}$, $x_m^{(p)} = b^{(p)}$. If η be an arbitrarily prescribed positive number, for each value 0, 1, 2, 3, ... m, of r, a finite polynomial $P^{(r)}(x^{(1)}, x^{(2)}, \ldots x^{(p-1)})$ can, in accordance with hypothesis, be so determined that

$$| P^{(r)}(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}) - f(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}, x_r^{(p)}) | < \eta.$$

All these polynomials can be included in a single expression

$$\Sigma \phi_{q_1, q_2, \ldots q_{p-1}}(x_r^{(p)})\, x^{(1)^{q_1}} x^{(2)^{q_2}}, \ldots x^{(p-1)^{q_{p-1}}};$$

where $q_1, q_2, \ldots q_{p-1}$ each has a finite set of integral values, including zero, and $\phi_{q_1, q_2, \ldots q_{p-1}}(x_r^{(p)})$ is zero when a particular term $x^{(1)^{q_1}} x^{(2)^{q_2}} \ldots x^{(p-1)^{q_{p-1}}}$ does not occur in $P^{(r)}(x^{(1)}, x^{(2)}, \ldots x^{(p-1)})$.

Let the functions $\phi_{q_1, q_2, \ldots q_{p-1}}(x^{(p)})$ be defined for each set of values of $q_1, q_2, \ldots q_{p-1}$ so as to be linear in each of the m intervals $(x_r^{(p)}, x_{r+1}^{(p)})$, and so as to have the prescribed values when $x^{(p)}$ has the values $x_r^{(p)}$, for $r = 0, 1, 2, \ldots m$. Since these functions are all continuous linear polygonal functions, in accordance with the theorem of § 159, if ζ be an arbitrarily prescribed positive number, finite polynomials $Q_{q_1, q_2, \ldots q_{p-1}}(x^{(p)})$ can be so determined that $| Q_{q_1, q_2, \ldots q_{p-1}}(x^{(p)}) - \phi_{q_1, q_2, \ldots q_{p-1}}(x^{(p)}) | < \zeta$, for all the sets of values of $q_1, q_2, \ldots q_{p-1}$. Let A denote the upper boundary of

$$\Sigma \,| x^{(1)^{q_1}} x^{(2)^{q_2}} \ldots x^{(p-1)^{q_{p-1}}} |$$

in the cell $(a^{(1)}, a^{(2)}, \ldots a^{(p-1)};\ b^{(1)}, b^{(2)}, \ldots b^{(p-1)})$. Let us consider the polynomial

$$R(x^{(1)}, x^{(2)}, \ldots x^{(p)}) \equiv \Sigma Q_{q_1, q_2, \ldots q_{p-1}}(x^{(p)})\, x^{(1)^{q_1}} x^{(2)^{q_2}} \ldots x^{(p-1)^{q_{p-1}}}.$$

We have

$$| R(x^{(1)}, \ldots x^{(p)}) - \Sigma \phi_{q_1, q_2, \ldots q_{p-1}}(x^{(p)})\, x^{(1)^{q_1}} x^{(2)^{q_2}}, \ldots x^{(p-1)^{q_{p-1}}} | \leq A\zeta.$$

If $x^{(p)}$ be in the interval $(x_r^{(p)}, x_{r+1}^{(p)})$, we have, for each set of indices of ϕ,

$$\phi(x^{(p)}) = \frac{x^{(p)} - x_r^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}} \phi(x_{r+1}^{(p)}) + \frac{x_{r+1}^{(p)} - x^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}} \phi(x_r^{(p)});$$

and therefore

$$\begin{aligned}
\Sigma \phi_{q_1, q_2, \ldots q_{p-1}}(x^{(p)})\, x^{(1)^{q_1}} x^{(2)^{q_2}} \ldots x^{(p-1)^{q_{p-1}}} &= \frac{x^{(p)} - x_r^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}} P^{(r+1)}(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}) \\
&\quad + \frac{x_{r+1}^{(p)} - x^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}} P^{(r)}(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}) \\
&= [f(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}, x_{r+1}^{(p)}) + \theta_1 \eta] \frac{x^{(p)} - x_r^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}} \\
&\quad + [f(x^{(1)}, x^{(2)}, \ldots x^{(p-1)}, x_r^{(p)}) + \theta_2 \eta] \frac{x_{r+1}^{(p)} - x^{(p)}}{x_{r+1}^{(p)} - x_r^{(p)}},
\end{aligned}$$

where $| \theta_1 | < 1, \quad | \theta_2 | < 1.$

Since $\quad f(x^{(1)}, x^{(2)}, \dots x^{(p-1)}, x_{r+1}^{(p)}) = f(x^{(1)}, x^{(2)}, \dots x^{(p)}) + \theta_3\delta$

and $\quad f(x^{(1)}, x^{(2)}, \dots x^{(p-1)}, x_r^{(p)}) = f(x^{(1)}, x^{(2)}, \dots x^{(p)}) + \theta_4\delta,$

where $|\theta_3| < 1$, $|\theta_4| < 1$, we have

$$\Sigma\, \phi_{q_1, q_2 \dots q_{p-1}}(x^{(p)})\, x^{(1)^{q_1}} x^{(2)^{q_2}} \dots x^{(p-1)^{q_{p-1}}}$$
$$= f(x^{(1)}, x^{(2)}, \dots x^{(p)}) + \theta_5\delta + \theta_6\delta + \theta_1'\eta + \theta_2'\eta,$$

where $\quad |\theta_5| < 1, \ |\theta_6| < 1, \ |\theta_1'| < 1, \ |\theta_2'| < 1.$

It now follows that

$$|R(x^{(1)}, x^{(2)}, \dots x^{(p)}) - f(x^{(1)}, x^{(2)}, \dots x^{(p)})| < A\zeta + 2\delta + 2\eta.$$

If ϵ be an arbitrarily prescribed positive number, let δ be chosen to have the value $\frac{1}{6}\epsilon$. The number δ having been fixed, the net can be determined, and η can be taken to have the value $\frac{1}{6}\epsilon$. The number ζ can then be chosen to have the value $\frac{\epsilon}{3A}$. It has been shewn that the polynomial

$$R(x^{(1)}, x^{(2)}, \dots x^{(p)})$$

is such that, everywhere in the p-dimensional cell, the condition

$$|f(x^{(1)}, x^{(2)}, \dots x^{(p)}) - R(x^{(1)}, x^{(2)}, \dots x^{(p)})| < \epsilon$$

is satisfied.

Since the theorem holds for $p = 1$, it is seen to hold for $p = 2, 3, \dots$.

We have thus proved the following theorem:

A continuous function of any number of variables, defined in a given closed cell, is such that a finite polynomial in the variables exists which differs from the function by less than a prescribed positive number, at all points of the cell.

161. It has been shewn in § 79 that the terms of a uniformly convergent series can be so bracketed that the new series converges absolutely in the whole interval. We have therefore the following result:

If $f(x)$ be continuous in an interval, or cell, (a, b), a series, of which the terms are finite polynomials, can be so determined that the series converges to $f(x)$ absolutely at every point of the interval, or cell, and uniformly in the whole interval, or cell.

It can be shewn that the sequence of polynomials $\{P_n(x)\}$ can be so chosen that it is monotone. For let us consider the continuous function $f(x) - \frac{1}{2^n}\epsilon$. A polynomial $P_n(x)$ can be so determined that

$$\left| f(x) - \frac{1}{2^n}\epsilon - P_n(x) \right| < \epsilon\left(\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\right),$$

for all points x in (a, b). It follows that $P_n(x)$ lies between the two numbers

$$f(x) - \frac{\epsilon}{2^n} \pm \epsilon\left(\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\right);$$

if we assume that this condition is satisfied by $P_n(x)$ for every value of n, and observe that

$$-\frac{\epsilon}{2^n}+\epsilon\left(\frac{1}{2^{n+1}}-\frac{1}{2^{n+2}}\right)<-\frac{\epsilon}{2^{n+1}}-\epsilon\left(\frac{1}{2^{n+2}}-\frac{1}{2^{n+3}}\right),$$

we see that $P_{n+1}(x) > P_n(x)$. We have accordingly the following theorem*:

If $f(x)$ be continuous in the interval, or cell, (a, b), a monotone sequence of finite polynomials can be so determined that the sequence converges uniformly to $f(x)$ in the interval (a, b).

It should be observed that, in Weierstrass' theorem, in the case of a function of a single variable, the polynomials may be so chosen that, at the points a and b, $P_n(x)$ has the same value as $f(x)$, for every value of n. For if $f(a) - P_n(a) = \eta_n$, $f(b) - P_n(b) = \eta_n'$, where $|\eta_n|$ and $|\eta_n'|$ are both less than the number ϵ_n, for which $|f(x) - P_n(x)| < \epsilon_n$, in (a, b), let $P_n'(x) = P_n(x) + Ax + B$, where A and B are so chosen that

$$P_n'(a) = f(a),\ P_n'(b) = f(b).$$

We have $Aa + B = \eta_n$, $Ab + B = \eta_n'$; and thus

$$A = \frac{\eta_n' - \eta_n}{b-a},\quad B = \frac{b\eta_n - a\eta_n'}{b-a},$$

whence we have $|Ax + B| < K\epsilon_n$, where K is a fixed number independent of n. It follows that $|f(x) - P_n'(x)| < \epsilon_n + K\epsilon_n$, and thus that the sequence $\{P_n'(x)\}$ converges uniformly to $f(x)$. Since $P_n'(a) = f(a)$, $P_n'(b) = f(b)$, the sequence $\{P_n'(x)\}$ satisfies the prescribed condition.

A sequence of polynomials which converges uniformly to the continuous function $f(x)$, in a given interval or cell (a, b), may be so chosen that each of the polynomials is less in absolute value, at any point of the cell or interval, than the upper boundary U, of $|f(x)|$ in (a, b). For, if the sequence $\{P_n(x)\}$ converges uniformly to $f(x)$, and thus

$$|f(x) - P_n(x)| < \epsilon_n,$$

for $n \geqq n_\epsilon$, in (a, b), let the sequence $\{k_n P_n(x)\}$ be considered; where $\{k_n\}$ is a sequence of increasing positive numbers which converges to the limit 1.

Since $|f(x) - k_n P_n(x)| < |f(x) - P_n(x)| + (1 - k_n)|P_n(x)|$

$$< \epsilon_n + (1 - k_n)(U + \epsilon_n) < 3\epsilon_n,\ \text{for } n \geqq n_\epsilon \text{ if } k_n = \frac{U - \epsilon_n}{U + \epsilon_n}.$$

It follows that, if this set of values of k_n be chosen, the sequence $\{k_n P_n(x)\}$ converges to $f(x)$, uniformly in (a, b). Further, we have

$$|k_n P_n(x)| < k_n |f(x)| + k_n \epsilon_n < U - \epsilon_n.$$

Thus the sequence $\{k_n\}$ may be so chosen that, for every value of x, $|k_n P_n(x)| < U - \epsilon_n$, and the sequence $\{k_n P_n(x)\}$ is a sequence of polynomials such as is required.

* See Hobson, *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 163.

162. Weierstrass' theorem may be applied to the case in which $f(x)$ is defined in any closed set G, in any number of dimensions, $f(x)$ being continuous in G. In accordance with a theorem given in § 108, if Δ be a closed cell, or interval, which contains G, the function $f(x)$ can be extended into a function $f_\Delta(x)$, continuous in Δ, and such that $f_\Delta(x) = f(x)$ at all points of G. If $P_n(x)$ be a finite polynomial such that $|f_\Delta(x) - P_n(x)| < \delta$, in Δ, then $|f(x) - P_n(x)| < \delta$, in G. It thus appears that $f(x)$ can be represented, in G, as the limit of a sequence of finite polynomials which converge uniformly to $f(x)$.

Let H be the outer limiting set of a sequence of closed sets $\{G_n\}$, each of which is contained in the next, and suppose $f(x)$ to be defined as a function that is continuous in H. Let $\{\epsilon_n\}$ denote a monotone sequence of positive numbers converging to zero; since $f(x)$ is continuous in H, it is continuous in G_n, and consequently a polynomial $P_n(x)$ can be so determined that $|f(x) - P_n(x)| < \epsilon_n$, at all points of G_n; and this for each value of n. The sequence $\{P_n(x)\}$ converges to $f(x)$ at every point of H; for, any point p, of H, belongs to all the sets $G_n, G_{n+1}, \ldots$ for some value of n, depending on p, and therefore the sequence $\{P_n(x)\}$ converges at p to the value $f(p)$.

In particular, any open set, whether bounded or not, is the outer limiting set of a sequence of closed sets; and all the points of the p-dimensional space form such an open set. Further, a set $D(O, G)$, which consists of the points which an open set and a closed set G have in common, is the outer limiting set of the sequence $\{D(g_n, G)\}$, of closed sets, where O is the outer limiting set of the sequence $\{g_n\}$, of closed sets.

The following theorem has now been established:

If a set E is either a closed set, or an open set, bounded or unbounded, or the set which a closed set and an open set have in common, and a function $f(x)$ be continuous in the set E, a sequence of finite polynomials can be determined which converges in E to $f(x)$. In particular, if $f(x)$ is continuous in the whole linear, or p-dimensional space, a sequence of finite polynomials can be determined which converges at every point x, to the value $f(x)$; moreover the convergence of the sequence is uniform in any finite cell, or interval.

For a discussion of the methods of Lagrange and Tchebicheff for the approximate representation of functions by series of polynomials, reference may be made to Borel's *Leçons sur les fonctions de variables réelles*, chapter IV. A considerable amount of attention has been paid recently by mathematicians to the question of the best approximations to a continuous function by polynomials. The question was first raised by Lebesgue* as to

* *Rend. del. cir. mat. di Palermo*, vol. XXVI (1908), p. 325. For a discussion of this and other questions see Dunham Jackson's *Preisschrift*, Göttingen, 1911, where many references to the literature of the subject will be found. Among these are Lebesgue, *Annales de Toulouse* (3), vol. I (1910), p. 25, de la Vallée Poussin, *Bull. de l'acad. roy. de Belgique* (1908), p. 403 and p. 193, and (1910), p. 808. See also de la Vallée Poussin's *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919.

the lowest degree of a polynomial $P(x)$ which satisfies the condition $|f(x) - P(x)| < \delta$ in the linear interval (a, b), $f(x)$ being an assigned continuous function and δ an assigned positive number.

UNBOUNDED CONTINUOUS FUNCTIONS

163. The theorem of Weierstrass may be extended so as to apply to the case in which the function $f(x)$, defined in a closed domain E, is continuous only in the extended sense of the term (see I, § 219), the two improper values ∞, $-\infty$, of the function being regarded as distinct from one another. Employing the transformation

$$\phi(x) = \frac{f(x)}{1 + |f(x)|},$$

the function $\phi(x)$ is continuous in E, in the ordinary sense. Accordingly, $\phi(x)$ is the limit of a sequence $\{Q_n(x)\}$, of finite polynomials, which converges uniformly to $\phi(x)$, and the sequence can be so chosen that $|Q_n(x)| < 1$, for all values of n and x (see § 150). Taking a sequence $\{k_n\}$ of positive numbers converging to the limit 1, the sequence $\{k_n Q_n(x)\}$ converges uniformly, in E, to $\phi(x)$. Since $\dfrac{k_n Q_n(x)}{1 - k_n |Q_n(x)|}$ is a continuous function, bounded in E, a finite polynomial $P_n(x)$ can be so determined that

$$\left| \frac{k_n Q_n(x)}{1 - k_n |Q_n(x)|} - P_n(x) \right| < \epsilon, \text{ in } E.$$

Let us consider the set of points in E for which $|f(x)| \leq A$; at these points we have $|\phi(x)| \leq \dfrac{A}{1 + A}$. We have also

$$|f(x) - P_n(x)| < \left| \frac{\phi(x)}{1 - |\phi(x)|} - \frac{k_n Q_n(x)}{1 - k_n |Q_n(x)|} \right| + \epsilon,$$

and $|\phi(x) - k_n Q_n(x)| < \epsilon$, for $n \geqq n_\epsilon$, at every point of E. In case $|\phi(x)| \geqq 2\epsilon$, $\phi(x)$ and $k_n Q_n(x)$ have the same sign, and $|k_n Q_n(x)| > \epsilon$; we have then, at all points of E at which $f(x) \leq A$,

$$|f(x) - P_n(x)| < \frac{\epsilon(1 + A)^2}{1 - \epsilon(1 + A)} + \epsilon < \frac{\eta}{1 - \dfrac{\eta}{1 + A}} + \frac{\eta}{(1 + A)^2},$$

if $\eta = \epsilon(1 + A)^2$; hence $|f(x) - P_n(x)| < \eta(1 + 2\eta) + \eta < 4\eta$, when $\eta < 1/2$, provided $n \geqq n_\epsilon$. If η be first chosen, ϵ and n_ϵ are determined. In case $|\phi(x)| < 2\epsilon$, we have, assuming that $\epsilon < 1/6$,

$$|f(x) - P_n(x)| < \epsilon + \frac{2\epsilon}{1 - 2\epsilon} + \frac{3\epsilon}{1 - 3\epsilon} < 32\epsilon < 32\eta;$$

it now follows that $|f(x) - P_n(x)| < 32\eta$, for all points of E at which $f(x) \leq A$, provided $n \geqq n_\epsilon$. Since $|f(x) - P_n(x)| < 32\eta$, in the set in which $f(x) \leq A$, the sequence $\{P_n(x)\}$ converges uniformly to $f(x)$ in that set of points.

Next, consider the points of E at which $f(x)$ is infinite; at these points $|\phi(x)| = 1$, $|k_n Q_n(x)| > 1 - \epsilon$, for $n \geqq n_\epsilon$; it follows that, in this set of points $\left|\dfrac{k_n Q_n(x)}{1 - k_n |Q_n(x)|}\right| > \dfrac{1-\epsilon}{\epsilon}$, and therefore $|P_n(x)| > \dfrac{1}{\epsilon}$, for $n \geqq n_\epsilon$. It has thus been shewn that the divergence of $\{P_n(x)\}$ is uniform in the closed set of points at which $f(x)$ is infinite. It is not necessarily the case that the approach of the sequence $\{P_n(x)\}$ to the function $f(x)$ is uniform in accordance with definition of uniform approach given in § 69. It can be shewn that each point at which $f(x)$ is infinite is a point of uniform divergence of the sequence, in accordance with the definition in § 73.

The following theorem has been proved:

If $f(x)$ be defined in a closed set of points E, of any number of dimensions, and infinite values of $f(x)$ are taken into account, the distinction between $+\infty$ and $-\infty$ being recognised, and the function be continuous, in E, in the extended sense, then a sequence of finite polynomials can be determined which converges uniformly to $f(x)$ in the set of points at which $|f(x)| \leqq A$, for every value of A; and diverges uniformly in the set at which $f(x)$ is infinite.

164. When no distinction between $+\infty$ and $-\infty$ is recognised, the following theorem is applicable:

If $f(x)$ is a bounded function, and a sequence of functions $\{f_n(x)\}$ converges to $f(x)$, uniformly, then $\dfrac{1}{f_n(x)}$ converges to $\dfrac{1}{f(x)}$ uniformly in the set of points at which $\dfrac{1}{|f(x)|} \leqq N$, and it diverges uniformly in the set in which $\dfrac{1}{|f(x)|}$ is infinite; $+\infty$ and $-\infty$ being regarded as not distinct from one another.

We have $|f(x) - f_n(x)| < \epsilon$, for all sufficiently large values of n. Let N be an arbitrarily chosen positive number, and consider those points at which $\dfrac{1}{|f(x)|} \leqq N$.

At the points at which $\dfrac{1}{|f(x)|} \leqq N$, we have, if $\eta = N^2\epsilon$, and n is sufficiently large,

$$\left|\frac{1}{f(x)} - \frac{1}{f_n(x)}\right| < \frac{\epsilon}{|f(x)|\,|f_n(x)|} < \frac{N\epsilon}{|f(x)| - \epsilon} < \frac{\eta}{1 - \dfrac{\eta}{N}}$$

$$< \eta\left(1 + \frac{2\eta}{N}\right) < 3\eta, \text{ if } N > 1, \text{ and } \eta/N < \tfrac{1}{2}.$$

It thus appears that $\dfrac{1}{f_n(x)}$ converges to $\dfrac{1}{f(x)}$, uniformly, in the set of points for which $\dfrac{1}{|f(x)|} \leqq N$.

At the points at which $\frac{1}{|f(x)|} = \infty$, we have $|f_n(x)| < \epsilon$, for $n \geqq n_\epsilon$; hence $\frac{1}{|f_n(x)|} > \frac{1}{\epsilon}$. It follows that the divergence of the sequence $\left\{\frac{1}{f_n(x)}\right\}$ is uniform in the set of points at which $\frac{1}{f(x)}$ is infinite.

165. Let us consider a function $f(x)$ defined in the finite interval (a, b), infinite values of $f(x)$ being admitted, but no distinction being made between $+\infty$ and $-\infty$. The function is taken to be continuous in (a, b) in the extended sense (I, § 219). The set of points at which $f(x)$ is infinite is closed; as follows from the condition of continuity. It will be assumed that this set is non-dense, so that the case in which the function is infinite in a whole sub-interval is left out of account. If P be a point at which $f(x)$ is infinite, there is an interval Δ_P enclosing P, at every point of which $|f(x)| \geqq N$. We can so choose Δ that at both its end-points $|f(x)|$ has the value N. A finite set of these intervals Δ can be so determined that every point at which $f(x)$ is infinite is interior to one of them. We thus obtain a finite set of intervals (α_r, β_r), where $r = 1, 2, 3, \ldots m$, such that $|f(x)| \geqq N$ in every point of all the intervals of the set, and such that $f(\alpha_r), f(\beta_r)$ both have one of the values $N, -N$.

Let the function $f_r(x)$ be defined by $f_r(x) = f(x)$, in (α_r, β_r); $f_r(x) = f(\alpha_r)$, in (a, α_r); and $f_r(x) = f(\beta_r)$, in (β_r, b). Thus $|f_r(x)| \geqq N$, at every point of (α_r, β_r). The function $1/f_r(x)$ is bounded and continuous in the whole interval (a, b). A sequence $\{P_{rs}(x)\}$ of polynomials can therefore be so determined as to converge uniformly to $1/f_r(x)$. By the last theorem it follows that $\left\{\frac{1}{P_{rs}(x)}\right\}$ converges and diverges to $f_r(x)$ in the mode specified in the theorem. The function $\sum_{r=1}^{r=n} f_r(x)$ differs from $f(x)$ only by a constant, in each of the intervals (α_r, β_r), and it is constant in each interval complementary to the set (α_r, β_r). Let $\sum_{r=1}^{r=n} f_r(x) = f(x) + k_r$, in the interval (α_r, β_r). In the interval (β_{r-1}, α_r) we have

$$f(\beta_{r-1}) + k_{r-1} = f(\alpha_r) + k_r,$$

say $= k_r'$. Let the continuous bounded function $f_{n+1}(x)$ be defined by the specifications $f_{n+1}(x) = f(x) - k_1'$, in (a, α_1); $f_{n+1}(x) = -k_1$, in (α_1, β_1); $f_{n+1}(x) = f(x) - k_2'$, in (β_1, α_2); $f_{n+1}(x) = -k_2$, in (α_2, β_2), etc. The function $f_{n+1}(x)$ is the limit of a sequence $P_{n+1,r}(x)$ of polynomials which converges uniformly. The function $f(x)$, or $\sum_{r=1}^{r=n} f_r(x) + f_{n+1}(x)$ is the limit of a sequence

$$\left\{\frac{1}{P_{1s}(x)} + \frac{1}{P_{2s}(x)} + \cdots + \frac{1}{P_{ns}(x)} + P_{n+1,s}(x)\right\},$$

which converges and diverges to the function $f(x)$ as in the theorem of § 163. The following theorem, due essentially to W. H. Young*, has thus been established:

If the function $f(x)$ is continuous in the interval (a, b), in the extended sense of the term, where ∞ and $-\infty$ are regarded as identical, and $f(x)$ is infinite only at a non-dense set of points, then $f(x)$ is the limit of a sequence of rational functions which converges to $f(x)$ uniformly in the set of points at which $f(x) \leqq A$, and diverges uniformly in the set at which $f(x)$ is infinite.

STANDARD SETS OF CONTINUOUS FUNCTIONS

166. Let a system of nets, with closed meshes, be fitted on to the finite interval (a, b). For any net, D_n, consider the set of continuous polygonal functions, each of which has a rational value at each end-point of each mesh, and is linear between the two end-points of each mesh. The totality of all these functions, for the net D_n is an enumerable set (see I, § 58). Further, when we consider the totality of all such enumerable sets of polygonal functions, for all the nets $D_1, D_2, \ldots$ of the system of nets, we have an enumerable set of continuous polygonal functions which may accordingly be denoted by $\{f_m(x)\}$, when arranged in enumerable order. This set of functions may be regarded as a standard set, and it has the property that, if $\phi(x)$ be any continuous function whatever, defined in (a, b), a subsequence of the functions $\{f_m(x)\}$ exists which converges uniformly to $\phi(x)$. To prove this, let $\{\epsilon_n\}$ be a diminishing sequence of positive numbers converging to zero. Let $f_{n_1}(x)$ be the first of the functions $\{f_m(x)\}$ which belongs to the net D_1 and also is such that, at each corner of the polygon which it represents, the value of $f_{n_1}(x)$ differs from $\phi(x)$ by less than ϵ_1. Next let $f_{n_2}(x)$ be the first function of the set, after $f_{n_1}(x)$, which belongs to the net D_2 and is such that, at each corner of the polygon which it represents, $f_{n_2}(x)$ differs from $\phi(x)$ by less than ϵ_2; and so on. We have then a subsequence $\{f_{n_p}(x)\}$ of the sequence $\{f_m(x)\}$, such that, at each corner of the polygon which $f_{n_p}(x)$ represents, $f_{n_p}(x)$ differs from $\phi(x)$ by less than ϵ_p; and this for all the values $1, 2, 3, \ldots$, of p; moreover, $f_{n_p}(x)$ belongs to the net D_p.

Let $D_{p'}$ be the first net for which $p' \geqq p$, and such that the fluctuation of $\phi(x)$ in each mesh is $< \epsilon_p$. Since, at each end-point x of each mesh of $D_{p'}$, we have $|\phi(x) - f_{n_{p'}}(x)| < \epsilon_{p'}$ and the fluctuations of $\phi(x)$ and of $f_{n_{p'}}(x)$ in such a mesh are $< \epsilon_p$, and $< 2\epsilon_p$, respectively, we have $|\phi(x) - f_{n_{p'}}(x)| < 3\epsilon_p$, at every point of (a, b). Since this holds good for every value of p, with the corresponding value of p', it follows that the sequence $\{f_{n_{p'}}(x)\}$ converges uniformly to $\phi(x)$, and $\{f_{n_{p'}}(x)\}$ is a sub-

* See *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 222.

sequence of the standard sequence $\{f_m(x)\}$. Instead of the sequence $\{f_m(x)\}$ we may employ a sequence of finite polynomials. Let $P_m(x)$ be a finite polynomial such that $|f_m(x) - P_m(x)| < \epsilon_m$, in (a, b). We then have $|\phi(x) - P_{n_p'}(x)| < 4\epsilon_p$; and consequently the sequence $\{P_{n_p'}(x)\}$ converges uniformly to $\phi(x)$.

The following theorem has now been established:

A standard sequence of continuous functions $\{f_n(x)$ *exists such that, if* $\phi(x)$ *be any continuous function whatever, defined in the interval* (a, b), *a subsequence* $\{f_{n_p'}(x)\}$ *is contained in* $\{f_n(x)\}$ *which converges uniformly to* $\phi(x)$. *Moreover, the standard functions* $\{f_n(x)\}$ *may be so chosen as to be finite polynomials.*

167. With but a slight modification, the foregoing proof may be employed to establish the corresponding theorem that a set of continuous functions of any number of variables exists, such that in a given cell, a subsequence of the functions (which may be taken to be polynomials) exists which converges to an assigned function of the variables which is continuous in the cell.

In the case of two-dimensional functions, instead of the polygonal functions employed in the one-dimensional case, we take in a mesh $(a_r^{(1)}, a_s^{(2)}; a_{r+1}^{(1)}, a_{s+1}^{(2)})$ the function

$$\frac{1}{(a_{r+1}^{(1)} - a_r^{(1)})(a_{s+1}^{(2)} - a_s^{(2)})}\Big[f(a_r^{(1)}, a_s^{(2)})(x^{(1)} - a_{r+1}^{(1)})(x^{(2)} - a_{s+1}^{(2)})$$
$$- f(a_{r+1}^{(1)}, a_s^{(2)})(x^{(1)} - a_r^{(1)})(x^{(2)} - a_{s+1}^{(2)})$$
$$- f(a_r^{(1)}, a_{s+1}^{(2)})(x^{(1)} - a_{r+1}^{(1)})(a^{(2)} - a_s^{(2)})$$
$$+ f(a_{r+1}^{(1)}, a_{s+1}^{(2)})(x^{(1)} - a_r^{(1)})(x^{(2)} - a_r^{(2)})\Big]$$

which is continuous, and such that, at each point of the mesh, its value is in the interval bounded by the greatest and least of the four numbers

$$f(a_r^{(1)}, a_s^{(2)}), f(a_r^{(1)}, a_{s+1}^{(2)}), f(a_{r+1}^{(1)}, a_s^{(1)}), f(a_{r+1}^{(1)}, a_{s+1}^{(2)}).$$

This function is the analogue, in two dimensions, of the polygonal function in one dimension; its form can easily be obtained in the case of any number of dimensions.

CONVERGENCE OF SEQUENCES ON THE AVERAGE

168. Let $\{s_n(x)\}$ be a sequence of measurable functions, defined in a measurable set E, of which the measure is either finite or infinite, and which is in any number of dimensions; each function is assumed to be finite almost everywhere in E.

Let the set of points x, of E, at which $|s_p(x) - s_q(x)| \geqq \epsilon$, be denoted by $e(\epsilon, p, q)$; where ϵ is any positive number. Let it be assumed that, for

each value of ϵ, $\lim\limits_{p\sim\infty,\, q\sim\infty} m\{e(\epsilon, p, q)\} = 0$; this is equivalent to the assumption that, when δ is an arbitrarily chosen positive number, the condition $m\{e(\epsilon, p, q)\} < \delta$ is satisfied, provided $p \geqq P$, $q \geqq Q$, where P, Q are integers dependent only on δ and ϵ. For two different pairs of values of p and q, the sets are in general different ones, although each of them has its measure $< \delta$.

A sequence $\{s_n(x)\}$ which satisfies this condition is said to *converge on the average in the set E*. The convergence of this type was first investigated* by Fischer and by F. Riesz, who employed the term *convergence en mesure*.

If the measurable set E have infinite measure, it is the outer limiting set of a sequence $\{E_n\}$ of measurable sets, each of which has finite measure (see I, § 134). The case in which E has finite measure may be included, by supposing that E_n is, for every value of n, identical with E.

It will be assumed that the sequence $\{s_n(x)\}$ is convergent on the average in each of the sets $E_1, E_2, \ldots E_n, \ldots$, but not necessarily in E, when $m(E)$ is infinite.

There exists, in E_1, a set of measure $> m(E_1) - \frac{1}{2}\eta$, at all points of which $|s_{n_1}(x) - s_{n_2}(x)| < \frac{1}{2}\eta$; where n_1, n_2 are fixed numbers, neither of which is less than a certain integer $N^{(1)}(\frac{1}{2}\eta)$. Similarly, there exists, in E_2, a set of points of which the measure is $> m(E_2) - \frac{1}{2^2}\eta$, at which

$$|s_{n_2}(x) - s_{n_3}(x)| < \frac{1}{2^2}\eta;$$

where n_2, n_3 are so fixed that neither of them is less than an integer $N^{(2)}\left(\frac{1}{2^2}\eta\right)$; the part of this set that is in E_1 has its measure $> m(E_1) - \frac{1}{2^2}\eta$. It is clear that n_1, n_2, n_3 may be so chosen that both the conditions are satisfied for the same set of values of these integers. We take for n_1 its least value $N^{(1)}\left(\frac{1}{2}\eta\right)$; for n_2 we take the least integer which is $> n_1$ and $\geqq N^{(2)}\left(\frac{1}{2^2}\eta\right)$. Similarly, if n_3 be the least integer which is $> n_2$ and $\geqq N^{(3)}\left(\frac{1}{2^3}\eta\right)$, there exists, in E_3, a set of points of measure

$$> m(E_3) - \frac{1}{2^3}\eta,$$

at which $|s_{n_3}(x) - s_{n_4}(x)| < \frac{1}{2^2}\eta$, provided n_4 is taken to be $> n_3$; the part of this set in E_1 has measure $> m(E_1) - \frac{1}{2^3}\eta$. Proceeding in this

* See Fischer, *Comptes Rendus*, vol. CXLIV (1907), pp. 1022, 1148, also Riesz, vol. CXLIII (1906), p. 738, and vol. CXLIV, pp. 615, 734, 1409.

manner, a sequence $\{n_p\}$ of increasing integers is defined, so that, at every point of a certain set, of measure $> m(E_r) - \frac{1}{2^r}\eta$, the condition

$$| s_{n_r}(x) - s_{n_{r+1}}(x) | < \frac{1}{2^r}\eta,$$

is satisfied; moreover the measure of the part of this set that is in E_1 is $> m(E_1) - \frac{1}{2^r}\eta$. As this holds for each set E_r, it follows that there exists, in E_1, a set F_1, of measure $> m(E_1) - \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^r} + \dots\right)\eta$, or $m(E_1) - \eta$, in which the conditions $| s_{n_r}(x) - s_{n_{r+1}}(x) | < \frac{1}{2^r}\eta$ is satisfied for every value of r.

In this set F_1, we have $| s_{n_r}(x) - s_{n_t}(x) | < \frac{1}{2^{r-1}}\eta$, for $t \geqq r$; it follows that, in F_1, the sequence $s_{n_1}(x), s_{n_2}(x), \dots$ is uniformly convergent.

If we omit n_1 and consider the sequence $s_{n_2}(x), s_{n_3}(x), \dots$, and let $\frac{1}{2}\eta$ take the place of η, it is seen that the sequence converges uniformly in a set F_2, contained in E_2, of measure $> m(E_2) - \frac{1}{2}\eta$. Generally, there is, in E_r, a set F_r, of measure $> m(E_r) - \frac{1}{2^{r-1}}\eta$, in which the sequence $s_{n_r}(x), s_{n_{r+1}}(x), \dots$, and therefore the sequence $\{s_{n_p}(x)\}$, converges uniformly. The part of F_r that is in E_1 has measure $> m(E_1) - \frac{1}{2^{r-1}}\eta$, and in this part, the sequence $\{s_{n_p}(x)\}$ converges uniformly. Since $\frac{1}{2^{r-1}}\eta$ is arbitrarily small, when r is increased, it follows that the sequence $\{s_{n_p}(x)\}$ converges almost everywhere in E_1. Similarly, it can be shewn that the sequence converges almost everywhere in each of the sets $E_2, E_3, \dots$, and therefore it converges almost everywhere in E. A function $s(x)$ is defined almost everywhere in E, as the limit of the sequence $\{s_{n_p}(x)\}$.

It will be shewn that the function $s(x)$, so defined, is unique, in the sense that two values obtained as in the above process, but employing different modes of determining the sequence $\{n_p\}$, can only differ from one another at points of a set of which the measure is zero; and thus that they are equivalent functions.

Let $s^{(1)}(x)$, $s^{(2)}(x)$ be two such values of $s(x)$, defined by sequences $\{s_{n_p}(x)\}$, $\{s_{n'_p}(x)\}$ respectively. If, in the set E_1, they are not equivalent to one another, there must exist two positive numbers λ, k, such that $| s^{(1)}(x) - s^{(2)}(x) | \geqq \lambda$, in a set of points of measure k, contained in E_1. If ϵ be an arbitrarily chosen positive number, we have $|s^{(1)}(x) - s_{n_p}(x)| < \epsilon$, in a certain set, contained in E_1, of measure $> m(E_1) - \zeta$, for a sufficiently

large value of p; similarly $|s^{(2)}(x) - s_{n'_p}(x)| < \epsilon$, in a certain set, of measure $> m(E_1) - \zeta'$, where p is sufficiently large.

It follows that both these inequalities are satisfied, for a sufficiently large value of p, in a set of measure $> m(E_1) - \zeta - \zeta'$, contained in E_1. Since $|s_{n_p}(x) - s_{n'_p}(x)| < \epsilon$, in a certain set of measure $> m(E_1) - \eta$, for a sufficiently large value of p; and since

$$|s^{(1)}(x) - s^{(2)}(x)| \leq |s^{(1)}(x) - s_{n_p}(x)| + |s^{(2)}(x) - s_{n'_p}(x)| + |s_{n_p}(x) - s_{n'_p}(x)|,$$

by choosing p sufficiently large, we see that, in a certain set, contained in E_1, of measure $> m(E_1) - \eta - \zeta - \zeta'$, we have $|s^{(1)}(x) - s^{(2)}(x)| < 3\epsilon$. Let ϵ be so chosen that 3ϵ is less than λ, and $\eta + \zeta + \zeta'$ is less than k; then $|s^{(1)}(x) - s^{(2)}(x)| < \lambda$, in a set of measure $> m(E_1) - k$. This is inconsistent with the assumption that, in E_1, $|s^{(1)}(x) - s^{(2)}(x)| \geqq \lambda$ in a set of points of measure k. It follows that, in E_1, $s^{(1)}(x)$ and $s^{(2)}(x)$ differ from one another only at points of a set of measure zero. The same argument applies in the case of each of the sets E_r; and therefore, in E, the two functions $s^{(1)}(x)$, $s^{(2)}(x)$ are equivalent.

The following theorem has now been established:

If a sequence $\{s_n(x)\}$, of measurable functions, is convergent on the average, in a measurable set E, of finite, or of infinite, measure, and of any number of dimensions; so that the measure of that part of E in which

$$|s_p(x) - s_q(x)| \geqq \epsilon,$$

for each fixed value of ϵ, converges to zero, as p and q are indefinitely increased, independently of one another, then a subsequence $\{s_{n_p}(x)\}$, of the sequence $\{s_n(x)\}$, can be defined, which converges to a single-valued function $s(x)$, almost everywhere in E. Moreover two functions $s(x)$ which satisfy this condition are equivalent to one another. In some set of points of measure zero, $s(x)$ may be undefined. Moreover, if E has infinite measure, the theorem is valid when $\{s_n(x)\}$ is convergent on the average in each part of E that has finite measure.

That the convergence of the sequence $\{s_{n_p}(x)\}$ to $s(x)$, almost everywhere in the set E_r, of finite measure, necessarily entails the uniform convergence of the sequence in some set of points of E_r, whose measure differs from $m(E_r)$ by less than an arbitrarily fixed number, follows from Egoroff's theorem (§ 99).

169. If $s(x)$ be a measurable function, defined almost everywhere in the measurable set E, of finite, or of infinite measure, and if there exists a sequence $\{s_n(x)\}$ of measurable functions, each of which is finite almost everywhere in E, such that the measure of the set $h(\epsilon, n)$, of points at which $|s(x) - s_n(x)| \geqq \epsilon$, converges, for each fixed value of ϵ, to zero, as $n \sim \infty$, then the sequence $\{s_n(x)\}$ is said to *converge on the average to $s(x)$, in the set E.*

The relation, which will be investigated below, between the two properties of "convergence of a sequence on the average" and "convergence of a sequence on the average to $s(x)$" is analogous to the relation between convergence of a sequence of numbers, and convergence of a sequence of numbers to a limit, leading to the General Principle of Convergence given in I, § 30.

In case $m(E)$ is infinite, the sequence may converge on the average to $s(x)$, in each part E_r, of E, and yet not necessarily converge on the average to $s(x)$ in E itself.

For, in E_r, we may have $m[h^{(r)}(\epsilon, n)] < \eta$, for $n \geqq N_r$, where N_r is an integer dependent on ϵ, η, and on the set E_r. Unless N_r is bounded, for all values of r, there exists no integer n, such that $m[h(\epsilon, n)] < \eta$, for $n \geqq N$, and if this is the case for all sufficiently small values of ϵ, η, the sequence does not converge on the average to $s(x)$, in E.

The following theorem will be established:

If, in the measurable set E, of finite, or of infinite, measure, a sequence $\{s_n(x)\}$ of measurable functions, finite almost everywhere, converges on the average to a measurable function $s(x)$, finite almost everywhere, then the sequence $\{s_n(x)\}$ is convergent on the average, in E. Moreover a partial sequence $\{s_{n_p}(x)\}$ can be defined which converges almost everywhere in E to a function equivalent to $s(x)$.

At every point of E not belonging to the set $h(\frac{1}{2}\epsilon, p)$, of the points at which $|s(x) - s_p(x)| \geqq \frac{1}{2}\epsilon$, nor to the set $h(\frac{1}{2}\epsilon, q)$, and at which $s(x)$ is defined, we have $|s(x) - s_p(x)| < \frac{1}{2}\epsilon$, and $|s(x) - s_q(x)| < \frac{1}{2}\epsilon$, and therefore $|s_p(x) - s_q(x)| < \epsilon$. Accordingly, at every point of E not belonging to a set of which the measure is

$$\leqq m[h(\tfrac{1}{2}\epsilon, p)] + m[h(\tfrac{1}{2}\epsilon, q)],$$

we have $|f_p(x) - f_q(x)| < \epsilon$. Therefore the set $e(\epsilon, p, q)$, of points of E, at which $|f_p(x) - f_q(x)| \geqq \epsilon$, has its measure

$$\leqq m[h(\tfrac{1}{2}\epsilon, p)] + m[h(\tfrac{1}{2}\epsilon, q)].$$

Thus $$\lim_{p\sim\infty,\, q\sim\infty} m[\epsilon, p, q] \leqq \lim_{p\sim\infty} m[h(\tfrac{1}{2}\epsilon, p)] + \lim_{q\sim\infty} m[h(\tfrac{1}{2}\epsilon, q)],$$

or the limit on the left hand side is zero. Therefore the sequence $\{s_n(x)\}$ is convergent on the average, in E.

By the theorem of § 168, a sequence $\{s_{n_p}(x)\}$ can be defined which converges almost everywhere in E to a function $\phi(x)$ defined almost everywhere in E. It will be shewn that $\phi(x)$ and $s(x)$ are equivalent to one another.

In the part E_r, of E, of finite measure, there exists a set of points of measure $> m(E_r) - \eta$, in which $\{s_{n_p}(x)\}$ converges uniformly to $\phi(x)$. Hence, in this set, $|\phi(x) - s_{n_p}(x)| < \epsilon$, for all sufficiently large values of

p. Moreover $| s(x) - s_{n_p}(x) | < \epsilon$, in a set, contained in E_r, of measure $> m(E_r) - \zeta$, provided p is sufficiently large. Hence, taking a sufficiently large value of p, we have $| s(x) - \phi(x) | < 2\epsilon$, in a set of points contained in E_r, of measure $> m(E_r) - \eta - \zeta$. Since η and ζ converge to zero with ϵ, it follows that $s(x) - \phi(x) = 0$, almost everywhere in E_r. Considering the sequence $\{E_r\}$, of which E is the outer limiting set; or in case $m(E)$ is finite, taking E_r to coincide with E, it follows that $\phi(x) = s(x)$ almost everywhere in E. Thus the second part of the theorem has been proved.

The following is the converse theorem:

If, in the measurable set E, of finite, or of infinite, measure, the sequence $\{s_n(x)\}$, of measurable functions, finite almost everywhere in E, is convergent on the average, the function $s(x)$, defined in accordance with the theorem of § 168, is such that the sequence $\{s_n(x)\}$ is convergent on the average to $s(x)$, in any part E_1, of E, of finite measure. If $m(E)$ is finite, $\{s_n(x)\}$ converges on the average to $s(x)$, in E.

Since $\{s_n(x)\}$ is convergent on the average, in E_1, $| s_p(x) - s_q(x) | < \eta$ in some set of measure $> m(E_1) - \zeta$, provided p and q are sufficiently large. Now let $q = n_r$, then $| s(x) - s_{n_r}(x) | < \eta$, in some set of measure $> m(E_1) - \zeta$, provided r is sufficiently large. It follows that

$$| s(x) - s_p(x) | < 2\eta,$$

in a set, contained in E_1, of measure $> m(E_1) - 2\zeta$, the fixed number p being sufficiently large. For this value of p, the set of points of E_1 at which $| s(x) - s_p(x) | \geqq 2\eta$ has its measure less than 2ζ. Since η and ζ converge together to zero, the condition is satisfied that $\{s_n(x)\}$ converges on the average to $s(x)$, in E_1. The set E_1 may be any measurable part of E, and in case $m(E)$ is finite, it may be identical with E.

A particular case of the last theorem arises when $\{s_n(x)\}$ converges, in the ordinary way, to $s(x)$, almost everywhere in E. If E_1 is a part of E, of finite measure, $\{s_n(x)\}$ converges uniformly in a part of E_1 of measure $> m(E_1) - \zeta$. If p and q have large enough values $| s_p(x) - s_q(x) | < \epsilon$, in a set of points of measure $> m(E_1) - \zeta$. Thus the points of E_1, at which $| s_p(x) - s_q(x) | \geqq \epsilon$, form a set of measure $< \zeta$, for each pair of values of p and q that are both large enough. Hence the sequence $\{s_n(x)\}$ converges on the average in each part E_1, of E, for which $m(E_1)$ is finite; accordingly $\{s_n(x)\}$ converges on the average to $s(x)$, in each part of E that has finite measure; if $m(E)$ is finite, $\{s_n(x)\}$ converges on the average to $s(x)$, in E. It has thus been shewn that:

If $\{s_n(x)\}$ converges almost everywhere to $s(x)$, it converges on the average to $s(x)$ in any part of E of which the measure is finite.

That the converse of this theorem does not hold good is seen by considering, as in § 169, the sequence $\{s_n(x)\}$, which is convergent on the average

to $s(x)$; the sequence itself does not necessarily converge to $s(x)$, but only the part sequence $\{s_{n_p}(x)\}$ converges to $s(x)$ almost everywhere in the set E.

For example, let $\{s_n(x)\}$ be defined in the linear interval $(0, 1)$ by the rule that, if $n = m^2 + r$, where $0 \leq r < 2m + 1$, then $s_n(x) = 1$, in the interval $\left(\frac{r}{2m+1}, \frac{r+1}{2m+1}\right)$, and everywhere else $s_n(x) = \frac{1}{m}$. The sequence $\{s_n(x)\}$ is convergent on the average in $(0, 1)$, but it is not a convergent sequence. The subsequence $\{s_{m^2}(x)\}$ converges to the function which has the value zero everywhere except at the point 0, where its value is 1.

170. If the measurable function $f(x)$, and the sequence of measurable functions $\{\phi_n(x)\}$, defined in the measurable set E, of finite, or of infinite measure, of any number of dimensions, be such that

$$\int_{(E)} |f(x) - \phi_n(x)|^k \, dx,$$

for a value of k (> 0), exists as an L-integral, for every value of n, and converges to zero as $n \sim \infty$, it is easily seen that $\{\phi_n(x)\}$ converges on the average to $f(x)$, in E. For, if n_ϵ be an integer, so great that

$$\int_{(E)} |f(x) - \phi_n(x)|^k \, dx < \epsilon^{k+1}, \text{ for } n \geqq n_\epsilon,$$

the set $h(\epsilon, n)$, of points of E at which $|f(x) - \phi_n(x)| \geqq \epsilon$ must have its measure $< \epsilon$. Since ϵ is arbitrary, it follows that $\{\phi_n(x)\}$ converges, on the average, to $f(x)$, in the set E. For, if $\epsilon' < \epsilon$, and $n \geqq n_{\epsilon'}$, the set $h(\epsilon, n)$ is contained in $h(\epsilon', n)$; and thus $m[h(\epsilon, n)] \leq m[h(\epsilon', n)] < \epsilon'$, for $n \geqq n_{\epsilon'}$; hence $m[h(\epsilon, n)]$ converges to zero, as $n \sim \infty$. It follows from the first theorem of § 169, that the sequence $\{\phi_n(x)\}$ is convergent on the average; and therefore, that, in accordance with the theorem of § 168, a partial sequence $\{\phi_{n_p}(x)\}$ exists which is convergent almost everywhere in E, and converges to $f(x)$, in accordance with the second theorem of § 169.

Similarly, if the sequence $\{\phi_n(x)\}$ be such that

$$\lim_{p\sim\infty,\, q\sim\infty} \int_{(E)} |\phi_p(x) - \phi_q(x)|^k \, dx = 0,$$

it is seen that the sequence $\{\phi_n(x)\}$ converges on the average in E. There then exists a sequence $\{\phi_{n_p}(x)\}$ which converges almost everywhere in E to a function $f(x)$, defined almost everywhere in E; and it converges on the average to $f(x)$ in any part of E which has finite measure.

171. The most important case for consideration, in view of applications in the theory of Fourier's and other series, is that in which $k = 2$. The more general case will be considered in § 177.

It will be shewn that:

If $\{\phi_n(x)\}$ be a sequence of functions, each of which has its square summable in the measurable set E, of finite, or of infinite, measure, in any number of dimensions, and if the functions are such that

$$\lim_{p\sim\infty,\, q\sim\infty} \int_{(E)} \{\phi_p(x) - \phi_q(x)\}^2 dx = 0,$$

then a sequence $\{n_p\}$, of integers, can be so determined that the sequence $\{\phi_{n_p}(x)\}$ converges, almost everywhere in E, to a function $f(x)$ whose square is summable in E. Moreover $f(x)$ is unique, in the sense that two values of it are equivalent to one another.

In accordance with what has been proved in § 170, the sequence $\{\phi_n(x)\}$ being convergent on the average, there exists a sequence $\{\phi_{n_p}(x)\}$ which converges, almost everywhere, in E, to a function $f(x)$.

It will first be shewn that $\int_{(E)} \{\phi_n(x)\}^2 dx$ converges, as $n \sim \infty$, to a definite limit.

We have

$$\int_{(E)} \{\phi_n(x)\}^2 dx < 2\int_{(E)} \{\phi_p(x)\}^2 dx + 2\int_{(E)} \{\phi_n(x) - \phi_p(x)\}^2 dx$$
$$< 2\int_{(E)} \{\phi_p(x)\}^2 dx + 2\epsilon,$$

where p is a fixed integer sufficiently large, for all sufficiently large values of n; therefore $\int_{(E)} \{\phi_n(x)\}^2 dx$ is bounded for all values of n. Again

$$\int_{(E)} |\{\phi_n(x)\}^2 - \{\phi_m(x)\}^2| \, dx < \left[\int_{(E)} \{\phi_n(x) - \phi_m(x)\}^2 dx \times\right.$$
$$\left.\int_{(E)} \{\phi_n(x) + \phi_m(x)\}^2 dx\right]^{\frac{1}{2}} < \epsilon^{\frac{1}{2}} \left[2\int_{(E)} \{\phi_n(x)\}^2 + 2\int_{(E)} \{\phi_m(x)\}^2 dx\right]^{\frac{1}{2}} < A\epsilon^{\frac{1}{2}},$$

where A is a fixed positive number, provided n and m are sufficiently large. It follows that $\int_{(E)} \{\phi_n(x)\}^2 dx$ converges to a definite limit, as $n \sim \infty$.

If the integration had been taken over any set G, contained in E, the same proof would shew that $\int_{(G)} \{\phi_n(x)\}^2 dx$ converges, as $n \sim \infty$, to a definite limit.

Let E be the outer limiting set of a sequence $\{E_n\}$, of measurable sets, all of finite measure, such that each set contains the preceding one. In case $m(E)$ is finite, E_n may be taken to coincide with E, for all values of n. In each set E_n there is a set G_n, of measure $> m(E_n) - \frac{\delta}{2^n}$, in which the sequence $\{\phi_{n_p}(x)\}$ converges uniformly to $f(x)$; the part of this set that

is in E_1 has its measure $> m(E_1) - \frac{\delta}{2^n}$. It is clear that the sets G_n may be so chosen that G_n is contained in G_{n+1}, for all values of n.

Since $\{\phi_{n_p}(x)\}$ converges uniformly to $f(x)$ in the set G_n, we have, in that set $|f(x)| < |\phi_{n_p}(x)| + \eta$, where η is a chosen positive number, provided p is sufficiently large. Since $|\phi_{n_p}(x)|^2$ is summable in G_n, it follows that $|f(x)|^2$ is also summable in G_n.

We have also

$$\left|\int_{(G_n)} [\{f(x)\}^2 - \{\phi_{n_p}(x)\}^2]\,dx\right| < \left[\int_{(G_n)} \{f(x) - \phi_{n_p}(x)\}^2\,dx \int_{(G_n)} \{f(x) + \phi_{n_p}(x)\}^2\,dx\right]^{\frac{1}{2}},$$

and, for all sufficiently large values of p, we have $|f(x) - \phi_{n_p}(x)| < \epsilon$, at all points of G_n; thus the integral on the left-hand side is less than

$$\epsilon\,[m(E_n)]^{\frac{1}{2}}\left[\int_{(G_n)} \{2\,|f(x)| + \epsilon\}^2\,dx\right]^{\frac{1}{2}}.$$

Since p becomes indefinitely great, as $\epsilon \sim 0$,

$$\begin{aligned}\int_{(G_n)} \{f(x)\}^2\,dx &= \lim_{p\sim\infty}\int_{(G_n)} \{\phi_{n_p}(x)\}^2\,dx\\ &= \lim_{m\sim\infty}\int_{(G_n)} \{\phi_m(x)\}^2\,dx\\ &\leqq \lim_{m\sim\infty}\int_{(E)} \{\phi_m(x)\}^2\,dx.\end{aligned}$$

Now $m(G_n) > m(E_n) - \frac{\delta}{2^n}$; hence, since δ is arbitrarily small, we have

$$\int_{(E_n)} \{f(x)\}^2\,dx \leqq \lim_{m\sim\infty}\int_{(E)} \{\phi_m(x)\}^2\,dx.$$

Thus $\int_{(E_n)} \{f(x)\}^2 dx$ is bounded for all values of n, and its limit, as $n \sim \infty$, accordingly exists as a definite number, since the values of the integral, as n increases indefinitely, form a monotone non-diminishing sequence. It follows that $\int_{(E)} \{f(x)\}^2 dx$ exists; and therefore $\{f(x)\}^2$ is summable in E.

172. We proceed to obtain further properties of the function $f(x)$. We have

$$\left|\int_{(G_p)} \{f(x) - \phi_m(x)\}^2\,dx - \int_{(G_p)} \{\phi_m(x) - \phi_{n_r}(x)\}^2\,dx\right|$$
$$\leqq \left[\int_{(G_p)} \{f(x) - \phi_{n_r}(x)\}^2\,dx \int_{(E)} \{f(x) + \phi_{n_r}(x) - 2\phi_m(x)\}^2\,dx\right]^{\frac{1}{2}}$$

It is easily seen that

$$\int_{(E)} \{f(x) + \phi_{n_r}(x) - 2\phi_m(x)\}^2\,dx$$

is less than a fixed number, independent of r and m; for the integrals of $\{\phi_{n_r}(x)\}^2$, $\{\phi_m(x)\}^2$ are less than fixed numbers; and since $\{f(x)\}^2$ is summable, the integrals of $f(x)\,\phi_{n_r}(x)$, $f(x)\,\phi_m(x)$, and $\phi_{n_r}(x)\,\phi_m(x)$, are seen, by employing Schwarz's inequality, to be numerically less than fixed positive numbers. We have therefore

$$\int_{(G_p)}\{f(x)-\phi_m(x)\}^2\,dx < \int_{(E)}\{\phi_m(x)-\phi_{n_r}(x)\}^2\,dx + K\left[\int_{(G_p)}\{f(x)-\phi_{n_r}(x)\}^2\,dx\right]^{\frac{1}{2}},$$

where K is independent of r, m and p.

If m be chosen sufficiently large, $\int_{(E)}\{\phi_m(x)-\phi_{n_r}(x)\}^2\,dx$ is less than an arbitrarily chosen positive number ϵ, for all sufficiently large values of r. Also $\{f(x)-\phi_{n_r}(x)\}^2$ is arbitrarily small ($<\eta^2$) everywhere in G_p, provided r be large enough. Therefore

$$\int_{(G_p)}\{f(x)-\phi_m(x)\}^2\,dx < \epsilon + K\eta\,\{m(G_p)\}^{\frac{1}{2}} < \epsilon + K\eta\,\{m(E_p)\}^{\frac{1}{2}}.$$

This holds for the set G_p, of measure $> m(E_p) - \dfrac{\delta}{2^p}$; hence, by diminishing δ indefinitely, it is seen that $\int_{(E_p)}\{f(x)-\phi_m(x)\}^2 dx \leq \epsilon$, since η is arbitrarily small; and this holds for all sufficiently large values of m.

Taking a fixed value of m sufficiently large, we see that, since, for sufficiently large values of p

$$\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx - \int_{(E_p)}\{f(x)-\phi_m(x)\}^2\,dx < \epsilon,$$

we have $\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx < 2\epsilon$, provided m is sufficiently large. Therefore $\lim\limits_{m\sim\infty}\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx = 0$.

Again, we have

$$\left|\int_{(E)}\{f(x)\}^2\,dx - \int_{(E)}\{\phi_m(x)\}^2\,dx\right| \leq \left[\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx \cdot \int_{(E)}\{f(x)+\phi_m(x)\}^2\,dx\right]^{\frac{1}{2}} \leq K'\left[\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx\right]^{\frac{1}{2}},$$

where K' is a fixed positive number.

It follows that

$$\int_{(E)}\{f(x)\}^2\,dx = \lim_{m\sim\infty}\int_{(E)}\{\phi_m(x)\}^2\,dx.$$

The following theorem has now been established:

The function $f(x)$, to which the sequence $\{\phi_{n_p}(x)\}$ converges in the measurable set E, of finite, or of infinite, measure is such that

$$\int_{(E)}\{f(x)\}^2\,dx = \lim_{m\sim\infty}\int_{(E)}\{\phi_m(x)\}^2\,dx,$$

and that

$$\lim_{m\sim\infty}\int_{(E)}\{f(x)-\phi_m(x)\}^2\,dx = 0,$$

where the functions $\{\phi_n(x)\}$ satisfy the conditions of the preceding theorem.

Conversely, it may be shewn that, if a function $f(x)$ whose square is summable in E, exists, and is such that

$$\lim_{n\sim\infty}\int_{(E)}\{f(x)-\phi_n(x)\}^2\,dx = 0,$$

then

$$\lim_{\substack{p\sim\infty\\ q\sim\infty}}\int_{(E)}\{\phi_p(x)-\phi_q(x)\}^2\,dx = 0.$$

For $\int_{(E)}\{\phi_p(x)-\phi_q(x)\}^2dx < 2\int_{(E)}\{f(x)-\phi_p(x)\}^2dx + 2\int_{(E)}\{f(x)-\phi_q(x)\}^2dx$.

A CLASSIFICATION OF SUMMABLE FUNCTIONS

173. If a measurable function $f(x)$, defined in the linear interval (a, b), or in a cell (a, b) of any number of dimensions, be such that $|f(x)|^p$, where $p \geqq 1$, be integrable (L) over (a, b), the function $f(x)$ is said to belong to the class $[L^p]$. If $p = 1$, the class consists of all summable functions; we shall therefore assume that $p > 1$. Let q be defined by $\frac{1}{p}+\frac{1}{q}=1$, so that $q > 1$. If $f_1(x)$ be of class $[L^p]$, and $f_2(x)$ be of class $[L^q]$, we have the fundamental relations given in I, § 435,

$$\left|\int_a^b f_1(x)f_2(x)\,dx\right| \leq \left[\int_a^b |f_1(x)|^p\,dx\right]^{\frac{1}{p}}\left[\int_a^b |f_2(x)|^q\,dx\right]^{\frac{1}{q}},$$

and if $f(x)$, $g(x)$ are both of class $[L^p]$, we have

$$\left|\int_a^b |f(x)+g(x)|^p\right|^{\frac{1}{p}} \leq \left[\int_a^b |f(x)|^p\,dx\right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^p\,dx\right]^{\frac{1}{p}}.$$

It follows from these relations that the product $f_1(x)f_2(x)$ of functions of classes $[L^p]$ and $[L^q]$ is summable, and that the sum of two functions, both of class $[L^p]$ is also of class $[L^p]$.

It may be proved* conversely that if, for all functions $f_1(x)$, of class $[L^p]$, the product $f_1(x)f_2(x)$ is summable, then $f_2(x)$ must be of class $[L^q]$.

* See F. Riesz, *Math. Annalen*, vol. LXIX (1910), p. 457. The theory of strong and weak convergence of functions of class $[L^p]$ is there given.

The following generalisation of the approximation theorem given in I, § 430 will be established:

If $|f(x)|^p$, *for a value of* p *that is* $\geqq 1$, *be summable in the interval, or cell,* (a, b), *a continuous function* $\phi(x)$ *can be so determined that*

$$\int_a^b |f(x) - \phi(x)|^p\, dx$$

is less than an arbitrarily assigned positive number. In case $f(x) \geqq 0$, *in* (a, b), *the function* $\phi(x)$ *can be so determined that* $\phi(x) \geqq 0$, *in* (a, b).

The proof of this theorem only requires a slight modification of the proof, given in I, § 433, for the case in which $p = 2$. Taking $f_1(x) \geqq 0$, the continuous function $\phi_1(x)$ ($\geqq 0$) can be so determined that

$$\int_a^b |f_1(x)^p - \phi_1(x)^p|\, dx < \eta.$$

For every value of x, we have

$$|f_1(x) - \phi_1(x)|^p \leqq |f_1(x)^p - \phi_1(x)^p|, \quad p > 1;$$

it follows that
$$\int_a^b |f_1(x) - \phi_1(x)|^p\, dx < \eta.$$

Taking $f(x) = f_1(x) - f_2(x)$, where $f_1(x) \geqq 0, f_2(x) \geqq 0$, and employing the inequality

$$\int_a^b |f(x) - \phi(x)|^p\, dx$$
$$\leqq 2^{p-1}\int_a^b |f_1(x) - \phi_1(x)|^p\, dx + 2^{p-1}\int_a^b |f_2(x) - \phi_2(x)|^p\, dx,$$

the result follows, as in I, § 433.

From this theorem there can be deduced a theorem established otherwise by F. Riesz (*loc. cit.*) for the case of a linear interval. The interval or cell (a, b) can be divided into a definite number of cells or intervals, such that in each of them the fluctuation of $\phi(x)$ is less than the prescribed positive number η. Let $\psi(x)$ be that function which has, within each cell or interval, a constant value equal to the value of $\phi(x)$ at the centre of the cell or interval; and let $\psi(x)$ have the value zero on the boundaries of the cells or intervals. It is seen then that $\int_a^b |\phi(x) - \psi(x)|^p\, dx$ is less than η^p multiplied by the measure of (a, b).

From the relation

$$\int_a^b |f(x) - \psi(x)|^p\, dx \leqq 2^{p-1}\int_a^b |f(x) - \phi(x)|^p\, dx$$
$$+ 2^{p-1}\int_a^b |\phi(x) - \psi(x)|^p\, dx$$

it is seen that $\int_a^b |f(x) - \psi(x)|^p\, dx$ is less than an arbitrarily assigned number, if $\phi(x)$ and $\psi(x)$ be properly chosen. Thus it has been shewn that:

If $f(x)$ be of class $[L^p]$, where $p \geqq 1$, a function $\psi(x)$, can be determined, which is constant within each cell or interval of a set into which (a, b) is divided, such that $\int_a^b |f(x) - \psi(x)|^p dx$ is less than an assigned positive number.

174. If a sequence $\{f_n(x)\}$ of functions belonging to the class $[L^p]$ satisfies the condition

$$\lim_{n\sim\infty} \int_a^b |f(x) - f_n(x)|^p dx = 0, \quad p > 1$$

the function $f(x)$ belonging also to the class $[L^p]$, the sequence $\{f_n(x)\}$ is said to converge *strongly* to the function $f(x)$, with exponent p. In case $p = 2$, strong convergence is identical with the convergence considered in §§ 171, 172.

If $g(x)$ be a function belonging to the class $[L^q]$, we have

$$\int_a^b \{f(x) - f_n(x)\} g(x)\, dx \leqq \left[\int_a^b |f(x) - f_n(x)|^p\right]^{\frac{1}{p}} \left[\int_a^b |g(x)|^q dx\right]^{\frac{1}{q}},$$

from which it follows that

$$\int_a^b f(x) g(x)\, dx = \lim_{n\sim\infty} \int_a^b f_n(x) g(x)\, dx. \qquad\text{(1)}$$

We have also

$$\left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} \leqq \left[\int_a^b |f(x) - f_n(x)|^p dx\right]^{\frac{1}{p}} + \left[\int_a^b |f_n(x)|^p dx\right]^{\frac{1}{p}},$$

from which it follows that

$$\int_a^b |f(x)|^p dx \leqq \underline{\lim}_{n\sim\infty} \int_a^b |f_n(x)|^p dx.$$

It can be shewn similarly, by interchanging $f(x)$ and $f_n(x)$ in the inequality, that

$$\overline{\lim}_{n\sim\infty} \int_a^b |f_n(x)|^p dx \leqq \int_a^b |f(x)|^p dx;$$

and it then follows that

$$\int_a^b |f(x)|^p dx = \lim_{n\sim\infty} \int_a^b |f_n(x)|^p dx. \qquad\text{(2)}$$

The relations (1) and (2) express cardinal properties of a sequence which converges strongly.

175. The sequence $\{f_n(x)\}$, of functions belonging to the class $[L^p]$, is said to converge *weakly*, with exponent p (> 1), to the function $f(x)$ of the same class, if (1), $\int_a^b |f_n(x)|^p dx < K$, for all values of n, and (2), $\lim_{n\sim\infty} \int_\Delta f_n(x)\, dx = \int_\Delta f(x)\, dx$, for every sub-interval, or sub-cell, of the given interval, or cell, (a, b).

If $g(x)$ be a function of class $[L^q]$, consider the function $\psi(x)$ which is constant within each cell or interval of a set into which (a, b) is divided, and is such that $\int_a^b |g(x) - \psi(x)|^q dx < \epsilon^q$; where ϵ is an arbitrarily chosen positive number. From the condition (2) in the definition, we have

$$\lim_{n\sim\infty} \int_a^b \{f(x) - f_n(x)\}\psi(x)\,dx = 0.$$

We have also,

$$\int_a^b \{f(x) - f_n(x)\} g(x)\,dx = \int_a^b \{f(x) - f_n(x)\}\{g(x) - \psi(x)\}\,dx + \int_a^b \{f(x) - f_n(x)\}\psi(x)\,dx;$$

the first integral on the right-hand side does not exceed in absolute value

$$\left[\int_a^b |f(x) - f_n(x)|^p dx\right]^{\frac{1}{p}} \left[\int_a^b |g(x) - \psi(x)|^q dx\right]^{\frac{1}{q}}$$

or $\epsilon\left[\int_a^b |f(x) - f_n(x)|^p dx\right]^{\frac{1}{p}}$, and it therefore does not exceed

$$\epsilon\left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} + \epsilon\left[\int_a^b |f_n(x)|^p dx\right]^{\frac{1}{p}};$$

and, in virtue of condition (1), this is less than a fixed multiple of ϵ. Hence $\overline{\lim\limits_{n\sim\infty}}\left|\int_a^b \{f(x) - f_n(x)\} g(x)\,dx\right|$ is less than a fixed multiple of ϵ; from which we have

$$\int_a^b f(x) g(x)\,dx = \lim_{n\sim\infty} \int_a^b f_n(x) g(x)\,dx \quad\quad (1)'$$

the same relation as in the case of strong convergence.

Next let $g(x) = \pm |f(x)|^{p-1}$, the upper or lower sign being taken according as $f(x) > 0$, or < 0; we have then $|g(x)|^q = |f(x)|^p$; and thus, from (1)′, we have

$$\int_a^b |f(x)|^p dx = \lim_{n\sim\infty} \int_a^b f_n(x) g(x)\,dx.$$

We have then

$$\left\{\int_a^b |f(x)|^p dx\right\}^p \leq \underline{\lim_{n\sim\infty}} \left[\int_a^b |f_n(x)|^p dx\right] \left[\int_a^b |g(x)|^q dx\right]^{\frac{p}{q}}$$

$$\leq \underline{\lim_{n\sim\infty}} \left[\int_a^b |f_n(x)|^p dx\right] \left[\int_a^b |f(x)|^p dx\right]^{p-1},$$

and therefore

$$\int_a^b |f(x)|^p dx \leq \underline{\lim_{n\sim\infty}} \int_a^b |f_n(x)|^p dx. \quad\quad (2)'$$

This inequality (2)′, for weak convergences, corresponds to the inequality (2), in the case of strong convergence.

176. The following theorem is fundamental in respect to weak convergence; it has however reference only to the case in which (a, b) is a linear interval:

If a family of functions $f(x)$ of a single variable, all of class $[L^p]$, contains an infinite (not necessarily enumerable) set of functions, and if

$$\int_a^b |f(x)|^p\, dx < K^p,$$

where K is independent of the particular function of the family, then the family contains at least one sequence $\{f_n(x)\}$ which converges weakly, with exponent p, to some function $\bar{f}(x)$ of class $[L^p]$.

For every function $f(x)$, of the family, the function $F(x) = \int_a^x f(x)\, dx$ can be formed. For all these functions we have

$$F(a) = 0, \quad |F(x_1) - F(x_2)| = \left|\int_{x_1}^{x_2} f(x)\, dx\right| \leq \left[\int_{x_1}^{x_2} |f(x)|^p\, dx\right]^{\frac{1}{p}} |x_1 - x_2|^{\frac{1}{q}}$$

or $|F(x_1) - F(x_2)| \leq K\, |x_1 - x_2|^{\frac{1}{q}}$. It follows that the family of functions $F(x)$ is equi-continuous, and since $|F(x)| \leq K(b-a)^{\frac{1}{q}}$, the conditions of Arzelà's theorem, given in § 120, are satisfied. It follows that a sequence $\{F_n(x)\}$ is contained in the family $\{F(x)\}$ which converges uniformly to a function $\bar{F}(x)$.

We may take $\{f_n(x)\}$ to be the sequence of functions of the given family which corresponds to the sequence $\{F_n(x)\}$. If the interval (a, b) be divided in any manner into a number m, of parts

$$(x_0, x_1), (x_1, x_2), \ldots (x_{m-1}, x_m),$$

where $x_1 = a$, $x_m = b$, we have (see I, § 452)

$$\sum_{r=0}^{r=m} \frac{|F_n(x_r) - F_n(x_{r-1})|^p}{(x_r - x_{r-1})^{p-1}} \leq \int_a^b |f_n(x)|^p\, dx \leq K^p.$$

By letting n increase indefinitely, we have

$$\sum_{r=0}^{r=n} \frac{|\bar{F}(x_r) - \bar{F}(x_{r-1})|^p}{(x_r - x_{r-1})^{p-1}} \leq K^p;$$

it has been shewn in I, §§ 451, 452 that this is the necessary and sufficient condition that $\bar{F}(x)$ should be the indefinite integral $\int_a^x \bar{f}(x)\, dx$, of a function $\bar{f}(x)$ which belongs to class $[L^p]$. It has thus been shewn that, to the sequence $\{f_n(x)\}$ there corresponds a function $\bar{f}(x)$, of the same class, such that $\lim\limits_{n\sim\infty} \int_a^x f_n(x)\, dx = \int_a^x \bar{f}(x)\, dx$; and thus that $\{f_n(x)\}$ converges weakly to $\bar{f}(x)$.

177. From the last theorem the following extension of the theorem of § 170, in the case of a linear interval (a, b), may be deduced:

If a sequence $\{f_n(x)\}$ of functions of class $[L^p]$ be such that

$$\underset{\substack{n\sim\infty\\ m\sim\infty}}{\int_a^b} [f_n(x) - f_m(x)]^p\, dx = 0$$

there exists a function $\bar{f}(x)$, of the same class, to which $\{f_n(x)\}$ converges strongly, with exponent p.

Since

$$\left[\int_a^b |f_n(x)|^p\, dx\right]^{\frac{1}{p}} \leqq \left[\int_a^b |f_n(x) - f_m(x)|^p\, dx\right]^{\frac{1}{p}} + \left[\int_a^b |f_m(x)|^p\, dx\right]^{\frac{1}{p}},$$

taking a fixed value of m, such that for $n \geqq m$, the first expression on the right-hand side is $< \eta$, we see that the expression on the left-hand side is, for all such values of n, less than a fixed number; and it follows that $\int_a^b |f_n(x)|^p\, dx$ is less than a number K^p, for all values of n. From the theorem of § 176 there exists a part $\{f_{n_r}(x)\}$ of the sequence $\{f_n(x)\}$ which converges weakly with index p, to a function $\bar{f}(x)$, of class $[L^p]$. It follows that the sequence $\{f_{n_r}(x) - f_n(x)\}$ converges weakly, with exponent p, to $\bar{f}(x) - f_n(x)$; we have therefore, from $(2)'$ of § 175,

$$\underline{\lim}_{r\sim\infty} \int_a^b |f_{n_r}(x) - f_n(x)|^p\, dx \geqq \int_a^b |\bar{f}(x) - f_n(x)|^p\, dx.$$

Letting n increase indefinitely, we have, from the condition in the enunciation, $\lim_{n\sim\infty} \int_a^b |\bar{f}(x) - f_n(x)|^p\, dx = 0$; and thus $\{f_n(x)\}$ converges strongly, with exponent p, to $\bar{f}(x)$.

PROPERTIES OF A MEASURABLE FUNCTION

178. In accordance with the fundamental approximation theorem given in I, § 430, if $f(x)$ be a summable function defined in a given cell Δ, there exists a continuous function $\phi(x)$ such that

$$\int_{(\Delta)} |f(x) - \phi(x)|\, dx < \epsilon.$$

If $f(x)$ be defined only in a bounded measurable set E, and be summable in E, we may suppose E to be contained in Δ. The function $f(x)$ may be extended to the whole cell Δ, by assuming that $f(x) = 0$ in $\Delta - E$; the extended function being summable in Δ. We have then

$$\int_{(E)} |f(x) - \phi(x)|\, dx + \int_{(\Delta - E)} |\phi(x)|\, dx < \epsilon,$$

and thus $\int_{(E)} | f(x) - \phi(x) | \, dx < \epsilon$. It thus appears that the approximation theorem of I, § 430, may be applied to a summable function $f(x)$ defined in a bounded and measurable set E. It also appears that the continuous function $\phi(x)$ may be taken to be not only continuous relatively to E, but also relatively to Δ, and consequently (see § 108) it can be extended so as to be continuous in all the space. In particular employing Weierstrass' theorem (§ 162), the function $\phi(x)$ may be taken to be a finite polynomial.

Taking $\epsilon = \eta^2$, it is seen that the part of E in which $| f(x) - \phi(x) | \geqq \eta$ is of measure less than η. Now let $f(x)$, although measurable in E, be no longer necessarily summable in E, but let it be finite almost everywhere in E. Employing the summable function $f^{(N)}(x)$, such that $f^{(N)}(x) = N$, or $-N$, according as $f(x)$ is positive or negative, at every point at which $| f(x) | \geqq N$, and $f^{(N)}(x) = f(x)$, when $| f(x) | < N$; and remembering that N can be so chosen that the set of points at which $f(x)$ and $f^{(N)}(x)$ are unequal has its measure less than η, we can determine a continuous function $\phi(x)$, which may be a finite polynomial, such that $| f^{(N)}(x) - \phi(x) | \geqq \eta$, only in points of a set, contained in E, of measure less than η. It follows that, at all points of a set of measure $> m(E) - 2\eta$, contained in E, the inequality $| f(x) - \phi(x) | < 2\eta$ is satisfied. It has thus been shewn that, $f(x)$ being any measurable function defined in the bounded and measurable set E, a function $\phi(x)$, continuous relatively to E, can be so determined that $| f(x) - \phi(x) | < \epsilon$, in a set contained in E, of measure $> m(E) - \epsilon$. Moreover the function $\phi(x)$ can be so chosen that it can be extended into a function that is continuous in all the space in which E is defined; and in particular, it may be a finite polynomial.

The following theorem has now been established:

If $f(x)$ be any measurable function (not necessarily summable), defined in the bounded and measurable set E, of any number of dimensions, and finite almost everywhere in E, then, if ϵ be a prescribed positive number, a function $\phi(x)$, continuous in the whole space in which E is defined, exists, such that $| f(x) - \phi(x) | < \epsilon$, at every point of E not belonging to a set of measure $< \epsilon$, contained in E. Moreover, the function $\phi(x)$ may be taken to be a finite polynomial.

179. Let E be a measurable set, not necessarily of finite measure. Taking E to be the outer limiting set of a sequence $\{E_n\}$ of sets each of which is of finite measure, let a function $\phi_n(x)$ be defined, which is continuous in all of the space, and such that, in E_n, $| f(x) - \phi_n(x) | < \frac{\epsilon}{2^n}$, at every point that does not belong to a certain set of measure $\frac{\epsilon}{2^n}$. It is

seen that, in E_1, the sequence $\{\phi_n(x)\}$ converges to $f(x)$, almost everywhere; for, in E_1, $|f(x) - \phi_n(x)| < \frac{\epsilon}{2^n}$, in a set of which the measure is

$$> m(E_1) - \frac{\epsilon}{2^n}.$$

The sets of which the measures are greater than

$$m(E_1) - \frac{1}{2}\epsilon,\ m(E_1) - \frac{1}{2^2}\epsilon, \dots m(E_1) - \frac{1}{2^n}\epsilon, \dots,$$

respectively, have a common part, of which the measure is $\geqq m(E_1) - \epsilon$, and, in this set, $|f(x) - \phi_m(x)| < \frac{1}{2}\epsilon$, for all values of m (1, 2, 3, ...). In the same set $|f(x) - \phi_m(x)| < \frac{\epsilon}{2^n}$, for all the values $n, n+1, \dots,$ of m. It follows that $\{\phi_m(x)\}$ converges to $f(x)$, almost everywhere in E_1, since ϵ is arbitrary. Since E_1 may be taken to be any set, of finite measure, contained in E, it follows that the sequence $\{\phi_m(x)\}$ converges to $f(x)$ almost everywhere in E.

The following theorem has now been established:

If $f(x)$ be a measurable function (not necessarily summable), defined in a measurable set E, of finite, or of infinite, measure (of any number of dimensions), there exists a sequence of functions $\{\phi_m(x)\}$, all of which are continuous in the whole of the space in which E is defined, such that $\{\phi_m(x)\}$ converges, as $m \sim \infty$, almost everywhere in E, to the function $f(x)$. Moreover, in particular, the sequence may be taken to be $\{P_m(x)\}$, where $P_m(x)$ denotes a finite polynomial.

It should be observed that, in the exceptional set, of measure zero, of points of E at which the sequence does not converge to $f(x)$, the sequence is not necessarily convergent.

When the set E is of finite measure, there exists, in E, a set of points of measure $> m(E) - \epsilon$, in which the sequence $\{\phi_n(x)\}$ converges uniformly to $f(x)$. This set may be so chosen as to be closed, or perfect.

Relatively to this set, the function $f(x)$ must be continuous (see § 86). Thus we have the following theorem:

If $f(x)$ be a measurable (not necessarily summable) function, defined in a set of points E, of finite measure, in any number of dimensions, and finite almost everywhere in E, there exists in E a perfect set of points, of measure arbitrarily near to $m(E)$, relatively to which the function $f(x)$ is continuous.

180. Let $f(x)$ be a function defined in the linear interval (a, b), and summable in that interval. It has been shewn in I, § 432, that

$$\int_{x_0}^{x} |f(x) - f(x_0)|\, dx$$

has a differential coefficient, equal to zero, at the point x_0, where x_0 is

any point of the interval (a, b), with the exception of points belonging to a set of measure zero. If x_0 be a point not belonging to the exceptional set, of measure zero, and η be an arbitrarily chosen positive number, h may be so chosen that $\frac{1}{2h}\int_{x_0-h}^{x_0+h} |f(x) - f(x_0)|\, dx < \eta$. The set of points in the interval $(x_0 - h, x_0 + h)$, at which $|f(x) - f(x_0)| \geqq \epsilon$, has its measure less than $\frac{2h\eta}{\epsilon}$; hence $|f(x) - f(x_0)| < \epsilon$, in a set contained in $(x_0 - h, x_0 + h)$, of measure $> 2h\left(1 - \frac{\eta}{\epsilon}\right)$. Keeping ϵ fixed, the number $1 - \frac{\eta}{\epsilon}$ converges to unity, as h and η converge together to zero. Therefore the metric density of the set of points at which $|f(x) - f(x_0)| < \epsilon$ is unity at the point x_0. Therefore $f(x)$ is approximately continuous at x_0 (see I, § 235).

Next, let $f(x)$, although measurable, not be summable in (a, b), but let it be finite almost everywhere. The summable function $f^{(N)}(x)$ may be defined as in § 179. Let N have successively the values in a monotone sequence $\{N_r\}$, such that N_r increases indefinitely with r. Then each of the functions $f^{(N_r)}(x)$ is approximately continuous almost everywhere in (a, b); and therefore, at almost every point of (a, b), all the functions $\{f^{(N_r)}(x)\}$ are approximately continuous; let x_0 be a point at which this is the case. Let s be a number such that $|f(x_0)| < N_s$; then the set of points at which $|f(x) - f(x_0)| < \epsilon$ is such that, for some value $t\,(> s)$, of r,

$$|f(x)| < N_t;$$

and it follows that, at all points of the set, $f(x) = f^{(N_t)}(x)$. Since the set of points at which $|f^{(N_t)}(x) - f^{(N_t)}(x_0)| < \epsilon$ has its metric density unity at x_0, the same holds for the set of points at which $|f(x) - f(x_0)| < \epsilon$. Hence $f(x)$ is approximately continuous at x_0.

The following theorem has now been established*:

If $f(x)$ be any measurable function, finite almost everywhere, defined in the linear interval (a, b), the function is approximately continuous almost everywhere in (a, b).

It is obvious that the function cannot be approximately continuous at a point of ordinary discontinuity. It may however be so at a point where the discontinuity is of the second kind. Thus, in a totally discontinuous function, all the points of ordinary discontinuity, if any, belong to the exceptional set; but at almost all the points at which the function has a discontinuity of the second kind the function must be approximately continuous.

A characterisation of the discontinuities of functions, based upon the notion of approximate continuity, has been made by M. H. A. Newman†.

* See Denjoy, *Bulletin de la soc. math. de France*, vol. XLIII (1915), p. 170.

† *Camb. Phil. Trans.* vol. XXIII (1923). See also Kempisty, *Fundamenta Mat.* vol. VI (1924), p. 6.

DESCRIPTIVE PROPERTIES OF SETS OF POINTS

181. It is convenient to give here an extension and amplification of the definitions relating to descriptive properties of sets of points. The aggregate of all points $(x^{(1)}, x^{(2)}, \ldots x^{(p)})$ of a space of p-dimensions will be denoted by S_p; it has been shewn in I, § 49, that the points of S_p correspond uniquely to the points of the space interior to the finite cell $(-1, -1, \ldots -1;\ 1, 1, \ldots 1)$, the relation of order being invariant for the transformation. The improper points at infinity, introduced in I, § 53, are points which correspond in order to the boundary points of the finite cell; when these improper points are adjoined to the set S_p, it becomes the closed set $\bar{S}_p$.

A set G is said to be closed in, or relatively to, a set E, when every limiting point of G that is in E belongs to G; also when G has no limiting point in E.

A closed set, in the ordinary sense, is a bounded set which is closed relatively to S_p; and such a set is also closed relatively to $\bar{S}_p$. A set is closed in the extended sense when it is closed relatively to S_p, but not to $\bar{S}_p$. It becomes closed in $\bar{S}_p$ when its improper limiting points are adjoined to the set.

For example, the set of points $1, 2, 3, \ldots n, \ldots$ is closed with respect to the open interval $(-\infty, \infty)$ because it has no limiting point in that interval. It is not closed with respect to the closed interval $(-\infty, \infty)$, but when the improper point ∞ is added to it, it becomes closed relatively to the closed interval $(-\infty, \infty)$.

A set G is said to be perfect in, or relatively to, a set E, when it is closed in E, and when further, every point of it in E is a limiting point of the set.

Thus a perfect set in the ordinary sense, being bounded, is perfect relatively to S_p and also to $\bar{S}_p$; it may be regarded as perfect, in the extended sense, when it is perfect relatively to S_p, but not relatively to $\bar{S}_p$. It becomes perfect in $\bar{S}_p$ when its improper limiting points are adjoined to it.

A set O is said to be open relatively to E if all the points of O that are in E are interior parts of O, relatively to E.

This is a slight extension of the definition given in I, p. 75, where the definition of an interior point of O, relatively to E, is given. It is easily seen that:

If a set G is closed in E, the part $D(G, E)$ of G, which is in E, is closed in E. If a set O is open in E, the part $D(O, E)$ is open in E.

For a limiting point of $D(G, E)$ that is in E belongs to G, and therefore to $D(G, E)$; thus $D(G, E)$ is closed in E. Again, if O is open in E, all the points of $D(O, E)$ are interior points of O relatively to E, and therefore interior points of $D(O, E)$, relatively to E.

The converse of this theorem is not in general true. For $D(G, E)$ may be closed in E, but a point of E may be a limiting point of G without belonging to G.

If a set G is closed in E, it is also closed in any set E_1 which is a part of E. Conversely, a set which is closed in E_1 is part of a set which is closed in E.

For every limiting point of $D(E, G)$ that is in E is also in $D(E, G)$; therefore every limiting point of $D(E, G)$ that is in E_1, and consequently in E, is contained in $D(E_1, G)$. Hence G is closed in E_1.

If G_1 be closed in E_1, every limiting point of G_1 that is in E_1 belongs to G_1. Consider the set G, obtained by adding to G_1 those of its limiting points that are in E but not in E_1. Every limiting point of G that is in E belongs to G; therefore G is closed in E, and it contains G_1.

If a set E be closed in G, its complement in G, namely $G - D(E, G)$, is open in G; and conversely.

For a point of $G - D(E, G)$ has no point of E that belongs to G in a sufficiently small neighbourhood, and is therefore an interior point of $G - D(E, G)$; therefore $G - D(E, G)$ is open relatively to G.

It should be observed that a closed, or an open, finite cell is a set of points which is both open and closed relatively to itself. The set S_p is both open and closed relatively to itself; it is open but not closed in $\bar{S}_p$.

If E be taken to be the set S_p, we have as a particular case:

Every set that is closed in S_p is closed in every set E contained in S_p; and conversely every set that is closed in a set E is part of a set that is closed in S_p.

The above theorems also hold good for open sets, where in each case a set open in E takes the place of a set, closed in E, and an absolutely open set takes the place of an absolutely closed set.

For, if O be open in E, the set $G \equiv O - D(O, E)$ is closed in E, and therefore in E_1, any part of E; thus $O - D(O, E_1)$ is closed in E_1, and therefore $D(O, E_1)$ is open in E_1. If O_1 is open in E_1, the set

$$O_1 - D(O_1, E_1) \equiv G_1$$

is closed in E_1.

The set G_1 is part of a set G which is closed in E, any set which contains E_1. Add to O_1 those points of E which do not belong to G or to O_1; we thus obtain a set O. This set O contains O_1, and since its complement in E is $D(G, E)$, which is closed in E, the set O is open in E. We have accordingly the following theorem.

Every set that is open in a set E is open in any part E_1, of E; and a set which is open in E_1 is a part of a set which is open in E. A set that is open in S_p is open in any set E contained in S_p; and a set which is open in E is part of a set open in S_p.

SETS OF POINTS OF ORDERS 1 AND 2

182. In the theory of the functions defined as the limits of sequences of continuous functions, sets of points of certain types are of importance.

If E be any set of points, in any number of dimensions, a set contained in E, and which is either closed relatively to E, or open relatively to E, is said to be *a set of the first order in E*. A set which is contained in E, and closed relatively to E, will be said to be *of type* $C_E^{(1)}$; and a set contained in E, and open relatively to E, will be said to be *of type* $O_E^{(1)}$. The sets of the first order in E thus consist of sets of types $C_E^{(1)}$ and $O_E^{(1)}$.

If $\{E_n\}$ be a sequence of sets contained in E, and such that each set is contained in the next, and if each of the sets E_n is of the first order in E, then the outer limiting set of $\{E_n\}$ is said to be *of type* $O_E^{(2)}$, whenever it is not of the first order in E.

In case each of the sets E_n contains the next, each set being of the first order in E, the inner limiting set of $\{E_n\}$ is said to be *of type* $C_E^{(2)}$, whenever it is not of the first order in E.

A set of either of the types $O_E^{(2)}$, $C_E^{(2)}$ is said to be *a set of the second order in E.*

The following properties of these sets are of importance:

A sequence $\{E_n\}$ of sets of the first order in E, each of which is contained in the next, has for its outer limiting set a set of the first order in E, provided an infinite number of the sets E_n are of type $O_E^{(1)}$.

For all the sets E_n which are not of type $O_E^{(1)}$ may be removed from the sequence, without affecting the outer limiting set. Thus the theorem is equivalent to the statement that the outer limiting set of a sequence of sets, open relatively to E, and each one of which is contained in the next, is open in E. This is a generalisation of the theorem relating to open sets in the continuum, given in I, § 56, and is proved in the same manner; it being observed that a set that is closed in E is complementary to a set that is open in E.

A sequence $\{E_n\}$ of sets of the first order in E, each of which contains the next, has for its inner limiting set a set of the first order in E, provided an infinite number of the sets E_n are of type $C_E^{(1)}$.

This follows also from a theorem given in I, § 56; it being observed that those of the sets E_n which are not of type $C_E^{(1)}$ may be removed from the sequence.

The complement with respect of E of a set of type $O_E^{(2)}$ is of type $C_E^{(2)}$. The complement with respect to E of a set of type $C_E^{(2)}$ is of type $O_E^{(2)}$.

For the complement of $M(E_1, E_2, \ldots E_n, \ldots)$, where each set is contained in the next, and is of type $C_E^{(1)}$, is the set $D(E - E_1, E - E_2, \ldots E - E_n \ldots)$. The sets $E - E_n$ are all of type $O_E^{(1)}$, and each contains the next, hence the complement of $M(E_1, E_2, \ldots E_n \ldots)$ is of type $C_E^{(2)}$, or of order 1. It cannot be of order 1, for then $M(E_1, E_2, \ldots)$ would be of order 1.

If a finite number of sets $H^{(1)}, H^{(2)}, \ldots H^{(r)}$ are all of type $O_E^{(2)}$, in the set E, the set $D(H^{(1)}, H^{(2)}, \ldots H^{(r)})$, of points common to all the r sets, is of type $O_E^{(2)}$, unless it is of order 1 *in E.*

Let $H^{(1)} = \lim_{n\sim\infty} G_n^{(1)}$, $H^{(2)} = \lim_{n\sim\infty} G_n^{(2)}$, $\ldots H^{(r)} = \lim_{n\sim\infty} G_n^{(r)}$, where $\{G_n^{(1)}\}$, $\{G_n^{(2)}\} \ldots$ are sequences of sets, closed in E, each of which is contained in the next. Any point of $D(H^{(1)}, H^{(2)}, \ldots H^{(r)})$ belongs to all the sets $G_n^{(1)}$, $G_n^{(2)}, \ldots G_n^{(r)}$, from and after some value of n depending upon the particular point, and therefore it belongs, for all such values of n, to

$$D(G_n^{(1)}, G_n^{(2)}, \ldots G_n^{(r)}).$$

The sets $D(G_n^{(1)}, G_n^{(2)}, \ldots G_n^{(r)})$, for $n = 1, 2, 3, \ldots$ form a sequence of sets, all closed in E; and each is contained in the next. Their outer limiting set is $D(H^{(1)}, H^{(2)}, \ldots H^{(r)})$, which is consequently of the type $O_E^{(2)}$, unless it is of order 1 in E.

The common part of two sets, A and B, each of which is either of type $O_E^{(2)}$, or else of the first order in E, is also of type $O_E^{(2)}$, or else of the first order in E.

If both sets are of type $O_E^{(2)}$, the theorem is a particular case of the preceding theorem. If one of the sets A is of type $O_E^{(2)}$, and the other of type $C_E^{(1)}$, since $A = \lim_{n\sim\infty} G_n$, where G_n is closed in E, we have

$$D(A, B) = \lim_{n\sim\infty} D(G_n, B),$$

and since B is closed in E, so also is $D(G_n, B)$; hence $D(A, B)$ is of type $O_E^{(2)}$, or else of the first order. If A is of type $O_E^{(2)}$, and B of type $O_E^{(1)}$, we have $B = \lim_{m\sim\infty} g_m$, where g_m is closed in E; then $D(A, B) = \lim_{n\sim\infty}\lim_{m\sim\infty} D(G_n, g_m)$, and this can be expressed as the limit of a simple sequence of sets closed in E; it follows that $D(A, B)$ is of type $O_E^{(2)}$, or else of the first order in E. If A is of type $O_E^{(1)}$ and B of type $C_E^{(1)}$, $D(A, B)$ is the set common to a set that is closed in E and one that is open in E; then $D(A, B)$ is expressible as the limit of a sequence of closed sets, and is therefore of type $O_E^{(2)}$, unless it is of the first order in E.

If $H^{(1)}, H^{(2)}, \ldots H^{(m)}, \ldots$ be a sequence of sets in E, all of type $O_E^{(2)}$, the set $M(H^{(1)}, H^{(2)}, \ldots H^{(m)}, \ldots)$ of points which belong to one or more of the

given sets, is also of type $O_E^{(2)}$, unless it is of the first order in E. Also the latter set is of the first category in E, in case all the sets of the sequence are so.

If $H^{(m)} = \lim\limits_{n\sim\infty} G_n^{(m)}$, where all the sets $G_n^{(m)}$ are closed in E, let us consider the sequence of sets, closed in E, $G_1^{(1)}$, $M(G_1^{(1)}, G_1^{(2)})$, $M(G_1^{(1)}, G_1^{(2)}, G_2^{(1)})$, $M(G_1^{(1)}, G_1^{(2)}, G_2^{(1)}, G_1^{(3)})$, $M(G_1^{(1)}, G_1^{(2)}, G_2^{(1)}, G_1^{(3)}, G_2^{(2)})$ .... Each of these is closed in E, and each is contained in the next, and every set $G_n^{(m)}$ occurs, from and after some fixed set of the sequence. It is clear that the outer limiting set is $M(H^{(1)}, H^{(2)}, \dots H^{(m)}, \dots)$ which is therefore of type $O_E^{(2)}$, if it is not of the first order. A set $H^{(m)}$ is of the first category in E, if all the sets $G_n^{(m)}$, for $n = 1, 2, 3, \dots$ are non-dense in E. If all the sets $H^{(m)}$ are of the first category in E, all the sets

$$G_1^{(1)},\ M(G_1^{(1)}, G_1^{(2)}),\ M(G_1^{(1)}, G_1^{(2)}, G_2^{(1)}), \dots$$

are non-dense in E, and therefore their outer limiting set is of the first category in E.

If E_1 be a part of E, the part of a set of type $O_E^{(2)}$ which is in E_1 is of type $O_{E_1}^{(2)}$, or else is of the first order in E_1. The corresponding result holds for a set of type $C_E^{(2)}$.

The set of type $O_E^{(2)}$ is the outer limiting set of a sequence $\{G_n\}$ of sets all closed relatively to E. The sets $D(E_1, G_n)$ are all closed in E_1, and thus the part of the given set which is in E_1 is the outer limiting set of a sequence of sets closed in E_1; thus the part is of type $O_{E_1}^{(2)}$, unless it be of the first order in E_1. The corresponding theorem for a set of type $C_E^{(2)}$ follows from the fact that the complement of such a set, relative to E, is of type $O_E^{(2)}$.

183. It has been shewn in I, § 96, that in case E be a perfect set, the outer limiting set of a sequence of non-dense closed sets, which is a set of the first category in E, has, for its complement in E, a set which is everywhere dense in E. It was in fact shewn that every cell or interval (α, β) containing points of E contains a point, defined by a sequence of cells or intervals (α, β), $(\alpha_1, \beta_1) \dots (\alpha_n, \beta_n) \dots$, each of which contains the next, which is a point of E, but not a point of the set of the first category. The argument is not in general applicable to a set E which is not closed, because the point defined by the sequence of cells or intervals may not be a point of E. The procedure is, however, applicable, in case E is an open set, because each of the cells or intervals (α_n, β_n) may then be taken to consist entirely of points of E. The set of the first category is then, in this case, diffuse (see I, § 55) in E, although it may be everywhere dense in E. The same remark applies to the case in which E consists of the points which an open set and a perfect set have in common.

We have thus the theorem:

If E be either a perfect set, or an open set, or consists of the points common to an open and a perfect set, the outer limiting set of a sequence of non-dense sets, all of which are closed relatively to E, and each one of which is contained in E, is diffuse in E. (See I, p. 76.)

As in I, § 94, it follows that, if E be an open set, the complement of the outer limiting set cannot be of the first category in E.

More generally, we have the theorem:

If E be either perfect, or open, or be the set of points which an open set and a perfect set have in common, and if $E_1, E_2, \ldots E_n, \ldots$ be a sequence of sets, all of the first category in E, then $M(E_1, E_2, \ldots)$ is of the first category in E; and thus it is impossible that $E = M(E_1, E_2, \ldots)$.

For if $E_n = M(G_n^{(1)}, G_n^{(2)}, \ldots)$, where $G_n^{(r)}$ is closed in E, we have

$$M(E_1, E_2, \ldots) = M(G_1^{(1)}, G_1^{(2)}, G_2^{(1)}, G_1^{(3)}, G_2^{(2)}, G_3^{(1)}, \ldots),$$

and the set on the right hand side is of the first category in E.

In particular E cannot be resolved into the sum of an enumerably infinite, or finite, series of sets, each of which is of the first category in E.

If E be identical with S_p, the aggregate of all points in p dimensions, the sets of type $O_{S_p}^{(1)}$ consist of all open sets, and the sets of type $C_{S_p}^{(1)}$ consist of all bounded closed sets and also of sets which contain all their finite limiting points. But if E be identical with $\bar{S}_p$, the absolute set in p dimensions, the sets of type $O_{\bar{S}_p}^{(1)}$ include all bounded open sets, and also all unbounded open sets with, or without, their limiting points at infinity; and the sets of type $C_{\bar{S}_p}^{(1)}$ consist of all closed sets, whether bounded or not. This is seen to be the case by employing the correspondence of S_p and of $\bar{S}_p$ with an open, or closed, finite cell. It is convenient to speak of sets closed relatively to $\bar{S}_p$ as of type $C^{(1)}$, and of sets that are open relatively to $\bar{S}_p$, as of type $O^{(1)}$.

Similarly, a set of type $O^{(2)}$ or $C^{(2)}$ means a set of type $O_{\bar{S}_p}^{(2)}$, or $C_{\bar{S}_p}^{(2)}$. It has been shewn in § 181 that the part of a set of type $O^{(1)}$ that is in E, is of type $O_E^{(1)}$, and that the part of a set of type $C^{(1)}$, in E, is of type $C_E^{(1)}$, whatever the set E may be; but the converse does not in general hold. It follows that the part of a set of type $O^{(2)}$ that is in E, is of type $O_E^{(2)}$, unless it is of the first order in E; and that a set of type $C^{(2)}$ has, for its part in E, a set of type $C_E^{(2)}$, unless it is of the first order in E.

184. It will be shewn that:

If E be either an open set, or a closed set, or a set which consists of the points which an open set and a closed set have in common, then a set, in E, of one of the types $O_E^{(2)}$, $O_E^{(1)}$, $C_E^{(1)}$ is also of one of the types $O^{(2)}$, $O^{(1)}$, $C^{(1)}$.

The theorem is obviously true in case E is a closed set. If E be an open set, it is the limit of a sequence $\{G_n\}$ of closed sets contained in it, and each of which is contained in the next. If $E = D(H, K)$, where H is open and K is closed, let $H = \lim_{n\sim\infty} g_n$, where g_n is a closed set; then $E = \lim_{n\sim\infty} D(g_n, K)$, and $D(g_n, K)$ is a closed set G_n; or $E = \lim_{n\sim\infty} G_n$. If F be a set of type $C_E^{(1)}$, we have, in either case, $F = \lim_{n\sim\infty} D(G_n, F)$; and it will be shewn that $D(G_n, F)$ is a closed set.

Any limiting point of $D(G_n, F)$ is in G_n, and therefore in E; also such a limiting point, being a limiting point of F which is in E, must belong to F, since F is closed in E; and it therefore belongs to $D(G_n, F)$, which is therefore closed. Therefore a set of type $C_E^{(1)}$ is of one of the types $O^{(2)}$, $C^{(1)}$, for if it were of type $O^{(1)}$ it would be of type $O_E^{(1)}$.

A set of type $O_E^{(1)}$ is the outer limiting set of a sequence of sets all of type $C_E^{(1)}$, that is one of the types $O^{(2)}$, $C^{(1)}$; hence it is of one of the types $O^{(2)}$, $O^{(1)}$. It follows that a set of type $O_E^{(1)}$ is of type $O^{(2)}$, unless it be of the first order.

A set of type $O_E^{(2)}$ is the outer limiting set of a sequence of sets all of which are of the first order in E, and consequently of one of the types $O^{(2)}$, $O^{(1)}$, $C^{(1)}$, hence the given set is of the type $O^{(2)}$ (see § 182). For it cannot be of one of the types $O^{(1)}$, $C^{(1)}$, since it would then be of type $O_E^{(1)}$ or $C_E^{(1)}$.

FUNCTIONS REPRESENTABLE BY SERIES OR SEQUENCES OF CONTINUOUS FUNCTIONS

185. The question as to the nature of the most general function that can be represented in a given interval, or cell, as the sum of a series of continuous functions, and therefore as the limit of a convergent sequence of such functions, received a complete answer from Baire, whose result is contained in the following remarkable theorem:

The necessary and sufficient condition that a function, defined in a closed interval, or cell, may be representable as the sum of a series of continuous functions which converges at every point of the interval, or cell, to the value of the function, is that the given function shall be at most pointwise discontinuous with respect to every perfect set of points in the given interval, or cell.

The theorem was first established by Baire* for the case of functions of a single variable, and was afterwards extended by Lebesgue† and by Baire‡ himself to the case of functions of any number of variables. Other

* In his memoir "Sur les fonctions de variables réelles," *Annali di Mat.* (3) A, vol. III (1899).

† *Comptes Rendus*, vol. CXXVIII (1899), p. 811.

‡ *Bull. de la soc. math. de France*, vol. XXVIII (1900), p. 173. See also Baire's treatise *Leçons sur les fonctions discontinues*, pp. 149–155.

proofs of the theorem have been given by Lebesgue*, Dell'-Agnola† and de la Vallée-Poussin‡.

The theorem will here be investigated by a method which is essentially that of de la Vallée Poussin, but in a somewhat generalised form.

If E denotes a set of points in any number of dimensions, a function defined over E which is continuous relative to E will be said to be of class 0, in E. By some writers, a more restricted definition of functions of class 0 is adopted; only those functions are said§ to be of class 0, in E, which are not only continuous relatively to E, but which are capable of being extended so as to be continuous in the closed set $M(E, E')$, obtained by adding to E those of its limiting points which do not belong to the set itself; such a function can then (see § 108) be further extended so as to be continuous in all the space. If a function, defined in E, is such that its value at each point is the limit of a sequence of functions, all of which are of class 0, in E, is said to be of class 1, in E, provided it is not of class 0, in E.

If $\{f_n(x)\}$, a sequence of functions, all continuous relatively to E, has for its limiting function $f(x)$, then $f(x)$ is of class ≤ 1, in E. The functions $f_n(x)$ need only be continuous in E in the extended sense of the term, and $f(x)$ may have an infinite value at a point at which the sequence $\{f_n(x)\}$ diverges to ∞, or to $-\infty$.

There is, however, no loss of generality in the theory if we assume that the functions $\{f_n(x)\}$ are all bounded, say in the interval $(-1, 1)$; in which case continuity is taken in the ordinary sense. For, if we employ the transformation $\phi_n(x) = \dfrac{f_n(x)}{1+|f_n(x)|}$, $\phi(x) = \dfrac{f(x)}{1+|f(x)|}$, and if $f_n(x)$ is continuous relatively to E, and has $f(x)$, of class 1, or 0, for its limiting function, it has been shewn in I, § 219, that the functions $\phi(x)$, $\{\phi_n(x)\}$, all of which are bounded and have their values confined to the interval $(-1, 1)$, are such that $\phi_n(x)$ is continuous relatively to E; moreover $\phi(x)$ has the same class 0, or 1, as $f(x)$, in the set E. The converse of this statement also holds good. It will accordingly be throughout assumed that all the functions $f_n(x)$, $f(x)$ are bounded.

186. It is clear that, if $f(x)$ is of class 1, in E, it is of class ≤ 1 in any part of E.

The following theorem is easily established:

If the functions $f_1(x), f_2(x), \ldots f_r(x)$ are all of class ≤ 1, in E, and the

* See Borel's *Leçons sur les fonctions de variables réelles*, pp. 149–155; also Lebesgue's memoir "Sur les fonctions représentables analytiquement," *Liouville's Journal* (6), vol. I (1905).

† *Atti Ven.* vol. LXVIII (1909), p. 775; *Rend. Lombardo*, vol. XLI (1908), pp. 287, 676.

‡ See his treatise *Intégrales de Lebesgue* (1916), pp. 121–125.

§ See, for example, Carathéodory's *Vorlesungen über reelle Funktionen*, p. 393.

function $F(f_1, f_2, \dots f_r)$ *is continuous with respect to* $(f_1, f_2, \dots f_r)$ *then* $F(f_1, f_2, \dots f_r)$ *is of class* $\leqq 1$, *in* E.

Let $f_s(x) = \lim_{n \sim \infty} f_{sn}(x)$, for $s = 1, 2, 3, \dots r$; where $f_{sn}(x)$ is of class 0, in E. The function $F(f_{1n}, f_{2n}, \dots f_{rn})$ is continuous, and thus of class 0, in E. Also $F(f_1, f_2, \dots f_r) = \lim_{n \sim \infty} F(f_{1n}, f_{2n}, \dots f_{rn})$, and therefore

$$F(f_1, f_2, \dots f_r)$$

is of class $\leqq 1$, in E.

The following special case of this theorem should be observed:

(1) *The sum, or the difference, or the product, of two functions each of which is of class* $\leqq 1$, *in* E, *is also of class* $\leqq 1$, *in* E.

(2) *If* $f(x)$ *is of class* $\leqq 1$, *in* E, *so also is* $|f(x)|$.

For $|f|$ is a continuous function of f.

(3) *If* $f_1(x), f_2(x), \dots f_r(x)$ *are all of class* $\leqq 1$, *in* E, *and* $\phi(x)$ *be the function which has, at each point, the value of the greatest of the given functions, then* $\phi(x)$ *is of class* $\leqq 1$, *in* E.

For $\phi(x)$ is a function of $f_1(x), f_2(x), \dots f_r(x)$ which is continuous in E, relatively to $(f_1 . f_2, \dots f_r)$.

(4) *If* $f(x)$ *is of class* 1, *in* E, *the function* $\phi(x)$ *which has the value* $f(x)$, *when* $A < f(x) < B$, *and has the value* A *when* $f(x) \leqq A$, *and the value* B *when* $f(x) \geqq B$, *is of class* $\leqq 1$, *in* E.

For $\phi(x)$ is a continuous function of $f(x)$.

(5) *If the function* $f(x)$, *of class* 1, *in* E, *is such that* $L \leqq f(x) \leqq U$, *in* E, *then* $f(x)$ *is the limiting function of a sequence* $\{\phi_n(x)\}$, *of functions of class* 0, *in* E, *such that* $L \leqq \phi_n(x) \leqq U$, *for every value of* n.

For if $f(x) = \lim_{n \sim \infty} \psi_n(x)$, and we take $\phi_n(x)$ to be the function which $= \psi_n(x)$ when $L \leqq \psi_n(x) \leqq U$, and which $= L$ when $\psi_n(x) < L$, and which $= U$ when $\psi_n(x) > U$, the function $\phi_n(x)$ is continuous in E, and $f(x) = \lim_{n \sim \infty} \phi_n(x)$.

187. The following general theorem will be established with a view to its application in the theory of functions of class $\leqq 1$:

Let E *be a set of points in* S_p *which consists of the points which belong to one or more of the sets* $\{E_n\}$ *of a sequence of sets, any two of which may have points in common; and let* G *be a set of points contained in* E, *which is either* (1), *perfect, or* (2), *open, or* (3), *a set which consists of the points which a perfect and an open set have in common. Then the necessary and sufficient condition that* G *should be the sum of sets* $\phi_1, \phi_2, \dots \phi_n, \dots$, *no two of which have a point in common, and such that* ϕ_n *is contained in* E_n, *for each value of* n *for which* ϕ_n *exists, and such that every* ϕ_n *is of type* $O_E^{(2)}$, *or of type* $O_E^{(1)}$, *or*

$C_E^{(1)}$, *is that, for every perfect set* Q, *contained in* G, *one at least of the sets* E_n *is compact in* Q. (See I, p. 76.)

This theorem was given by de la Vallée Poussin* for case (1), in which G is perfect and bounded. It is not necessary that the set G should be bounded; it is sufficient that it be perfect or open in $\bar{S}_p$ or the set common to a perfect and an open set. But, by employing the mode of correlation referred to in § 181, it is seen that there is no loss of generality if the set E be contained in a finite closed cell.

It has been shewn in § 182 that a set of one of the types $O^{(2)}$, $O^{(1)}$, $C^{(1)}$, is also of one of the types $O_E^{(2)}$, $O_E^{(1)}$, $C_E^{(1)}$; and the converse holds good in case E is either closed, or open, or a set which consists of the points common to a closed and an open set (see § 184). If H be any set of points whatever, contained in E, the set H will be said to be *decomposable* if it can be expressed as the sum $\phi_1 + \phi_2 + \ldots + \phi_n + \ldots$, of sets which satisfy the conditions laid down in the statement of the theorem. The set H will be said to be *decomposable at a point* p if a closed neighbourhood Δ, of p, exists such that the set $D(H, \Delta)$ is decomposable.

In order to prove the theorem, a number of subsidiary theorems will be established:

(*a*) *If* H *is decomposable, and* H_1, *a set in* E, *is of one of the types* $O_E^{(1)}$, $C_E^{(1)}$, $O_E^{(2)}$, *then* $D(H, H_1)$ *is decomposable.*

For if $H = \sum_{n=1}^{\infty} \phi_n$, we have $D(H, H_1) = \sum_{n=1}^{\infty} D(\phi_n, H_1)$, and each set $D(\phi_n, H_1)$ is of type $O_E^{(2)}$, $O_E^{(1)}$, or $C_E^{(1)}$ (see § 182); therefore $D(H, H_1)$ is decomposable.

(*b*) *If* H *is the sum of a finite, or infinite, number of sets* H_n, *each of which is decomposable, and no two of which have a point in common, then* H *is decomposable. If a finite, or infinite, number of sets* H_n *are all closed and all decomposable, but may have points common to two or more of them, the set* $M(H_1, H_2, \ldots H_n, \ldots)$ *is decomposable.*

If $H = \Sigma H_m$, and $H_m = \phi_1^{(m)} + \phi_2^{(m)} + \ldots$, we have

$$H = \sum_{n=1} \left(\sum_{m=1} \phi_n^{(m)} \right);$$

thus H is decomposable, since $\sum_{m=1} \phi_n^{(m)}$ is of one of the types $O_E^{(2)}$, $O_E^{(1)}$, $C_E^{(1)}$ (see § 182).

To prove the second part of the theorem, let $H = M(H_1, H_2, \ldots)$, where $H_1, H_2, \ldots$ are all closed and decomposable; then the sets

$$E - H_1,\ E - M(H_1, H_2),\ E - M(H_1, H_2, H_3), \ldots$$

* *Intégrales de Lebesgue*, p. 108.

If each of the sets $E_1, E_2, \ldots E_n, \ldots$, is of type $O^{(2)}$, or $O^{(1)}$, or $C^{(1)}$, any set G, contained in E, which is either perfect, or open, or a set of points common to a perfect and an open set, is decomposable.

Each set E_n is the outer limiting set of a sequence of closed sets $F_{n1}, F_{n2}, \ldots$; thus E is equivalent to $M(F_{11}, F_{12}, F_{21}, F_{31}, F_{22}, F_{13}, \ldots)$. This set is compact in Q, and therefore one at least of the sets F_{nm} is compact in Q. If F_{nm} is compact in Q, so also is E_n; thus the condition of the general theorem is satisfied.

188. The following theorem, a generalisation of a theorem due to Lebesgue, will now be established:

The necessary and sufficient condition that a function $f(x)$, defined in a set E, of points in any number of dimensions, and which is either perfect, or open, or the set of points which an open and a perfect set have in common, should be of class $\leqq 1$, in E, is that, for every number A, the sets of points of E at which $f(x) > A$, and $f(x) < A$, should be of type $O_E^{(2)}$, or else of the first order in E.

It will be observed that the theorem includes the case in which E consists of the whole space R_p. It has been shewn in § 184, that a set of type $O_E^{(2)}$, or of the first order in E, is of type $O^{(2)}$ or else closed, or open. The necessity of the condition will first be proved. Let

$$f(x) = \lim_{n \sim \infty} f_n(x),$$

where the function $f_n(x)$ are all continuous in E; and let the sets of points at which $f_n(x) \geqq A + \epsilon_k$ be denoted by $E_n^{(k)}$, where $\{\epsilon_k\}$ is a monotone decreasing sequence of numbers converging to zero. Since $f_n(x)$ is continuous in E, the sets $E_n^{(k)}$ are closed in E. Let the set

$$D(E_n^{(k)}, E_{n+1}^{(k)}, \ldots),$$

which is closed in E, be denoted by $F_n^{(k)}$. A point at which $f(x) > A$ belongs, for some value of k, to all those of the sets $F_n^{(k)}$ for which $n \geqq n_k$, an integer dependent on k. At a point x, of $F_n^{(k)}$, we have $f_{n+m}(x) \geqq A + \epsilon_k$, for $m = 0, 1, 2, 3, \ldots$; and therefore $f(x) > A$.

The set $M(F_1^{(1)}, F_1^{(2)}, F_2^{(1)}, F_1^{(3)}, F_2^{(2)}, F_3^{(1)}, \ldots)$ is such that every point of it belongs to $F_n^{(k)}$, for some values of n and k; and therefore, at every point of the set, we have $f(x) > A$. Conversely, since every point at which $f(x) > A$ belongs to all the sets $F_n^{(k)}$ for suitable values of n and k, it is seen that the set consists of all points at which $f(x) > A$; and this set is of type $O_E^{(2)}$, unless it is of order 1 in E. The proof of the necessity of the part of the theorem relating to the set for which $f(x) < A$ is precisely similar.

In order to prove the sufficiency of the conditions in the theorem, the special case will, in the first instance, be considered, in which $f(x)$ has the value 1, in a part E_1, of E, and has the value O in $E - E_1$. In accordance with the condition in the theorem, E_1 and E_2 are taken to be each of one of the types $O_E^{(2)}$, $O_E^{(1)}$, $C_E^{(1)}$.

Let it be assumed that $E_1 = \lim_{n \sim \infty} G_n^{(1)}$, $E - E_1 = \lim_{n \sim \infty} G_n^{(2)}$, where $\{G_n^{(1)}\}$ and $\{G_n^{(2)}\}$ are sequences of sets, closed in E, each of which is contained in the next set of the sequence. Let $f_n(x) = 1$, in $G_n^{(1)}$; and let $f_n(x) = 0$, in $G_n^{(2)}$; at any point of E which does not belong to $G_n^{(1)}$ or $G_n^{(2)}$, let $f_n(x) = \dfrac{d_2}{d_1 + d_2}$, where d_1, d_2 are the distances of the point from the sets $G_n^{(1)}$, $G_n^{(2)}$. The function $f_n(x)$ is continuous in E, and $f(x) = \lim_{n \sim \infty} f_n(x)$; and therefore $f(x)$ is of class $\leqq 1$, in E. The sufficiency of the conditions has thus been established in the special case considered.

Next, let $f(x) = c_1$, in E_1; $f(x) = c_2$, in E_2; ... $f(x) = c_r$, in E_r; when $E = E_1 + E_2 + \ldots + E_r$; no two of the sets having a point in common.

It will be shewn that, if each of the sets $E_1, E_2, \ldots E_r$ is of type $O_E^{(2)}$, $O_E^{(1)}$, $C_E^{(1)}$, $f(x)$ is of class $\leqq 1$, in E. For, let $f^{(s)}(x)$ be the function defined by the specifications $f^{(s)}(x) = 1$, in E_s; $f^{(s)}(x) = 0$, in $E - E_s$; for $s = 1, 2, 3, \ldots r$. By what has been proved above, $f^{(s)}(x)$ is of class $\leqq 1$ in E; and since $f(x) = c_1 f^{(1)}(x) + c_2 f^{(2)}(x) + \ldots + c_r f^{(r)}(x)$, it follows that $f(x)$ is of class $\leqq 1$, in E.

In the general case, let U and L be the upper and lower boundaries of $f(x)$. It has already been pointed out that there is no loss of generality in taking U and L to have finite values. Let a mesh $(a_1, a_2, \ldots a_m)$ be fitted on to the linear interval (L, U), where $a_1 = L$, $a_m = U$, and let every mesh of the net have breadth $< \epsilon$. Let δ be any positive number $< \epsilon$, and let e_1 denote the set of points of E at which $a_1 - \delta < f(x) < a_2$; let e_2 denote the set at which $a_2 - \delta < f(x) < a_3$, ... and let e_m denote the set at which $a_m - \delta < f(x)$. The set e_1 consists of the points common to the two sets for which $f(x) > a_1 - \delta$, $f(x) < a_2$; therefore e_1 is of type $O_E^{(2)}$, unless it is of order 1, in E. Similarly it is seen that $e_2, e_3, \ldots e_m$ are all of type $O_E^{(2)}$, or else of order 1, in E. Since $E = M(e_1, e_2, \ldots e_m)$, we have $Q = M\{D(Q, e_1), D(Q, e_2), \ldots D(Q, e_m)\}$, where Q is any perfect set contained in E. The sets $D(Q, e_r)$ are all of type $O_Q^{(2)}$, or else of the first order in the perfect set Q, and are all consequently of type $O^{(2)}$, or else open or closed.

Since the sets $D(Q, e_1)$, $D(Q, e_2)$, ..., $D(Q, e_m)$ are all of type $O^{(2)}$, or of the first order, one at least of them must be compact in Q (see § 187 (*i*)). Thus the condition of the general theorem of § 187 is satisfied, in relation

to the sets $e_1, e_2, \dots e_m$. It follows that E may be resolved into a sum $\phi_1 + \phi_2 + \dots + \phi_m$, where ϕ_r is contained in e_r, and is of type $O^{(2)}$, or else of order 1.

Let $\phi(x)$ be defined by the specifications, $\phi(x) = a_1$, in ϕ_1; $\phi(x) = a_2$, in ϕ_2; ... $\phi(x) = a_m$, in ϕ_m. Then $\phi(x)$ is of class ≤ 1, in E; moreover $|\phi(x) - f(x)| < 2\epsilon$. Let ϵ have the values $\epsilon_1, \epsilon_2, \dots$, in a decreasing sequence which converges to zero; and let $\phi_r(x)$ be the value of $\phi(x)$ which corresponds to the value ϵ_r, of ϵ. If the sequence $\{\epsilon_r\}$ be so chosen that the series $\sum\limits_{r=1} \epsilon_r$ is convergent, $f(x)$ is the sum of the absolutely convergent series

$$\phi_1(x) + \{\phi_2(x) - \phi_1(x)\} + \{\phi_3(x) - \phi_2(x)\} + \dots.$$

Since the function $\phi_{r+1}(x) - \phi_r(x)$ is of class ≤ 1, in E, and takes only a finite number of values, it is the limit of a sequence $\chi_{rs}(x)$, of continuous functions; so that $\lim\limits_{s\sim\infty} \chi_{rs}(x) = \phi_{r+1}(x) - \phi_r(x)$. Since

$$|\phi_{r+1}(x) - \phi_r(x)| < 4\epsilon_r,$$

the sequence $\{\chi_{rs}(x)\}$ can be so chosen that $|\chi_{rs}(x)| < 4\epsilon_r$, for all values of s.

The continuous function $\chi_{1s}(x) + \chi_{2s}(x) + \dots + \chi_{ps}(x)$ is less, in absolute value, than $4(\epsilon_1 + \epsilon_2 + \dots + \epsilon_p)$; moreover

$$\lim_{s\sim\infty} \{\chi_{1s}(x) + \chi_{2s}(x) + \dots + \chi_{ps}(x)\} = \phi_{p+1}(x) - \phi_1(x).$$

Let s_p be the smallest value of s, such that

$$\chi_{1s}(x) + \chi_{2s}(x) + \dots + \chi_{ps}(x)$$

differs from $\phi_{p+1}(x) - \phi_1(x)$ by less than ϵ_{p+1}, and therefore from $f(x) - \phi_1(x)$ by less than $3\epsilon_{p+1}$. The number s_p can be determined for each value of p; we have then a sequence

$$\{\chi_{1s_p}(x) + \chi_{2s_p}(x) + \dots + \chi_{ps_p}(x)\}$$

of continuous functions, which converges to $f(x) - \phi_1(x)$. Therefore $f(x) - \phi_1(x)$ is of class ≤ 1, in E, and consequently $f(x)$ is of class ≤ 1, in E.

189. It will now be shewn that:

The necessary and sufficient condition that a function $f(x)$ defined in a set E, which is either perfect, or open, or the set of points common to a perfect and an open set, should be of class ≤ 1, in E, is that one at least of the two sets $[E, f(x) > A]$, $[E, f(x) < B]$ should be compact in every perfect set Q, contained in E, whatever values A and B may have, such that $A < B$.

To shew that the condition is necessary, let E_1, E_2 denote the two sets $[E, f(x) > A]$, $[E, f(x) < B]$; then E_1, E_2 are, in accordance with the theorem of § 188, each of type $O^{(2)}$, or else of the first order. Since E, or

$M(E_1, E_2)$, contains Q, and is therefore compact in Q; so also must be the set $M\{D(E_1, Q), D(E_2, Q)\}$. The two sets $D(E_1, Q)$, $D(E_2, Q)$ being both of type $O^{(2)}$, or else of order 1, cannot both be of the first category with respect to Q (see § 183), therefore one at least of them is compact in Q, and consequently one at least of the sets E_1, E_2 is compact in Q.

To prove that the condition is sufficient, let $\{B_n\}$ be a sequence of decreasing values of B which converges to A. Assuming that the condition of the theorem is satisfied for A and B_n, every point of E belongs to one at least of the sets, both of type $O_E^{(2)}$, or else of the first order, for which $f(x) > A$, and $f(x) < B_n$. In accordance with the general theorem of § 187 if the condition of the present theorem be satisfied, since every point of E belongs to one at least of the two sets $[E, f(x) > A]$, $[E, f(x) < B_n]$, E can be resolved into the sum of two sets X_n, Y_n, both of type $O^{(2)}$, or else of the first order; where X_n is contained in the set $[E, f(x) > A]$, and Y_n is contained in the set $[E, f(x) < B_n]$. Each point for which $f(x) > A$ belongs to X_n, from and after some particular value of n, and consequently every point for which $f(x) > A$ is contained in the set $M(X_1, X_2, \ldots X_n, \ldots)$. Therefore the set $[E, f(x) > A]$ is either of type $O^{(2)}$ or of the first order. Similarly, it may be shewn that the set $[E, f(x) < A]$ is either of type $O^{(2)}$ or of the first order. Therefore, by the theorem of § 188, $f(x)$ is of class $\leqq 1$.

190. We are now in a position to establish, in a generalized form, the theorem of Baire, referred to in § 185.

The necessary and sufficient condition that a function $f(x)$, defined in a set E, in any number of dimensions, which is either perfect, or open, or the set of points common to a perfect and an open set, is the limit of a sequence of functions, all of which are continuous in E, is that $f(x)$ be, at most, pointwise discontinuous with respect to every perfect set contained in E.

To prove the necessity of the condition, it will be shewn that, if a perfect set Q, contained in E, is such that $f(x)$ is neither pointwise discontinuous nor continuous, with respect to Q, then $f(x)$ cannot be of class $\leqq 1$, in E.

The set of points of Q at which the saltus of $f(x)$, with respect to Q, is $\geqq \epsilon$, cannot, for every value of ϵ, be non-dense in Q; otherwise $f(x)$ would be pointwise discontinuous, or continuous, in Q. Therefore ϵ can be so chosen that the set is compact in Q; and consequently a cell Δ exists, such that, at every point of the perfect set $D(Q, \Delta)$, the saltus of $f(x)$, with respect to Q, is $\geqq \epsilon$. If A and B are any two numbers such that

$$0 < B - A < \epsilon,$$

the set of points $[E, A < f(x) < B]$ cannot be compact in $D(Q, \Delta)$, for otherwise the saltus of $f(x)$, with respect to Q, in $D(Q, \Delta)$ could not exceed $B - A$. A set of intervals (A_1, B_1), (A_2, B_2), ... (A_r, B_r) each of which is

of measure $< \epsilon$, and such that any two of them may overlap, can be so determined that every value of $f(x)$ lies within one or more of the intervals. It is here assumed that $f(x)$ is bounded; it having been shewn in § 185 that this involves no real restriction upon the generality of the theorem. Let e_s denote the set $[E, A_s < f(x) < B_s]$, for $s = 1, 2, 3, \ldots r$; then $E = M(e_1, e_2, \ldots e_r)$. If all the sets $e_1, e_2, \ldots e_r$ were either of type $O^{(2)}$, or of order 1, one at least of them would be compact in $D(Q, \Delta)$; as this is not the case, there must be at least one set e_s which is neither of type $O^{(2)}$, nor of order 1. Consequently, from the theorem of § 189, $f(x)$ cannot be of class ≤ 1, in E; since the sets $[E, f(x) > A_s]$, $[E, f(x) < B_s]$ are not both of type $O^{(2)}$ or of the first order.

To prove that the condition in the theorem is sufficient, it will be shewn that if $f(x)$ is not of class ≤ 1, there is contained in E a perfect set Q, with respect to which $f(x)$ is neither pointwise discontinuous nor continuous. If no such set Q exists, it will then follow that $f(x)$ is of class ≤ 1, in E.

It follows from the theorem of § 189 that, if $f(x)$ is not of class ≤ 1, in E, a perfect set Q, and two numbers A, B, where $A < B$, exist, such that neither of the sets $[E, f(x) > A]$, $[E, f(x) < B]$ is compact in Q. If q be any point of Q, an interval or cell Δ, containing q within it, can be so determined that there exist in Δ points of Q which do not belong to the set $[E, f(x) > A]$, and also points of Q which do not belong to the set $[E, f(x) < B]$. There are therefore, in Δ, points at which $f(x) \leq A$, and also points at which $f(x) \geq B$. Since Δ is an arbitrary neighbourhood of q, it follows that the saltus of $f(x)$, at q, with respect to Q, is $\geq B - A$. Therefore every point q, of Q, is a point of discontinuity of $f(x)$; and thus $f(x)$ is totally discontinuous in Q. The sufficiency of the condition in the theorem has accordingly been established.

It may be observed that, when the function $f(x)$ satisfies the condition of the theorem, it follows from the results of § 161, that $f(x)$ may be exhibited as the sum of a series of finite polynomials, the series converging to $f(x)$ at all points of E.

THE CONVERGENCE OF MONOTONE SEQUENCES OF FUNCTIONS

191. A special case of the convergence of a sequence of functions continuous in a given set E, which we may take to be perfect, or open, or the set common to a perfect and an open set, is that which arises when the sequence is monotone. In such a case the limit of the sequence is an l-function, or a u-function, according as the sequence is non-diminishing or non-increasing (see § 103).

The following theorem has reference to this case:

The necessary and sufficient condition that a function $f(x)$, defined in a

set E, open, or perfect, or consisting of the points which an open and a perfect set have in common, is an l-function in E is that, for every value of A, the set $[E, f(x) > A]$ which consists of those points of E at which $f(x) > A$, is open in E. The necessary and sufficient condition that $f(x)$ is a u-function is that the set $[E, f(x) < A]$ is, for every value of A, open in E.

If $[E, f(x) > A]$ is open in E then $[E, f(x) \leqq A]$ is closed in E. It is sufficient to establish the first part of the theorem, as the second part can be deduced by changing the signs of $f(x)$ and of A.

Let ξ be a limiting point, in E, of the set $[E, f(x) \leqq A]$; then, if ϵ be an arbitrary positive number, there is a neighbourhood of ξ, such that at every point of E, in that neighbourhood, $f(x) > f(\xi) - \epsilon$. Since x may be a point of the set $[E, f(x) \leqq A]$, we have $f(\xi) < \epsilon + A$; and since ϵ is arbitrary, it follows that $f(\xi) \leqq A$; thus ξ belongs to the set $[E, f(x) \leqq A]$, which is therefore closed in E; and consequently the complementary set $[E, f(x) > A]$ is open in E. The necessity of the condition has thus been established.

To prove its sufficiency, let it be assumed that the set $[E, f(x) \leqq A]$ is closed in E, for every value of A. If, at a point ξ, of E, the function is not lower semi-continuous in E, there exists a positive number α such that, in every neighbourhood of ξ, there are points of E at which $f(x) \leqq f(\xi) - \alpha$, and ξ must be a limiting point of the set of all such points. Therefore the set $[E, f(x) \leqq f(\xi) - \alpha]$ is not closed in E, which is contrary to the hypothesis. Accordingly $f(x)$ is lower semi-continuous with respect to E, at every point of E.

192. We proceed to give the corresponding theorems for *lu*-functions and for *ul*-functions.

The necessary and sufficient condition that, in a set E, of the same character as before, the function $f(x)$ is an lu-function is that the set $[E, f(x) > A]$ should be a set of type $O_E^{(2)}$, for every value of A. The necessary and sufficient condition that $f(x)$ should be a ul-function in E is that the set $[E, f(x) < A]$ should be, for every value of A, of type $O_E^{(2)}$.

It is sufficient to prove the first part of this theorem, as the second part is immediately deducible from the first. To prove the necessity of the theorem, since an *lu*-function $f(x)$ is the limit of a monotone increasing sequence of *u*-functions $f_n(x)$, the set of points $[E, f(x) \leqq A]$ is the inner limiting set of the sequence of sets $[E, f_n(x) < A]$, for $n = 1, 2, 3, \ldots$, and all these sets are of the type $O_E^{(1)}$. Therefore the set $[E, f(x) \leqq A]$ is of type $C_E^{(2)}$, and consequently the set $[E, f(x) > A]$ is of type $O_E^{(2)}$.

To prove the sufficiency of the condition, let U and L denote the upper and lower boundaries of $f(x)$ in E. Let a system of nets be fitted on to the linear interval $(L - \epsilon, U + \epsilon)$; and let $a_0, a_1, a_2, \ldots a_{n_m}$ be the end-

points of the meshes of the net D_n, of the system. Let the function $f_n(x)$ be defined by the rule that $f_n(x) = a_{r-1}$ at all points of the set $[E, a_{r-1} < f(x) \leq a_r]$, for $r = 1, 2, 3, \ldots, n_m$; then the monotone sequence $\{f_n(x)\}$ converges uniformly in E to $f(x)$. The function $f_n(x)$ may be expressed in the form $f_{n0}(x) + f_{n1}(x) + \ldots + f_{n, n_m-1}(x)$, where $f_{n0}(x) = a_0$, in E; $f_{n1}(x) = a_1 - a_0$, in $[E, f(x) > a_1]$, $f_{n1}(x) = 0$ in $[E, f(x) \leq a_1]$; and generally $f_{nr}(x) = a_r - a_{r-1}$, in $[E, f(x) > a_r]$, and $f_{nr}(x) = 0$, in $[E, f(x) \leq a_r]$. By hypothesis the set $[E, f(x) > a_r]$ is of type $O_E^{(2)}$, and is therefore the outer limiting set of a sequence of sets all of which are of type $C_E^{(1)}$; it follows that $f_{nr}(x)$ is the limit of a monotone increasing sequence of functions $\{\phi_m(x)\}$ each of which has the value $a_r - a_{r-1}$ in a set of type $C_E^{(1)}$ contained in the set $[E, f(x) > a_r]$; and each of the sets $\phi_m(x)$ is a u-function. It follows that the function $f_{nr}(x)$ is, for each value of r, an lu-function. Therefore $f_n(x)$ is also an lu-function. Hence, since $\{f_n(x)\}$ converges uniformly to $f(x)$, $f(x)$ is also an lu-function (see § 113).

By the theorem of § 188, if $f(x)$ is of class ≤ 1, both the sets $[E, f(x) > A]$, $[E, f(x) < A]$ are of type $O_E^{(2)}$, or else of the first order. It thus appears from the above theorem that *a function of class* ≤ 1, *in the set* E, *which is open, or perfect, or the set common to an open or perfect set, is both a ul-function and an lu-function, unless it be a u-function or an l-function.*

BAIRE'S CLASSIFICATION OF FUNCTIONS

193. A classification of functions was introduced by Baire*, depending upon the properties of the functions in relation to their representation as limits of sequences of functions. In § 185, functions continuous relative to a given domain E were defined to be of class 0, in E; and any function which is the limit of a sequence of functions of class 0, in E, was defined to be of class 1, in case it is not of class 0, in E.

Functions of class 2 can similarly be defined as functions which are, in E, the limits of sequences of functions of class < 2, provided they are not themselves of class < 2, in E. It can be shewn, by means of an example, that functions of class 2 exist. Consider the function $f(x)$, in a continuous linear interval, which has the value 1, for all rational values of x, and the value 0 for every irrational value of x. This function is totally discontinuous, and is therefore not of class 0, or of class 1, but it can be seen to be the limit of a sequence of functions, all of which are of class 1. Let $f_n(x)$ be defined as having the value 1 at every point at which the value of x is rational and has for its denominator, when expressed in its lowest terms, an integer not exceeding n, and let $f_n(x)$ have the value 0 at all other points; this function has then only a finite number of discontinuities in any given

* *Comptes Rendus*, vol. CXXIX (1899), p. 1010 and *Annali di Mat.* (3) A, vol. III (1899); also Baire's treatise, *Leçons sur les fonctions discontinues*.

interval, and therefore belongs to class 1. The function $f(x)$ is the limit of the sequence $\{f_n(x)\}$, and is of class 2.

It is capable of the analytical representation

$$f(x) = \lim_{m \sim \infty} \lim_{n \sim \infty} (\cos m!\, \pi x)^{2n}.$$

A function which is of class 2, in a given cell or interval, can be represented by a double series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{m,n}(x)$, where $P_{m,n}(x)$ denotes a finite polynomial. This double series cannot be reduced to a single one, the terms of which are continuous, for the function would then not be of class 2.

The definition can be extended by induction so as to apply to a function of class p, in the domain E, where p is a finite ordinal. A function is of class p, in E, if it is the limit of a sequence of functions all of which are of class $< p$, provided it is not itself of class $< p$. The definition can still further be extended, by transfinite induction, to apply to a function of class β, where β is an ordinal number of the second class. A function is of class β when it is the limit of a sequence of functions, all of which are of class $< \beta$, provided it be not itself of class $< \beta$.

A proof has been given by Lebesgue* that functions of class γ exist, where γ is an ordinal number of the first, or of the second, class. A simpler form of this proof has been given by de la Vallée Poussin†. Baire's classification of functions is of importance in relation to the question as to the characteristics of a function which is representable analytically. A function that can be constructed by carrying out, according to a norm, a finite, or enumerable, set of additions, multiplications, and of passages to the limit, operating with variables and constants, may be said to be representable analytically. The other operations employed in Analysis are reducible to those here enumerated. This definition will include cases in which the function is multiple valued. It has been shewn by Lebesgue that every single-valued function, that is representable analytically, in a cell, or interval, is not only measurable, but measurable (B), in the sense that the set of points at which the function exceeds in value an arbitrarily prescribed number is measurable (B). Lebesgue has further shewn that every such function belongs to one of Baire's classes‡.

It can be shewn that the totality of functions of all classes, in a given domain, has the cardinal number of the continuum. This can be proved by induction and transfinite induction. Let it be assumed to be true for all the functions of class $< \gamma$, where γ is a number of the first or second class. If $f_n(x)$ be a function of class $\leq \gamma$, we have

$$f(x) = \lim_{n \sim \infty} f_n(x),$$

* *Liouville's Journal* (6), vol. I (1905), p. 139. † *Intégrales de Lebesgue* (1916), pp. 145–151

‡ *Loc. cit.* p. 170.

where the functions $f_n(x)$ are all of class $< \gamma$. Each such function $f(x)$ corresponds to an association of the integers n with a particular number of the aggregate of functions of class $\leqq \beta$, the totality of which has, by hypothesis, the cardinal number c; it follows that the aggregate of all such functions $f(x)$ has the cardinal number $c^{\aleph_0}$, or c (see I, § 183). The aggregate of functions of class 0 has cardinal number c (see I, § 216); hence it follows, by finite and transfinite induction, that the aggregate of all the functions of class $\leqq \gamma$ has the cardinal number c.

It has been shewn, in I, § 216, that the aggregate of all functions has its cardinal number $> c$; it was accordingly affirmed by Baire that functions exist which do not belong to any class, either of finite, or of transfinite order. Lebesgue has shewn* how to define effectively functions which do not belong to any class, and which are consequently not representable analytically. It should be observed that, in the whole theory of functions of Baire's classes, all the functions may be taken to be bounded: as this entails no real loss of generality.

194. The following is the generalization of the theorem given in § 186, relating to functions of class $\leqq 1$:

If the functions $f_1(x), f_2(x), \dots f_r(x)$ are all of class $\leqq \gamma$, in E, and the function $F(f_1, f_2, \dots f_r)$ is continuous with respect to $(f_1, f_2, \dots f_r)$, in E, then $F(f_1, f_2, \dots f_r)$ is of class $\leqq \gamma$, in E.

It will be shewn that, if the theorem holds for all ordinal numbers less than γ, it holds also for γ. We have $f_s(x) = \lim_{n\sim\infty} f_{sn}(x)$, for $s = 1, 2, 3, \dots r$, where all the functions $f_{sn}(x)$ are of class $< \gamma$. On account of the continuity of the function F we have $F = \lim_{n\sim\infty} F(f_{1n}, f_{2n}, \dots f_{rn})$, hence F must be of class $\leqq \gamma$, in E. The theorem has already been proved for the case $\gamma = 1$, hence, by ordinary and transfinite induction, it holds for every number of the first, or the second, class.

As in § 186, the following results follow from the above theorem:

(1) *The sum, or the difference, or the product, of two functions, each of which is of class $\leqq \gamma$, in E, is also of class $\leqq \gamma$, in E.*

(2) *If $f(x)$ is of class $\leqq \gamma$, in E, so also is $|f(x)|$.*

(3) *If $f_1(x), f_2(x), \dots f_r(x)$ are all of class $\leqq \gamma$, in E, and $\phi(x)$ be the function which has, at each point, the value of the greatest of the given functions, then $\phi(x)$ is of class $\leqq \gamma$, in E.*

(4) *If $f(x)$ is of class γ, in E, the function $\phi(x)$ which has the value $f(x)$ when $A < f(x) < B$, and has the value A when $f(x) \leqq A$, and the value B when $f(x) \geqq B$, is of class $\leqq \gamma$, in E.*

* *Loc. cit.* pp. 213–216. An example of such a function, defined without the use of transfinite numbers, has been given by Sierpinski, *Fundamenta Mat.* vol. v (1924), p. 87.

(5) *If the function $f(x)$, of class γ, in E, is such that $L \leqq f(x) \leqq U$, in E, then $f(x)$ is the limiting function of a sequence $\{\phi_n(x)\}$, of functions of class $< \gamma$, in E, and such that $L \leqq \phi_n(x) \leqq U$, for every value of n.*

195. The following theorem will be established:

If a sequence $\{f_n(x)\}$ converges uniformly to $f(x)$, in E, and all the functions are of class $\leqq \gamma$, then $f(x)$ is of class $\leqq \gamma$, in E.

A monotone increasing sequence $\{n_r\}$ of integers exists, such that $|f(x) - f_{n_r}(x)| < \frac{1}{2^r}$, for $r = 1, 2, 3, \ldots$, and for all points of E. Let $f_{n_1}(x) = \lim\limits_{m \sim \infty} \phi_{0,m}(x)$, and let $f_{n_{r+1}}(x) - f_{n_r}(x) = \lim\limits_{m \sim \infty} \phi_{r,m}(x)$, for $r = 1, 2, 3, \ldots$; where $\phi_{0,m}(x)$ and all the functions $\phi_{r,m}(x)$ are of class $< \gamma$. We have $|f_{n_{r+1}}(x) - f_{n_r}(x)| < \frac{1}{2^r} + \frac{1}{2^{r+1}} < \frac{1}{2^{r-1}}$; therefore, in accordance with theorem (5), we may take $|\phi_{r,m}(x)| \leqq \frac{1}{2^{r-1}}$, for all values of r (> 1) and m.

$$\begin{aligned} \text{We have } & |f(x) - \phi_{0,m}(x) - \phi_{1,m}(x) - \ldots - \phi_{m,m}(x)| \\ & \leqq |f(x) - \{\phi_{0,m}(x) + \phi_{1,m}(x) + \ldots + \phi_{r-1,m}(x)\}| \\ & + |\phi_{m+1,m}(x) + \phi_{m+2,m}(x) + \ldots + \phi_{r-1,m}(x)|, \end{aligned}$$

where $r > m$. The expression on the right-hand side is

$$\leqq |f(x) - f_{n_r}(x)| + \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots,$$

and this is $< \frac{1}{2^r} + \frac{1}{2^{m-1}}$, or $< \frac{1}{2^{m-2}}$. We thus have

$$f(x) = \lim_{m \sim \infty} \{\phi_{0,m}(x) + \phi_{1,m}(x) + \ldots + \phi_{m,m}(x)\};$$

and it then follows that $f(x)$ is of class $\leqq \gamma$.

196. The theory of Baire's classes is closely connected with that of certain species of sets of points obtained by a generalization of the sets considered in § 111. The definition given in § 182, of sets of points, of orders, 1 and 2, contained in a given set E, may be extended in such a manner that sets of points, of order γ, are introduced, where γ is any ordinal number, of the first, or the second, class. Starting with the definitions of sets of types $C_E^{(1)}, C_E^{(2)}, O_E^{(1)}, O_E^{(2)}$, we define a set of types $O_E^{(\gamma)}, C_E^{(\gamma)}$, as follows:

The outer limiting set of a sequence $\{E_n\}$ of sets contained in E, such that each set of the sequence is contained in the next, and such that all of them are of type $O_E^{(\gamma')}$ or of type $C_E^{(\gamma')}$, where γ' is any number $< \gamma$, is said to be of type $O_E^{(\gamma)}$, in case it is not of any type $O_E^{(\gamma')}$ or $C_E^{(\gamma')}$. It is here assumed that the definition has already been given for types of order $< \gamma$, and the method of induction is then employed.

A set of type $C_E^{(\gamma)}$ is defined in a similar manner, except that it is taken to be the inner limiting set of a sequence $\{E_n\}$, each set of which contains the next.

A set of order γ, in E, is a set which is of either of the types $O_E^{(\gamma)}$, $C_E^{(\gamma)}$. The whole class of sets of points of any order are known as Borel-sets in E; their properties have been investigated by Hausdorff*.

The following generalizations of theorems given in § 182 are of importance:

A sequence $\{E_n\}$ of sets of order $< \gamma$, each of which is contained in the next, has for its outer limiting set a set of order $< \gamma$, provided an infinite number of the sets E_n are of any type $O_E^{(\gamma')}$, where $\gamma' < \gamma$.

A sequence $\{E_n\}$ of sets of order $< \gamma$, each of which contains the next, has for its inner limiting set a set of order $< \gamma$, provided an infinite number of the sets E_n are of any type $C_E^{(\gamma')}$, where $\gamma' < \gamma$.

Thus a set of type $O_E^{(\gamma)}$ may always be generated as the outer limiting set of a sequence, all of whose members are of some type $C_E^{(\gamma')}$, where $\gamma' < \gamma$; and a set of type $C_E^{(\gamma)}$ may always be generated as the inner limiting set of a sequence, all of whose members are of some type $O_E^{(\gamma')}$, where $\gamma' < \gamma$.

The theorems have already been established for the cases $\gamma = 2$; and it may readily be proved by induction that the outer limiting set of a sequence of sets, each of which is of some type $O_E^{(\gamma')}$, where $\gamma' < \gamma$, is of type $< \gamma$; the first theorem then follows immediately. The second theorem can be proved in a similar manner.

By induction, starting with the theorems given in § 182, we have the following theorems:

If a finite number of sets $H^{(1)}, H^{(2)}, \ldots H^{(r)}$ are all of types $O_E^{(\gamma')}$, where $\gamma' \leq \gamma$, the set $D(H^{(1)}, H^{(2)}, \ldots H^{(r)})$, of points common to all the r sets, is of type $O_E^{(\gamma')}$, where $\gamma' \leq \gamma$.

If $H^{(1)}, H^{(2)}, \ldots H^{(n)}, \ldots$, be a sequence of sets in E, all of some type $O_E^{(\gamma')}$, where $\gamma' \leq \gamma$, the set $M(H^{(1)}, H^{(2)}, \ldots H^{(m)}, \ldots)$, of points which belong to one or more of the given sets, is also of type $O_E^{(\gamma')}$, where $\gamma' \leq \gamma$.

It has been proved by Hausdorff† that if E be closed, a Borel-set of any order, in E, has the cardinal number C, unless the Borel-set be enumerable.

When the set E consists of the absolute set $\bar{S}_p$, the types $O_E^{(\gamma)}$, $C_E^{(\gamma)}$ may be denoted simply by $O^{(\gamma)}$, $C^{(\gamma)}$.

* *Math. Annalen*, vol. LXXVII (1916), p. 430; also *Math. Zeitschr.* vol. V (1919), p. 307, where the theory is connected with that of monotone sequences. See also W. H. Young, *Proc. Lond. Math. Soc* (2) vol. XII (1912), p. 260.

† *Math. Annalen*, vol. LXXVII (1916), p. 433.

The following are generalizations of theorems given in § 182:

The complement, with respect to E, of a set of type $C_E^{(\gamma)}$ is of type $O_E^{(\gamma)}$; and the converse.

This theorem has been proved in § 182, for the case $\gamma = 2$; and it may be deduced by finite and transfinite deduction. Assume that the theorem holds for every number $\gamma' < \gamma$. A set of type $O_E^{(\gamma)}$ is the outer limiting set of a sequence of sets, all of order $< \gamma$, in E. The complement in E of the given set is accordingly the inner limiting set of a sequence of sets, all of which are of order $< \gamma$; therefore this complement is of type $C_E^{(\gamma)}$, or else of order $< \gamma$. It cannot be of order $< \gamma$, for if it were so, its complement, the given set, would be, by hypothesis, of order $< \gamma$. Hence the complementary set is of type $C_E^{(\gamma)}$.

The part of a set of type $O_E^{(\gamma)}$ which is in E_1, a part of E, is of type $O_{E_1}^{(\gamma)}$, or else of order $< \gamma$, in E_1. In particular, the part of a set of type $O^{(\gamma)}$ which is in any set E_1, is of type $O_{E_1}^{(\gamma)}$, or else of order $< \gamma$, in E_1.

The corresponding result holds for a set of type $C_E^{(\gamma)}$.

This may be proved by induction, commencing with the case $\gamma = 2$, for which the theorem has been proved in § 182. Let it be assumed that the theorem is true for all numbers $\gamma' < \gamma$.

The given set is the outer limiting set of a sequence of sets, all of order $< \gamma$; the part of each of these that is in E_1 is of order $< \gamma$. Therefore the part of the given set that is in E_1 is the outer limiting set of a sequence of sets, all of which are of order $< \gamma$; from which the result follows. Since the theorem holds for $\gamma = 2$, it holds generally. The corresponding theorem for a set of type $C_E^{(2)}$ follows from the fact that the complement, with respect to E, of such a set, is of type $O_E^{(2)}$.

If E consist of a perfect set, or an open set, or of the points which an open and a closed set have in common, a set of type $O_E^{(\gamma)}$ is of type $O^{(\gamma)}$; and a set of type $C_E^{(\gamma)}$ is of type $C^{(\gamma)}$.

The theorem has already been proved in § 184, for the case $\gamma = 2$. Let it be assumed to be true for every number $\gamma' < \gamma$. A set of type $O_E^{(\gamma)}$ is the outer limiting set of a sequence of sets, all of which are of type $< \gamma$, in E. By hypothesis these are all of type $< \gamma$, in $\bar{S}_p$. It follows that the given set is of type $O^{(\gamma)}$, or else of order less than γ. The latter cannot be the case, for if the set were of order less than γ, in $\bar{S}_p$, it would be of order $< \gamma$, in E. Therefore the given set is of type $O^{(\gamma)}$. Since the theorem holds for $\gamma = 2$, it holds generally.

The necessary and sufficient conditions that a function defined in a given set of points should be of class γ, where γ is any number of the first and second class, have been obtained as generalizations of the theorems

of Lebesgue and of Baire, given in §§ 188–191, which correspond to the case $\gamma = 1$. For the investigation of these generalizations, reference must be made to the memoir of Lebesgue*, and the treatise of de la Vallée Poussin†. A full treatment of the matter has been given by Hahn‡, who has freed Lebesgue's theorem, but not that of Baire, from restrictions on the nature of the set in which the function is defined.

PROPERTY OF A MEASURABLE FUNCTION

197. It will be shewn that, if $f(x)$ be a measurable function, defined in a measurable set E, there exists a function, of class $\leqq 2$, which has the same value of $f(x)$ almost everywhere in E. This theorem was given by Vitali§, for the case in which E is a closed linear interval. It follows from the theorem that a function of any of Baire's classes differs from some function of class $\leqq 2$, only at the points of a set of measure zero. It thus appears that, in the processes of Analysis, in which L-integrals are employed, a function of any class may be replaced by a function of class not greater than two; hence Baire's general classification of functions, although of much theoretical interest, is of somewhat less importance in general Analysis than might have been anticipated. The same conclusion might be drawn from the results in §§ 178, 179.

Let a measurable function $f(x)$ be defined in the closed cell or interval Δ; we may, without loss of generality, assume $f(x)$ to be bounded. In accordance with the theorem in § 179, there exists a sequence $\{G_n\}$, of perfect sets, each one of which is contained in the next, and for which $\lim_{n \sim \infty} m(G_n) = m(\Delta)$, such that $f(x)$ is continuous relatively to each of the sets G_n.

Let $\phi_n(x)$ be, for each value of n, the function which has the value of $f(x)$ at each point of G_n, and the value zero at each point of $\Delta - G_n$. It will be shewn that $\phi_n(x)$ is pointwise discontinuous with respect to every perfect set H, contained in Δ, and is therefore of class $\leqq 1$, in Δ.

Let Δ_1 be a cell, or interval, containing points of H; if the points of H within Δ_1 are all points of G_n, these points are all points in which $\phi_n(x)$ is continuous with respect to H. If, however, there is a point of $\Delta - G_n$ within Δ_1, which belongs to H, a neighbourhood Δ_2 of that point can be determined, which is interior to Δ_1, and which contains no points of G_n, since $\Delta - G_n$ is open relatively to Δ; all the points of H that are in Δ_2 are points of $\Delta - G_n$, and thus points of continuity of $\phi_n(x)$; hence they are also points of continuity of $\phi_n(x)$ relatively to H. It has thus been shewn that the set of points of continuity of $\phi_n(x)$ relatively to H is everywhere

* *Liouville's Journal* (6), vol. I (1905). † *Intégrales de Lebesgue* (1916), pp. 126–151.

‡ *Theorie der reellen Funktionen*, vol. I (1921), pp. 318–392.

§ *Rendi. Lombardo* (2), vol. XXXVIII (1905), p. 599.

dense in H; and thus that $\phi_n(x)$ is at most pointwise discontinuous with respect to H. From the theorem of § 190, it follows that $\phi_n(x)$ is of class $\leqq 1$, in Δ.

If K be the outer limiting set of $\{G_n\}$, we have $m(K) = m(\Delta)$, and in K we have $\lim_{n\sim\infty} \phi_n(x) = f(x)$, whereas in $\Delta - K$ we have $\lim_{n\sim\infty} \phi_n(x) = 0$. If we denote by $\psi(x)$ the function defined, in Δ, as $\lim_{n\sim\infty} \phi_n(x)$, $f(x) - \psi(x)$ differs from zero only at points belonging to the set $\Delta - K$, of measure zero; moreover $\psi(x)$ is of class $\leqq 2$.

Next, let the measurable function $f(x)$ be defined everywhere in the space S_p; and let $\{\Delta_m\}$ be a sequence of cells, or intervals, each of which is contained in the next, and such that the span of Δ_m increases indefinitely with m. Let $\{\epsilon_m\}$ denote a sequence of positive numbers converging to zero; and let G_m be a perfect set contained in Δ_m, and such that

$$m(\Delta_m) - m(G_m) < \epsilon_m;$$

and this for each value of m. It is easily seen that the sequence $\{G_m\}$ may be so chosen that each set is contained in the next. For if G_2 does not contain G_1, we can replace G_2 by $G_2 + D(G_1, \Delta_2 - G_2)$, which is also a perfect set. To see this we observe that every point of it which belongs to G_2 is a limiting point of the set, and every point which belongs to $D(G_1, \Delta_2 - G_2)$ is the limit of a sequence of points of G_1, all of which belong either to G_2 or to $D(G_1, \Delta_2 - G_2)$, and thus the set is dense in itself. In any neighbourhood of a limiting point of the set there is an infinite set of points of G_2, or an infinite set of points of G_1 that do not belong to G_2; thus the point must belong either to G_1 or to G_2, and thus belongs to $G_2 + D(G_1, \Delta_2 - G_2)$; therefore the set is closed. Since it is dense in itself, and closed, it is perfect. We can proceed in a similar manner to ensure that G_2 is contained in G_3, and so on.

Let $\phi_m(x) = f(x)$, in G_m, and $\phi_m(x) = 0$, in the rest of the space S_p; then $\phi_m(x)$ is of class $\leqq 1$, in Δ_m, and therefore in S_p. The sequence $\{\phi_m(x)\}$ converges in Δ_m, almost everywhere to $f(x)$; for if we consider the sequence $\phi_m(x), \phi_{m+1}(x), \dots \phi_{m+r}(x), \dots$, the function $\phi_{m+r}(x)$ is continuous relatively to G_{m+r}, and therefore relatively to G_m. Moreover we have $m(\Delta_{m+r}) - m(G_{m+r}) < \epsilon_{m+r}$; hence if g_{m+r} denote the perfect part of G_{m+r} that is contained in Δ_m, we have

$$m(\Delta_m) - m(g_{m+r}) < \epsilon_{m+r}.$$

Thus $G_m, g_{m+1}, g_{m+2}, \dots$, is a sequence of perfect sets, all contained in Δ_m, each one contained in the next, and such that $\lim_{r\sim\infty} m(g_{m+r}) = m(\Delta_m)$; it follows that, in Δ_m, the sequence $\phi_m(x), \phi_{m+1}(x), \dots \phi_{m+r}(x), \dots$, converges almost everywhere to $f(x)$. Since m may have all integral values, it follows that the sequence $\{\phi_n(x)\}$ converges almost everywhere in S_p to the value of $f(x)$, and in the remaining points it converges to zero.

In case the function $f(x)$ is defined in a measurable set E, of finite, or of infinite, measure, we may suppose $f(x)$ to be extended to the whole space S_p, by assuming that its value is zero at all points of the complement of E; it remains a measurable function when so extended. The sequence $\{\phi_n(x)\}$ corresponding to this extended function may then be applied only to E.

We have thus established the following theorem:

If $f(x)$ be a measurable function, defined in a measurable set E, of finite, or of infinite, measure, and in any number of dimensions, there exists in E a function $\phi(x)$, of class ≤ 2, such that its value differs from that of $f(x)$ only at points of E belonging to a set of measure zero.

In comparing this theorem with that given in § 179, we observe that, in the present case $f(x)$ is representable almost everywhere by a sequence of functions, of class ≤ 1, which is convergent everywhere in E, without exception, although it may not converge to $f(x)$ everywhere in E; whereas in the theorem of § 179, $f(x)$ is represented almost everywhere in E as the limit of a sequence of functions of class 0, but this sequence is not necessarily convergent everywhere, without exception, in E.

THE PRIMITIVES OF A FUNCTION IN A FINITE INTERVAL

198. If a measurable function $f(x)$ be defined in a given linear interval (a, b), infinite values being admitted, the question arises whether a continuous function $F(x)$ exists which has, almost everywhere in (a, b), a differential coefficient of which the value is $f(x)$. Such a function $F(x)$ is said to be, in a more general sense than that employed in I, § 343, a primitive of $f(x)$. It has been shewn in I, § 298, that the function $F(x)$ cannot have an infinite differential coefficient at all points of a set which has its measure > 0; accordingly, the question can only admit of an affirmative answer in case $f(x)$ is almost everywhere finite, and it will therefore be assumed that this is the case. The answer to the question, with this restriction, is contained in a theorem due to Lusin*, which may be stated as follows:

If a function $f(x)$ be measurable in the finite linear interval (a, b), and be finite almost everywhere, there exists a continuous function $F(x)$, such that, almost everywhere in (a, b), $F'(x)$ exists, and has the value $f(x)$. The function $F(x)$ is, in general, not unique; two values of $F(x)$ not, in general, differing from one another by a constant.

In case the function $f(x)$ has an L-integral, or a D-integral, in (a, b), $\int_a^x f(x)\,dx$ is a primitive which has, almost everywhere in (a, b), a differential coefficient equal to $f(x)$ (see I, § 470); thus $\int_a^x f(x)\,dx + C$ belongs to the

* *Comptes Rendus*, vol. CLXII (1916), p. 975.

class of primitives of $f(x)$. Further, if $f(x)$ be everywhere finite in (a, b), and it is known that $f(x)$ is a differential coefficient, then (I, § 471) $f(x)$ has a D-integral in (a, b). Moreover, the only primitives which have a differential coefficient equal to $f(x)$, everywhere in (a, b), with the possible exception of an enumerable set of points, are the primitives

$$\int_a^x f(x)\,dx + C;$$

this is the case in virtue of the theorem given in I, § 267.

199. In order to prove Lusin's theorem, two lemmas will be required:

Lemma I. *If $f(x)$ be a continuous function, defined in the interval (a, b), a continuous function $\phi(x)$ can be so determined that $|\phi(x) - f(x)| < \epsilon$, in (a, b); $\phi(a) = f(a)$, $\phi(b) = f(b)$; and $\phi'(x) = 0$, almost everywhere in (a, b); where ϵ is an arbitrarily chosen positive number.*

Let (α, β) be any sub-interval in (a, b), and let G be a perfect non-dense set of points in (α, β), of measure zero. A correspondence exists between the points of (α, β), and the points and contiguous intervals of G, such that to all points P' of an interval contiguous to G there corresponds a single point P of (α, β), and to a point P', of G, there corresponds a single point P of (α, β); the relation of order of all points P being the same as for the corresponding points of G or of contiguous intervals.

Let $\phi(x)$ have at each point P' the value of $f(x)$ at the corresponding point P; thus $\phi(x)$ is constant in each interval contiguous to G. Since G has measure zero, $\phi'(x)$ exists almost everywhere in (α, β), and has the value zero; also $\phi(\alpha) = f(\alpha)$, $\phi(\beta) = f(\beta)$.

The interval (a, b) can be divided into a finite number of parts (α, β), in each of which the fluctuation of $f(x)$ is $< \epsilon$. In each part, $\phi(x)$ is defined as above, and $\phi(x)$ has in (α, β) the same range of values as $f(x)$; thus it is clear that $|\phi(x) - f(x)| < \epsilon$, in (a, b); moreover $\phi'(x) = 0$, almost everywhere in (a, b), and is such that $\phi(a) = f(a)$, $\phi(b) = f(b)$.

Lemma II. *If $f(x)$ be measurable in (a, b), and finite almost everywhere, an enumerable set of non-dense perfect sets $\{G_n\}$ can be so determined that no two of the sets have a point in common, that the sum of their measures is equal to that of the whole interval, and that $f(x)$ is bounded in each set. Moreover the sets can be so determined that the points a, b do not belong to any of them.*

If N be a positive number, and e_N be the set of points at which $|f(x)| < N$, we have $\lim\limits_{N \to \infty} m(e_N) = b - a \equiv l$. A value N_1, of N, can be so chosen that $m(e_{N_1}) > \frac{1}{2}l$, and a non-dense perfect set G_1 can be chosen as a part of e_{N_1} so that $m(G_1) > \frac{1}{2}l$; the complementary set $C(G_1)$ has measure

$< \frac{1}{2}l$. As before, a non-dense perfect set G_2, contained in $C(G_1)$, can be determined, such that $|f(x)|$ is, in G_2, $< N_2$, where $N_2 > N_1$, and such that

$$m\{C(G_2)\} < \frac{1}{2^2}l.$$

Proceeding in this manner, every point of (a, b) belongs to one of the sets $G_1, G_2, \ldots, G_n, C(G_n)$; and $m\{C(G_n)\} < \frac{1}{2^n}l$. Thus $\sum_{r=1}^{r=n} m(G_r)$ converges to l, as $n \sim \infty$; moreover in any set G_n, we have $|f(x)| < N_n$.

200. We can now proceed to the proof of Lusin's theorem. Let $\{G_n\}$ be the sequence of perfect sets, constructed so that the conditions of Lemma II are satisfied.

Let $\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}, \ldots$, denote the intervals contiguous to G_n; the integer λ_n can be so chosen that $\sum_{\lambda_n+1}^{\infty} m(\delta_r^{(n)}) < m(G_n)$. There are $\lambda_n + 1$ intervals $\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots \Delta_{\lambda_n+1}^{(n)}$ complementary to the λ_n intervals

$$\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}; \text{ and } \sum_{r=1}^{\lambda_n+1} m(\Delta_r^{(n)}) < 2m(G_n).$$

Let $f_n(x) = f(x)$, in the points of G_n, and $f_n(x) = 0$, in $C(G_n)$; let $\phi_n(x) = \int_a^x f_n(x)\,dx$, the integral existing as an L-integral, because $f(x)$ is bounded in G_n. The function $\phi_n(x)$ is continuous in (a, b), and is constant in each of the intervals contiguous to G_n. Let $\psi_n(x)$ be any function, continuous in (a, b), and such that $\psi_n(x) = \phi_n(x)$, in the intervals $\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}$, and such that $\psi'_n(x) = 0$, almost everywhere in (a, b), and satisfies the condition $|\psi_n(x) - \phi_n(x)| < \frac{g_n}{2^n}$, where g_n is the length of the least of the intervals $\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots \Delta_{\lambda_n+1}^{(n)}$. That $\psi_n(x)$ can be determined so as to satisfy these conditions, follows from Lemma I.

Let $F_n(x) = \phi_n(x) - \psi_n(x)$; then $F_n(x)$ is continuous in (a, b); it is such that $|F_n(x)| < \frac{g_n}{2^n}$, in (a, b); it vanishes in each of the intervals $\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}$; and $F_n'(x) = f(x)$, at almost all points of G_n.

Let us consider the function $F(x) = \sum_{n=1}^{\infty} F_n(x)$. Since

$$|F_n(x)| < \frac{g_n}{2^n} < \frac{l}{2^n},$$

the series converges uniformly and absolutely in (a, b), and therefore $F(x)$ is continuous in (a, b); it will be shewn to be a primitive, such as is required.

Since $F_n'(x) = 0$, almost everywhere in $C(G_n)$, and $F_n'(x) = f(x)$, almost everywhere in G_n, there exists a set of points S, of measure zero, such that each of the functions $\{F_n(x)\}$ has a differential coefficient that is not infinite, for all points not in S; also $F_1'(x)$, $F_2'(x)$, ..., are in $C(S)$ all zero, except that one of them has the value $f(x)$. If $\iota \leqq \lambda_n + 1$, add to $\Delta_\iota^{(n)}$ two intervals, of length g_n, one on the right and the other on the left; and let the interval so extended be denoted by $U_\iota^{(n)}$, where $\iota = 1, 2, 3, \ldots \lambda_n + 1$. Consider the set E_n, of points which belong to one or more of the intervals $U_1^{(n)}, U_2^{(n)}, \ldots U_{\lambda_n+1}^{(n)}$; we have

$$m(E_n) \leqq \sum_{\iota=1}^{\lambda_n+1} m(U_\iota^{(n)}) < 3 \sum_{\iota=1}^{\lambda_n+1} m(\Delta_\iota^{(n)}) \leqq 6m(G_n);$$

consequently the series $\sum\limits_{n=1} m(E_n)$ is convergent. It follows that the set T of points, each of which belongs to an infinite number of the sets E_1, E_2, ..., has its measure zero.

Let R be the set of points of (a, b) that do not belong to S or to T; then $m(R) = l$. Each point of R belongs only to a finite number of the sets $\{E_n\}$. It will be shewn that, at every point x, of R, $F'(x) = f(x)$.

Let ξ be a point of R, then

$$\frac{F(\xi+h) - F(\xi)}{h} = \sum_{n=1}^{\infty} \frac{F_n(\xi+h) - F_n(\xi)}{h}.$$

The number N may be so chosen that ξ does not belong to $E_{N+1}, E_{N+2}, \ldots$; thus, if $n > N$, ξ is interior to one of the intervals $\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}$; and therefore $F_n(\xi) = 0$, for $n > N$; thus

$$\sum_{n=N+1}^{\infty} \frac{F_n(\xi+h) - F_n(\xi)}{h} = \sum_{n=N+1}^{\infty} \frac{F_n(\xi+h)}{h}.$$

If h be such that $F_n(\xi+h) \neq 0$, $\xi + h$ is outside $\delta_1^{(n)}, \delta_2^{(n)}, \ldots \delta_{\lambda_n}^{(n)}$, that is interior to an interval $\Delta_1^{(n)}, \Delta_2^{(n)}, \ldots \Delta_{\lambda_n+1}^{(n)}$; but ξ is exterior to E_n, and therefore to all the intervals $U_1^{(n)}, U_2^{(n)}, \ldots U_{\lambda_n+1}^{(n)}$. It follows that $|h| > g_n$, and thus that

$$\left| \sum_{n=N+1}^{\infty} \frac{F_n(\xi+h)}{h} \right| < \sum_{n=N+1}^{\infty} \frac{|F_n(\xi+h)|}{g_n}.$$

Since $|F_n(x)| < \frac{g_n}{2^n}$, if $n > N$, we have, in the whole of (a, b),

$$\sum_{n=N+1}^{\infty} \frac{|F_n(\xi+h)|}{g_n} < \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{1}{2^N};$$

and therefore

$$\left| \frac{F(\xi+h) - F(\xi)}{h} - \sum_{r=1}^{r=N} \frac{F_r(\xi+h) - F_r(\xi)}{h} \right| < \frac{1}{2^N},$$

where N is fixed, and h is arbitrary. It follows that each of the four derivatives of $F(x)$, at ξ, differs from $F_1'(\xi) + F_2'(\xi) + \ldots + F_N'(\xi)$ by less than $\frac{1}{2^N}$. A number N_1 can be so fixed that, for $N > N_1$,

$$F_1'(\xi) + F_2'(\xi) + \ldots + F_N'(\xi) = f(\xi);$$

therefore $F(x)$ has a differential coefficient at ξ, which has the value $f(\xi)$; this is the case everywhere in (a, b), except at points of S or T.

The theorem has accordingly been established.

It will be observed that the set R is of the first category; the question whether $F(x)$ can be so determined that $F'(x) = f(x)$, at all points of a set of the second category, remains open.

CHAPTER V

SEQUENCES OF INTEGRALS

THE INTEGRATION OF SERIES AND SEQUENCES

201. When a function is represented in some given field of the variable or variables by a series which converges to the value of the function, it is important to be in possession of conditions which shall ensure that the sum of the integrals of the terms of the series shall converge to the integral of the sum-function, the integrals being taken over the given field, or over some part of it.

If $s_n(x)$ denote the nth partial sum of a series $f_1(x) + f_2(x) + \ldots$, which converges to $s(x)$, in a field E, the condition is identical with that for the convergence of $\int s_n(x)\,dx$ to $\int s(x)\,dx$. Thus, when $\{s_n(x)\}$ represents a sequence of functions of one or more variables, typified by x, which converges to $s(x)$, in a field E, and $\lim\limits_{n\sim\infty}\int_{(E)} s_n(x)\,dx = \int_{(E)} s(x)\,dx$, the sequence $\{s_n(x)\}$ is said to be *an integrable sequence* in the field E, which is taken to be a measurable set of points in one or more dimensions; the integrals being assumed to exist in accordance with the definition of Lebesgue, which implies that the set E is measurable. The measure of E may be finite or infinite.

In case, $\lim\limits_{n\sim\infty}\int s_n(x)\,dx = \int s(x)\,dx$, not only when the field of integration is E, but also when the integrals are taken over any measurable part F, of E, the sequence is said to be *completely integrable* over the set E. This terminology was introduced* by Vitali.

Proofs that the convergence of a series to its sum-function is sufficient to ensure the validity of term by term integration were advanced by Cauchy and by Moigno. These proofs, although they are invalid, may be accepted as signs that Mathematicians had, early in the nineteenth century, become conscious that the validity of the process was in need of investigation.

202. A very important criterion of a general character is contained in the following theorem, which is a generalization of a theorem due† to Lebesgue.

If a sequence $\{s_n(x)\}$ of functions, all of which are summable in a measurable set E, of one or more dimensions, and of finite or infinite measure, con-

* *Rendiconti del Circ. Mat. di Palermo*, vol. XXIII (1907), p. 140.

† *Leçons sur l'intégration*, p. 114.

verges at all points of E to the values of a function $s(x)$, it is sufficient, in order that $\int_{(E)} s(x)\,dx$ may exist, and that the sequence $\{s_n(x)\}$ be completely integrable over the set E, that a non-negative function $\phi(x)$ should exist which is summable over E, and such that $|s_n(x)| \leq \phi(x)$, for all values of n and x (in E).

Moreover the convergence of the integrals $\int_{(F)} s_n(x)\,dx$ to $\int_{(F)} s(x)\,dx$ is uniform for all measurable sets F contained in E.

It should be observed that if, in E, there is an exceptional set of points, of measure zero, at which the sequence $\{s_n(x)\}$ does not converge, the theorem still holds good. For it may be applied to the set obtained by removing the exceptional set from E, and this makes no difference to the values of the L-integrals.

To prove the theorem, it is seen from the relation $|s_n(x)| \leq \phi(x)$, that $|s(x)| \leq \phi(x)$, and thus that $|s(x)|$ is summable in E; hence $s(x)$ is summable over E. In particular, if $m(E)$ is infinite, $s(x)$ is absolutely summable over E. It is only when $m(E)$ is infinite that summability of a function over E does not necessarily imply absolute summability. First let it be assumed that $m(E)$ is finite, and let ϵ be an arbitrarily chosen positive number. Let e_n be that set of points, in E, for all of which $|s(x) - s_{n+m}(x)| \leq \epsilon$, for $m = 0, 1, 2, 3, \ldots$; e_n certainly exists if n be sufficiently large. If F be a measurable part of E, let f_n be the part of e_n which is contained in F. We have then

$$\left|\int_{(F)} \{s(x) - s_n(x)\}\,dx\right| \leq \int_{(f_n)} |s(x) - s_n(x)|\,dx + \int_{(F-f_n)} |s(x) - s_n(x)|\,dx$$
$$\leq \epsilon m(e_n) + 2\int_{(E-e_n)} \phi(x)\,dx,$$

since $$|s(x) - s_n(x)| \leq 2\phi(x).$$

It is clear that the set e_n is contained in the set e_{n+1}, and that $E - e_n$ contains the set $E - e_{n+1}$. Moreover there exists no point common to all the sets $E - e_{n+m}$, for $m = 0, 1, 2, 3, \ldots$, for if $\bar{x}$ were such a point, we should have $|s(\bar{x}) - s_{n+m}(\bar{x})| > \epsilon$, for all values of m; and this is inconsistent with the convergence of the sequence of numbers $\{s_{n+m}(\bar{x})\}$ to the limit $s(\bar{x})$. Employing a theorem given in I, § 131, it now follows that $\lim_{n\sim\infty} m(E - e_n) = 0$.

If n be greater than or equal to a fixed integer n_ϵ, it now follows that $\left|\int_{(F)} \{s(x) - s_n(x)\}\,dx\right| < \epsilon m(E) + \epsilon$, whatever measurable set F may be, contained in E. Since ϵ is arbitrary, $\int_{(F)} s_m(x)\,dx$ converges to $\int_{(F)} s(x)\,dx$, uniformly for all such sets F.

Next, let $m(E)$ be infinite; a part E_1, of E, of finite measure, may be so determined that $\int_{(E-E_1)} \phi(x)\,dx < \epsilon$. If F be any measurable part of E, it consists of a part F_1 of E_1, and a part $F - F_1$ of $E - E_1$. We now have

$$\left|\int_{(F)} \{s(x) - s_n(x)\}\,dx\right| \leqq \left|\int_{(F_1)} \{s(x) - s_n(x)\}\,dx\right| + \left|\int_{(F-F_1)} \{s(x) - s_n(x)\}\,dx\right|.$$

The second integral on the right-hand side is less than $2\int_{(E-E_1)} \phi(x)\,dx$, or 2ϵ; and the first integral converges to zero, as $n \sim \infty$, uniformly for all the sets F_1 contained in E_1; thus $\left|\int_{(F)} \{s(x) - s_n(x)\}\,dx\right| < 3\epsilon$, provided n is not less than some number dependent on ϵ; therefore

$$\int_{(F)} s_n(x)\,dx$$

converges to $\int_{(E)} s(x)\,dx$, uniformly for all measurable parts F, of E.

The above theorem is equivalent to the following:

If the sequence $\{s_n(x)\}$ of functions, all summable in the measurable set E, and all absolutely summable in case E has infinite measure, converge to $s(x)$, everywhere (or almost everywhere) in E; and if a non-negative function $\psi(x)$, summable in E, exists and is such that $|s(x) - s_n(x)| \leqq \psi(x)$, for all values of n, and of x (in E), then $\int_{(E)} s(x)\,dx$ exists, and the sequence $\{s_n(x)\}$ is completely integrable in E. Moreover, the convergence of $\int_{(F)} s_n(x)\,dx$ to $\int_{(F)} s(x)\,dx$ is uniform for all measurable sets F, contained in E.

For we have

$$|s(x)| \leqq |s(x) - s_n(x)| + |s_n(x)| \leqq \psi(x) + |s_n(x)|,$$

and therefore $s(x)$ is absolutely summable in E. Moreover

$$|s_n(x)| \leqq \psi(x) + |s(x)|,$$

which is a summable non-negative function; and thus the condition of the theorem, in its first form, is satisfied.

It may be observed that:

In the first theorem, the condition $|s_n(x)| \leqq \phi(x)$ may be replaced by the condition $\phi_1(x) \leqq s_n(x) \leqq \phi_2(x)$, where $\phi_1(x)$, $\phi_2(x)$ are two functions, each of which is absolutely summable in the set E, of finite, or infinite, measure.

For if $\phi_1(x) \leqq s_n(x) \leqq \phi_2(x)$, $|s_n(x)|$ is at each point not greater than the larger of the two numbers $|\phi_1(x)|$, $|\phi_2(x)|$, and we may take $\phi(x)$ to be the function which has this value at each point x. It is easily seen that $\phi(x)$ is summable in E.

203. The theorem of § 202 can be extended to the case in which the functions $\{s_n(x)\}$, $s(x)$ involve a parameter α which may typify a point in a given set of points K, in any number of dimensions. We consider a sequence $\{s_n(x, \alpha)\}$ which, at each point x, of the set E, converges uniformly for all points α, in K, to the value $s(x, \alpha)$.

Let e_n denote the set of points in E, for all of which

$$|s(x, \alpha) - s_{n+m}(x, \alpha)| \leq \epsilon, \text{ for } m = 0, 1, 2, 3, \ldots,$$

and for all points α, in K. It can be shewn that there exists no point of E which belongs to all the sets $E - e_n, E - e_{n+1}, E - e_{n+2}, \ldots$. For, if $\bar{x}$ were such a point, we should have $|s(\bar{x}, \alpha_m) - s_{n+m}(\bar{x}, \alpha_m)| > \epsilon$, for all values of m, where $\{\alpha_m\}$ is some sequence of points belonging to K; and therefore, at the point $\bar{x}$, $s_n(\bar{x}, \alpha)$ does not converge to $s(\bar{x}, \alpha)$ uniformly for all points of K, which is contrary to hypothesis. It now follows that $\lim_{n\sim\infty} m(E - e_n) = 0$. Assuming that there exists a non-negative function $\phi(x)$, summable in E, such that $|s_n(x, \alpha)| \leq \phi(x)$, for all points x in E, and for all points α in K, the proof of the theorem given in § 202 is applicable when the sets $\{e_n\}$, as here defined, are employed. We have, accordingly, the following theorem:

Let $\{s_n(x, \alpha)\}$ be a sequence of functions, defined for each point x, in a measurable domain E, of one or more dimensions, and of finite, or infinite, measure, the sequence existing for each point α, in a set K, of one or more dimensions; and let it be assumed that $\{s_n(x, \alpha)\}$ converges to the value of a function $s(x, \alpha)$, at each point x, in E, uniformly for all points α, of K. It is sufficient, in order that $\int_{(E)} s(x, \alpha)\,dx$ may exist, and that the sequence $\{s_n(x, \alpha)\}$ be completely integrable in E, for all values of α in K, the sequence of integrals being uniformly convergent in K, that a non-negative function $\phi(x)$ should exist, which is summable over E, and is such that $|s_n(x, \alpha)| \leq \phi(x)$, for all values of n, x (in E), and α (in K).

As in § 202, the condition in this theorem may be replaced by the condition that $|s(x, \alpha) - s_n(x, \alpha)| \leq \psi(x)$, where $\psi(x)$ is a non-negative function, summable in E, for all values of n, and of α (in K). When E has infinite measure the functions $s_n(x, \alpha)$ must be taken to be absolutely summable in E.

204. Important particular criteria are obtained, for the case of a set E, of finite measure, by assuming that the functions $\phi(x)$, $\psi(x)$, employed in the two forms of the theorem of § 202, are both constant in E.

We find, from the first form of the theorem, the following:

If a sequence $\{s_n(x)\}$ of functions, all measurable in a set E, of finite measure, be convergent in that set, and if $s_n(x)$ is bounded, for all the values

of n and x, so that $| s_n(x) | \leq K$, where K is independent of n and x, then the sequence $\{s_n(x)\}$ is completely integrable in E. Moreover the convergence of $\int_{(F)} s_n(x)\,dx$ to $\int_{(F)} s(x)\,dx$ is uniform with respect to every measurable set F, contained in E.

As in the general case, it makes no difference if there be an exceptional set of points in E, of measure zero, at each of which $\{s_n(x)\}$ does not converge, or at which the condition $| s_n(x) | \leq K$ is not satisfied.

This theorem has been given, in a somewhat less general form, in I, § 398.

From the second form of the theorem in § 202, we obtain the following:

If a sequence of functions, summable in a set E, of finite measure, be convergent in that set, and if $| s(x) - s_n(x) | \leq K$, for all values of n, and of x (in E), then the sequence $\{s_n(x)\}$ is completely integrable in E. Moreover the convergence of $\int_{(F)} s_n(x)\,dx$ to $\int_{(F)} s(x)\,dx$ is uniform with respect to every measurable set F, contained in E.

It should be observed that the condition $| s(x) - s_n(x) | \leq K$ is (see § 94), in the case in which E is closed, equivalent to the condition that there are in E no points at which the measure of non-uniform convergence is infinite. Thus:

It is sufficient for the complete integrability of the sequence $\{s_n(x)\}$ in a closed and bounded set E that there be in E no points at which the measure of non-uniform convergence is infinite.

This theorem is a development of a theorem first given* by Osgood, for the case of a linear interval in which $s(x)$ and $s_n(x)$ are all continuous. The case for a linear interval in which $s(x)$ is not necessarily continuous was obtained† by Hobson, and was also investigated by W. H. Young‡, and by Arzelà§.

If $\{s_n(x)\}$ converge uniformly in the set E, of finite measure, we have $| s(x) - s_n(x) | < \epsilon$, for all sufficiently large values of n, and for all the values of x. Thus we have the theorem:

It is sufficient for the complete integrability of the sequence $\{s_n(x)\}$, in a set E, of finite measure, that the sequence converge uniformly on E. The convergence of the integrals is then uniform with respect to all measurable sets contained in E.

The results obtained here may clearly be extended to the case in which the functions involve a parameter, as in § 203.

* *Amer. Journal of Math.* vol. XIX (1897), p. 182.

† *Proc. Lond. Math. Soc.* (1), vol. XXXIV (1901), p. 254.

‡ *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 89.

§ *Mem. d. R. Acad. Bologna* (5), vol. VIII (1900), p. 703.

205. An important criterion is obtained by applying the second theorem of § 202 to the case in which the sequence $\{s_n(x)\}$ is monotone; either non-diminishing, so that $s_{n+1}(x) \geqq s_n(x)$, or non-increasing, so that $s_{n+1}(x) \leqq s_n(x)$, for all the values of x and n. In these cases, we have

$$0 \leqq |s(x) - s_n(x)| \leqq |s(x) - s_1(x)| \leqq |s(x)| + |s_1(x)|.$$

It is sufficient, in order to apply the theorem, that $s(x)$, and $s_1(x)$ should be summable in E, when $m(E)$ is finite, and that they should be absolutely summable in E, when $m(E)$ is infinite.

We thus obtain the following theorem:

If, in a measurable set E, of finite, or infinite, measure, the monotone sequence $\{s_n(x)\}$ converges (almost everywhere) to a function $s(x)$ which is summable in E when $m(E)$ is finite, and absolutely summable in E when $m(E)$ is infinite, and if $s_1(x)$ satisfies the same condition; then the sequence $\{s_n(x)\}$ is completely integrable in E. Moreover the convergence of $\int_{(F)} s_n(x)\,dx$ to $\int_{(F)} s(x)\,dx$ is uniform for all measurable sets F, contained in E.

This theorem may be extended, as in § 203, to the case in which the functions involve a parameter α. The functions $s_n(x, \alpha)$ must then be taken to be monotone for each value of the parameter α; and $|s(x, \alpha)|$, $|s_1(x, \alpha)|$ must be taken not to exceed positive functions which are both summable in E.

206. Other criteria for the complete integrability of a sequence may be obtained, which depend upon conditions involving integrals of the functions in the sequence.

The following theorem will be first established:

Let the sequence $\{s_n(x)\}$ converge everywhere (or almost everywhere) in a measurable set of points E (of one or more dimensions) of finite measure, to the values of a function $s(x)$ which is summable in E. If the condition $\lim_{n\sim\infty} \int_{(e_n)} |s_n(x)|\,dx = 0$ is satisfied for every sequence $\{e_n\}$ of measurable sets contained in E, such that each set of the sequence contains the next, and such that $\lim_{n\sim\infty} m(e_n) = 0$, then the sequence $\{s_n(x)\}$ is completely integrable in E. Moreover the convergence of the integrals over measurable components of E is uniform for all such components.

Let e_n denote that set of points of E, for each of which

$$|s(x) - s_{n+m}(x)| > \epsilon,$$

for one or more values of m in the sequence $0, 1, 2, 3, \ldots$; so that

$$|s(x) - s_{n+m}(x)| \leqq \epsilon,$$

in the set $E - e_n$, and the conditions are satisfied that e_{n+1} is contained

in e_n, and $\lim\limits_{n\sim\infty} m(e_n) = 0$. If F be any measurable set contained in E, let f_n, $F - f_n$ denote the parts of e_n, $E - e_n$, respectively, that are contained in F.

We have now

$$\left|\int_{(F)}\{s(x) - s_n(x)\}\,dx\right| \leq \int_{(f_n)}|s(x) - s_n(x)|\,dx + \int_{(F-f_n)}|s(x) - s_n(x)|\,dx$$
$$\leq \int_{(f_n)}|s(x) - s_n(x)|\,dx + \epsilon m(E).$$

Also $$\int_{(f_n)}|s(x) - s_n(x)|\,dx \leq \int_{(e_n)}|s(x)|\,dx + \int_{(e_n)}|s_n(x)|\,dx,$$

and since both the integrals on the right-hand side converge to zero, as $n \sim \infty$, it is seen that the integral on the left-hand side is $< \epsilon$, provided n is not less than some integer n_ϵ. We now have

$$\left|\int_{(F)}\{s(x) - s_n(x)\}\,dx\right| < \epsilon + \epsilon m(E), \text{ for } n \geqq n_\epsilon,$$

and for every measurable set F, contained in E. Therefore $\int_{(F)} s_n(x)\,dx$ converges to $\int_{(F)} s(x)\,dx$, uniformly for all such sets F.

It is easily seen that:

The sufficient condition in the theorem may be replaced by the less stringent condition $\lim\limits_{n\sim\infty}\int_{(e_n)} s_n(x)\,dx = 0$. *The sequence is then completely integrable in E, but the condition that the integrals of $s_n(x)$ over a measurable part of E to the integral of $s(x)$ over that part is uniform for all such parts is not necessarily satisfied.*

For the inequality employed in the above proof may be replaced by

$$\left|\int_{(F)}\{s(x) - s_n(x)\}\,dx\right| \leqq \left|\int_{(f_n)}\{s(x) - s_n(x)\}\,dx\right| + \epsilon m(E),$$

now $\int_{(f_n)} s(x)\,dx$ converges uniformly to zero, as $n \sim \infty$ (see I, § 392), and $\int_{(f_n)} s_n(x)\,dx$, by hypothesis, converges to zero, as $m(f_n)$ converges to zero, but it cannot be asserted that $\left|\int_{(f_n)} s_n(x)\,dx\right| < \epsilon$, for $\eta \geqq n_\epsilon$, where n_ϵ is independent of the particular sequence $\{f_n\}$, and therefore independent of F.

207. If E have infinite measure, we have the following theorem:

If $\{s_n(x)\}$ converge everywhere (or almost everywhere) in a measurable set, of infinite measure, to the values of a function $s(x)$, absolutely summable in E, then, provided the condition of the last theorem is satisfied in every part

E_1, of E, which has finite measure, and provided that, if ϵ be arbitrarily assigned, E_1 can be so determined that

$$\int_{(E_2-E_1)} s_n(x)\,dx < \epsilon, \text{ for } n \geqq n_\epsilon,$$

and for every measurable set E_2, which contains E_1, and is contained in E, the sequence $\{s_n(x)\}$ is integrable in E.

The set E_2, of finite measure, can be so determined that

$$\left|\int_{(E-E_2)} s(x)\,dx\right| < \epsilon, \text{ and also } \left|\int_{(E-E_2)} s_n(x)\,dx\right| < 2\epsilon, \text{ for } n \geqq n_\epsilon.$$

We have now

$$\int_{(E)}\{s(x)-s_n(x)\}\,dx = \int_{(E-E_2)}\{s(x)-s_n(x)\}\,dx + \int_{(E_2)}\{s(x)-s_n(x)\}\,dx,$$

and since the sequence $\int_{(E_2)} s_n(x)\,dx$ satisfies the condition of the last theorem, we have $\left|\int_{(E_2)}\{s(x)-s_n(x)\}\,dx\right| < \epsilon$, for $n \geqq n_\epsilon'$. It follows that $\left|\int_{(E)}\{s(x)-s_n(x)\}\,dx\right| < 4\epsilon$, for $n \geqq n_\epsilon''$, where n_ϵ'' is the greater of the numbers n_ϵ, n_ϵ'. Since ϵ is arbitrary, the sequence $\int_{(E)} s_n(x)\,dx$ converges to $\int_{(E)} s(x)\,dx$.

The theorem might be so stated as to involve the complete integrability of the sequence, and also so that this is uniform with respect to all sets.

208. Let E be a set of points of finite measure.

If, corresponding to an arbitrarily chosen positive number ϵ, another number η exists, such that $\left|\int_{(e)} s_n(x)\,dx\right| < \epsilon$, where e is any measurable set contained in E, provided $m(e) < \eta$, for every value of n, the integrals $\left\{\int_{(E)} s_n(x)\,dx\right\}$ of the sequence are said to be *equi-convergent*. The term equi-absolutely continuous is sometimes used instead of equi-convergent, and sometimes the term uniformly convergent is employed.

If the integrals $\int_{(E)} s_n(x)\,dx$ are equi-convergent, so also are the integrals $\int_{(E)} |s_n(x)|\,dx$, and conversely.

For, let e_1 and e_2 be the two parts of e in which $s_n(x)$ is $\geqq 0$, and in which $s_n(x) < 0$; we have then $\int_{(e_1)} s_n(x)\,dx < \epsilon$, $-\int_{(e_2)} s_n(x)\,dx < \epsilon$, and therefore $\int_{(e)} |s_n(x)|\,dx < 2\epsilon$, for every value of n; since 2ϵ is arbitrary, the integrals $\int_{(E)} |s_n(x)|\,dx$ are equi-convergent.

Conversely, if $\int_{(E)} |s_n(x)|\,dx$ are equi-convergent, we have

$$\int_{(e)} |s_n(x)|\,dx < \epsilon, \text{ if } m(e) < \eta,$$

for every value of n; hence

$$\int_{(e_1)} s_n(x)\,dx < \epsilon, \quad -\int_{(e_2)} s_n(x)\,dx < \epsilon,$$

therefore
$$\left|\int_{(e)} s_n(x)\,dx\right| < \epsilon.$$

If E be of infinite measure, and if the condition is satisfied, that for every set E_1, of finite measure, contained in E, the sequence of integrals $\int_{(E_1)} s_n(x)\,dx$ is equi-convergent, and provided also E_1 can be so determined that $\left|\int_{(E_2-E_1)} s_n(x)\,dx\right| < \epsilon$, for all values of n, and for all sets E_2 of finite measure, contained in E, and containing E_1, then the integrals $\int_{(E)} s_n(x)\,dx$ are said to be equi-convergent in the set E, of infinite measure.

In case all the functions $s_n(x)$ are absolutely summable in E, it is easily seen that the equi-convergence of the integrals $\int_{(E)} s_n(x)\,dx$ involves that of the integrals $\int_{(E)} |s_n(x)|\,dx$; and conversely. For, since $E_2 - E_1$ is of finite measure, the sequence of integrals is equi-convergent in $E_2 - E_1$, and therefore we have, as before, $\int_{(E_2-E_1)} |s_n(x)|\,dx < 2\epsilon$.

209. It will be shewn that, when the integrals $\int_{(E)} s_n(x)\,dx$ are equi-convergent, and when $m(E)$ is finite, the condition $\lim\limits_{n\sim\infty}\int_{e_n} s_n(x)\,dx = 0$, of the theorem of § 206, is satisfied. Let e_r be the first set of the sequence $\{e_n\}$ for which $m(e_r) < \eta$, then $\left|\int_{(e_{r+m})} s_n(x)\,dx\right| < \epsilon$, for all values of n, and for $m = 0, 1, 2, 3, \ldots$. Thus we have $\left|\int_{(e_{r+m})} s_{r+m}(x)\,dx\right| < \epsilon$, for $m = 0, 1, 2, 3, \ldots$, and hence $\overline{\lim\limits_{n\sim\infty}}\left|\int_{(e_n)} s_n(x)\,dx\right| < \epsilon$; and since ϵ is arbitrary, it follows that $\lim\limits_{n\sim\infty}\int_{(e_n)} s_n(x)\,dx = 0$. We thus obtain Vitali's theorem* that:

It is sufficient for the complete integrability of a sequence $\{s_n(x)\}$ which converges almost everywhere in a set E, of finite measure, to a function $s(x)$, summable in E, that the integrals of $s_n(x)$ in E should be equi-convergent.

* *Rendiconti del Circ. Mat. di Palermo*, vol. XXIII (1907), p. 137. Vitali has further proved that the condition of equi-absolute continuity is necessary as well as sufficient.

Vitali's theorem may be extended to the case in which the set E has infinite measure.

Let it be assumed that the sequence $\int_{(E)} s_n(x)\,dx$ is equi-convergent in E. Let F be a measurable part of E; then if $m(F)$ is finite, the integrability of the sequence in F follows from Vitali's theorem. We need therefore only consider the case in which $m(F)$ is infinite. A set E_1, contained in E, of finite measure, can be so determined that $\left|\int_{(E_2-E_1)} s_n(x)\,dx\right| < \epsilon$, for every set E_2, of finite measure, containing E_1, and contained in E, and for all values of n. It then follows, as in § 208, that $\int_{(E_2-E_1)} |s_n(x)|\,dx < 2\epsilon$. Let F_1 be the part of F that is in E_1, and F_2 the part that is in E_2; we have then $\left|\int_{(F_2-F_1)} s_n(x)\,dx\right| < 2\epsilon$. Let now $m(E_2)$ increase indefinitely, so that E is the outer limiting set of the sequence of sets $\{E_2\}$, then F is the outer limiting set of the sequence of sets F_2; we have thus $\left|\int_{(F-F_1)} s_n(x)\,dx\right| \leq 2\epsilon$, and it then follows that $\left|\int_{(F-F_1)} \{s(x) - s_n(x)\}\,dx\right| < 3\epsilon$, provided $m(F_1)$ is sufficiently large. Hence

$$\left|\int_{(F)} \{s(x) - s_n(x)\}\,dx\right| \leqq \left|\int_{(F_1)} \{s(x) - s_n(x)\}\,dx\right| + \left|\int_{(F-F_1)} \{s(x) - s_n(x)\}\,dx\right|,$$

and the expression on the right-hand side is $< 4\epsilon$, if $n \geqq n_\epsilon$; it follows that $\int_{(F)} s_n(x)\,dx$ converges to $\int_{(F)} s(x)\,dx$.

We have accordingly established the following theorem:

It is sufficient for the complete integrability of a sequence $\{s_n(x)\}$, of functions, summable in a set E, of infinite measure, which converges almost everywhere in E, to a summable function $s(x)$, that the integrals of $s_n(x)$ in E should be equi-convergent.

210. In case $s_n(x) \geqq 0$, for all values of n and x, in E, the condition $\lim_{n\sim\infty} \int_{(e_n)} s_n(x)\,dx = 0$, of the theorem of § 206, can be shewn to be necessary for the complete integrability of the convergent sequence $\{s_n(x)\}$ over the set E, of finite measure. An integer n_1 can be so determined that $\int_{(e_{n_1})} s(x)\,dx$ is less than an arbitrarily prescribed positive number δ; moreover $n_\delta (\geqq n_1)$ can be so determined that $\int_{(e_{n_1})} s_n(x)\,dx$ differs from $\int_{(e_{n_1})} s(x)\,dx$ by less than δ, provided $n \geqq n_\delta \geqq n_1$. We have now, for $n \geqq n_\delta$,

$$\int_{(e_n)} s_n(x)\,dx \leqq \int_{(e_{n_1})} s_n(x)\,dx < \int_{(e_{n_1})} s(x)\,dx + \delta < 2\delta.$$

Thus $\overline{\lim_{n\sim\infty}} \int_{(e_n)} s_n(x)\,dx < 2\delta$; and since δ is arbitrary, $\lim_{n\sim\infty} \int_{(e_n)} s_n(x)\,dx = 0$. Hence we have the following theorem:

If the sequence $\{s_n(x)\}$ is almost everywhere convergent in the set E, of finite measure, and $s_n(x) \geqq 0$, for all the values of n and x, it is necessary and sufficient for the complete integrability of the sequence in E, that

$$\lim_{n\sim\infty} \int_{(e_n)} s_n(x)\,dx = 0,$$

where $\{e_n\}$ is any sequence of measurable sets, contained in E, such that each set contains the next, and $m(e_n)$ converges to zero, as $n \sim \infty$.

Let it be now assumed that a sequence $\{s_n(x)\}$, which converges almost everywhere in E, is such that $0 \leqq s_n(x) \leqq f_n(x)$, for almost all values of x, and for all values of n; where $\{f_n(x)\}$ is a completely integrable sequence in E. We have then $0 \leqq \int_{(e_n)} s_n(x)\,dx \leqq \int_{(e_n)} f_n(x)\,dx$, and since $\lim_{n\sim\infty} \int_{(e_n)} f_n(x)\,dx = 0$, it follows that $\lim_{n\sim\infty} \int_{(e_n)} s_n(x)\,dx = 0$, and therefore the sequence $\{s_n(x)\}$ is completely integrable in E. It has thus been proved that:

If E be of finite measure, and $\{s_n(x)\}$ be almost everywhere convergent in E, and $0 \leqq s_n(x) \leqq f_n(x)$, where $\{f_n(x)\}$ is completely integrable in E, then $\{s_n(x)\}$ is completely integrable in E.

Next, let it be assumed that $\{f_n(x)\}$, $\{g_n(x)\}$ are any two sequences, completely integrable in E, and that, almost everywhere in E,

$$f_n(x) \leqq s_n(x) \leqq g_n(x),$$

where $\{s_n(x)\}$ is convergent almost everywhere in E. The two sequences $\{s_n(x) - f_n(x)\}$, $\{g_n(x) - f_n(x)\}$ are both convergent, almost everywhere in E, and $s_n(x) - f_n(x)$, $g_n(x) - f_n(x)$ are both $\geqq 0$, almost everywhere. Since $0 \leqq s_n(x) - f_n(x) \leqq g_n(x) - f_n(x)$, it follows that the sequence $\{s_n(x) - f_n(x)\}$ is completely integrable in E, and thence that the sequence $\{s_n(x)\}$ is completely integrable in E. Thus the theorem* has been proved that:

A sequence $\{s_n(x)\}$, convergent almost everywhere in the set E, of finite measure, is completely integrable in E if two other sequences $\{f_n(x)\}$, $\{g_n(x)\}$ exist, both completely integrable in E, and such that $f_n(x) \leqq s_n(x) \leqq g_n(x)$.

That the condition is necessary as well as sufficient, is seen by taking $f_n(x) = -1 + s_n(x)$, $g_n(x) = 1 + s_n(x)$.

211. To extend the above theorems to the case in which $m(E)$ is infinite, let $\{E_r\}$ be a sequence of sets of finite measure, each one contained

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 315.

by an inequality given in I, § 435. It follows that $\lim\limits_{n\sim\infty}\int_{(e_n)}|s_n(x)|\,dx = 0$, and thus the theorem of § 206, is applicable.

213. The following theorem, which was given in a less general form in I, § 399, will now be established:

Let $\{s_n(x)\}$ be a sequence of functions, summable in a measurable set E, of finite, or infinite, measure, and of any number of dimensions, be such that $s_n(x) \geqq 0$, for all values of n, and of x, in E; these functions need not be bounded above. If, for each value of x, the sequence $\{s_n(x)\}$ is monotone non-diminishing (a set of measure zero being possibly disregarded), and if $\lim\limits_{n\sim\infty}\int_{(E)} s_n(x)\,dx$ has a definite value, then (1), the points of E at which $\{s_n(x)\}$ does not converge form a set of measure zero, and (2), the function $s(x)$ having the value of $\lim\limits_{n\sim\infty} s_n(x)$, where this limit exists, is summable in E, and the sequence $\{s_n(x)\}$ is completely integrable in E; the convergence of the integral of $s_n(x)$ to that of $s(x)$ being uniform in all measurable sets contained in E.

First, let $m(E)$ be finite; and let g_n be the set of points at which $s_n(x) > A$. The set g_n is contained in g_{n+1}, and thus $\lim\limits_{n\sim\infty} m(g_n)$ is the measure of the set of points for which $s_m(x) > A$, for all values of m, from and after some integer depending on x. Let it be assumed, if possible, that $\{s_n(x)\}$ diverges at the points of a set h, of positive measure; then

$$\lim_{n\sim\infty} m(g_n) \geqq m(h) > 0.$$

An integer n_1 can be determined so that $s_{n_1}(x) > A$, in a set of points of measure $\frac{1}{2}m(h)$; hence $\int_{(E)} s_{n_1}(x)\,dx > \frac{1}{2}Am(h)$; and therefore

$$\lim_{n\sim\infty}\int_{(E)} s_n(x)\,dx > \tfrac{1}{2}Am(h).$$

Since A is arbitrarily great, this is inconsistent with the hypothesis that the limit on the left-hand side has a finite value. It follows that $\{s_n(x)\}$ converges almost everywhere in E.

If $m(E)$ be not finite, a sequence of measurable sets $\{E_r\}$ exists, each of which is contained in the next, and is of finite measure, such that E is its outer limiting set. In E_r, the set of points e_r, at which $\{s_n(x)\}$ is divergent, has measure zero. The outer limiting set of $\{e_r\}$ is the set of points of divergence of $\{s_n(x)\}$ in E; and in virtue of a theorem established in I, § 131, the measure of this set is zero.

If λ_n be the set of points of E_r for which $s(x) - s_n(x) < \epsilon$, we have $\int_{(\lambda_n)}\{s(x) - s_n(x)\}\,dx < \epsilon m(\lambda_n)$, or $\int_{(\lambda_n)} s(x)\,dx < \int_{(\lambda_n)} s_n(x)\,dx + \epsilon m(E_r)$;

hence $\lim_{n\sim\infty}\int_{(\lambda_n)} s(x)\,dx \leqq \lim_{n\sim\infty}\int_{(E_r)} s_n(x)\,dx$. Since $m(\lambda_n)$ converges to $m(E_r)$, it is seen that $\int_{(E_r)} s(x)\,dx = \lim_{n\sim\infty}\int_{(\lambda_n)} s(x)\,dx$; and thus $\int_{(E_r)} s(x)\,dx$ exists, and is $\leqq \lim_{n\sim\infty}\int_{(E_r)} s_n(x)\,dx$, or is $\leqq \lim_{n\sim\infty}\int_{(E)} s_n(x)\,dx$.

It now follows that $\int_{(E)} s(x)\,dx = \lim_{r\sim\infty}\int_{(E_r)} s(x)\,dx$ exists, and is $\leqq \lim_{n\sim\infty}\int_{(E)} s_n(x)\,dx$. Thus $s(x)$ is summable in E, whether $m(E)$ be finite, or not. The last part of the theorem now follows by applying the theorem of § 205.

In case E has finite measure, instead of the condition $s_n(x) \geqq 0$ we may assume that $s_n(x) \geqq -K$, where K is independent of n and x. For the theorem may be applied to the sequence $\{s_n(x) + K\}$, and since K is summable in any set of finite measure, the result follows.

INTEGRATION OF SERIES DEFINED IN AN INTERVAL

214. If $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ converge to a function $s(x)$ everywhere, or almost everywhere, in a linear interval (a, b), and $s(x)$ be summable in (a, b), it is of importance to possess criteria sufficient to secure that $\sum_{r=1}^{r=n}\int_a^x u_r(x)\,dx$, or $\int_a^x s_n(x)\,dx$, converges to $\int_a^x s(x)\,dx$, for all values of x in the interval (a, b). When this convergence takes place, the series is said to be integrable in (a, b) in the ordinary sense; whereas it is said (see § 201) to be completely integrable in (a, b), when $\int_{(e)} s(x)\,dx$ is the limit, as $n \sim \infty$, of $\int_{(e)} s_n(x)\,dx$, for every measurable set of points e, in the interval (a, b). A series may be integrable in the ordinary sense in (a, b) when it is not completely integrable therein, but many of the criteria sufficient to ensure ordinary integrability are also sufficient to secure complete integrability.

The following criteria are obtained as special cases of the theorems in §§ 202–213, which were established for integration over sets of points in any number of dimensions:

(1) *If, in the finite interval (a, b), the functions $u_n(x)$ are summable, and the series $\Sigma u_n(x)$ converges uniformly in the interval to the values of $s(x)$, then $\int_a^x s(x)\,dx = \sum_{n=1}^{\infty}\int_a^x u_n(x)\,dx$, and the convergence is uniform for all values of x in (a, b).*

This has been proved in § 204.

(2) *If, in an interval (a, b), the functions $u_n(x)$ are all summable, and such that, almost everywhere in (a, b), the series $\sum_1^\infty u_n(x)$ converges to $s(x)$, and a non-negative function $\phi(x)$ exists, which is summable in (a, b) and such that $|s_n(x)| \leq \phi(x)$, for all values of n and x, then $\int_a^x s(x)\,dx$ exists, and the series $\sum_{n=1}^\infty \int_a^x u_n(x)\,dx$ converges to it, uniformly for all values of x in (a, b). This theorem also holds for an indefinitely great interval (a, ∞), or $(-\infty, \infty)$, provided $\phi(x)$ is summable in the interval.*

This has been proved in § 202.

(3) *In the case of a finite interval (a, b), the condition may be replaced by the condition $|R_n(x)| \leq \psi(x)$, where $\psi(x)$ is non-negative, and summable in (a, b). As in (2), $\int_a^x s(x)\,dx$ then exists, and the series $\sum_{n=1}^\infty \int_a^x u_n(x)\,dx$ converges to it, uniformly in (a, b). This holds also for (a, ∞), or for $(-\infty, \infty)$, provided the terms $u_n(x)$ are absolutely summable in (a, ∞), or in $(-\infty, \infty)$.*

(4) *Whether (a, b) be finite or infinite, the condition in (2) may be replaced by the condition $\phi_1(x) \leqq s_n(x) \leqq \phi_2(x)$, for all the values of n and x, where $\phi_1(x)$, $\phi_2(x)$ are two functions, each of which is absolutely summable in the interval.*

(5) *A particular case of (3) is the condition, in the case of a finite interval, that $|R_n(x)|$ should be bounded for all the values of n and x. This is equivalent to the condition that the series $\sum_{n=1}^\infty u_n(x)$ has no points of infinite measure of non-uniform convergence.*

(6) *If $\sum_{n=1} u_n(x)$ converges, almost everywhere in a finite, or infinite, interval to the values of a function $s(x)$, summable in the interval, and if all the terms $u_n(x)$ are $\geqq 0$, for all (or almost all) the values of n and x, then $\sum_{n=1} \int_a^x u_n(x)\,dx$ converges to $\int_a^x s(x)\,dx$, uniformly in the interval.*

This has been proved in § 205.

(7) *If $\sum_{n=1} u_n(x)$ is almost everywhere convergent in a finite, or infinite, interval (a, b), and $0 \leq u_n(x) \leq v_n(x)$, where $\sum_{n=1} v_n(x)$ is a series such that $\sum_{n=1} \int_a^x v_n(x)\,dx$ converges everywhere to $\int_a^x \Sigma(x)\,dx$, where $\sum_{n=1} v_n(x)$ converges to $\Sigma(x)$, then $\sum_{n=1} \int_a^x u_n(x)\,dx$ converges uniformly to $\int_a^x s(x)\,dx$.*

(8) *If $v_n(x) \leqq u_n(x) \leqq w_n(x)$, where both the sequences $\{v_n(x)\}$, $\{w_n(x)\}$ are integrable in (a, b), and $\sum_{n=1} u_n(x)$ is convergent almost everywhere in the*

finite, or infinite, interval (a, b), *then* $\sum_{n=1} \int_a^x u_n(x)\,dx$ *converges uniformly in* (a, b) *to* $\int_a^x s(x)\,dx$.

(7) and (8) have been proved in §§ 210, 211. In particular if $\sum_{n=1} |u_n(x)|$ is integrable term by term, so also is $\sum_{n=1} u_n(x)$, for

$$-|u_n(x)| \leq u_n(x) \leq |u_n(x)|.$$

(9) *If, in the finite interval* (a, b), $\sum_{n=1} u_n(x)$ *converges almost everywhere to* $s(x)$, *and* $\int_a^b \{s_n(x)\}^p\,dx$ *is bounded for all values of* n, *where* p *is a number* > 1, *then* $s(x)$ *is summable in* (a, b), *and* $\sum_{n=1} \int_a^x u_n(x)\,dx$ *converges uniformly in* (a, b) *to* $\int_a^x s(x)\,dx$.

(10) *If, in a finite, or infinite, interval* (a, b), $u_n(x) \geqq 0$ *for all values of* n *and* x *(except possibly at a set of points of measure zero), and if* $\sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx$ *has a definite value, then the series* $\sum_{n=1} u_n(x)$ *is almost everywhere convergent, and* $s(x)$ *is summable in* (a, b); *moreover* $\sum_{n=1} \int_a^x u_n(x)\,dx$ *converges uniformly in* (a, b) *to* $\int_a^x s(x)\,dx$.

In all these cases there is complete integrability of the series, provided in (7) and (8) the integrability of the sequences $\{v_n(x)\}$, $\{w_n(x)\}$ is assumed to be complete.

In case (1), if it be assumed that the convergence of $\sum_{n=1} u(x)$ to $s(x)$ is simply uniformly convergent only, this is sufficient to ensure that $s(x)$ is summable, but it is then not necessarily true that $\sum_{n=1} \int_a^x u_n(x)\,dx$ is a convergent series. It can, however, be shewn that, whenever this series is convergent, it converges to the value of $\int_a^x s(x)\,dx$. In fact we know that, by bracketing the terms of the simply uniformly convergent series $\sum_{n=1} u_n(x)$ in a suitable manner, the series is converted into a uniformly convergent series $\sum_{m=1} v_m(x)$, and the result (1) is then applicable to this series, and thus $\sum_{m=1} \int_a^x v_m(x)\,dx$ converges uniformly to the value of $\int_a^x s(x)\,dx$. It is clear that, whenever $\sum_{n=1} \int_a^x u_n(x)\,dx$ converges, it must converge to the same value as does the series $\sum_{m=1} \int_a^x v_m(x)\,dx$. We thus obtain the following theorem:

(1 *a*) If the series $\sum\limits_{n=1} u(x)$ converges simply-uniformly in the finite interval (a, b) to $s(x)$, and all the terms $u_n(x)$ are summable in the interval, then*, (1) if the series $\sum\limits_{n=1}\int_a^x u(x)\,dx$ be convergent it converges to the value $\int_a^x s(x)\,dx$, and (2) if the series be not convergent, it may by suitably bracketing the terms, and amalgamating the terms in each bracket, be converted into a series which converges to $\int_a^x s(x)\,dx$. The convergence is uniform with respect to x.

A practical test that $\sum\limits_{n=1}^{\infty}\int_0^{\infty} u_n(x)\,dx = \int_0^{\infty}\sum\limits_{n=1}^{\infty} u_n(x)\,dx$ which may be applied in many cases is the following:

If the series $\sum\limits_{n=1} |u_n(x)|$ *converges everywhere to a sum-function which is summable in the infinite interval* $(0, \infty)$, *then*

$$\sum_{n=1}^{\infty}\int_0^{\infty} u_n(x)\,dx = \int_0^{\infty}\sum_{n=1}^{\infty} u_n(x)\,dx.$$

This theorem is a particular case of (2), for if $\phi(x)$ is the function to which $\sum\limits_{n=1} |u_n(x)|$ converges, we have $\left|\sum\limits_{n=1}^{n} u_n(x)\right| \leqq \phi(x)$ for all values of n and x.

215. When, in the finite interval (a, b), the condition in (5), that the series $\sum\limits_{n=1} u_n(x)$ has no points at which the measure of non-uniform convergence is infinite is not satisfied, there exists a set G of such points which (see § 94) is necessarily closed, and may be finite. In this case the theorem may fail to hold good either (1), when $\sum\limits_{n=1}\int_a^x u_n(x)\,dx$ is not everywhere convergent in (a, b), or (2), when its sum is not continuous in the interval. It may also happen that, in these circumstances, the continuous function $U(x)$ to which the sum of the integrals converges, is not equal to $\int_a^x s(x)\,dx$; this last integral being assumed to exist.

The following theorem will however be established:

If the series $\sum\limits_{n=1} u_n(x)$, *of which all the terms are summable in the finite interval* (a, b), *converges to the summable function* $s(x)$, *and if, further, the series* $\sum\limits_{n=1}\int_a^x u_n(x)\,dx$ *converges everywhere to the values of a function which is continuous in* (a, b), *it is a sufficient condition*† *that this function be equal to*

* The first part of this theorem was given by Bendixson, for the case in which the functions $u_n(x)$ are all continuous; see *Stockholm Öfv.* vol. LIV (1897), p. 609.

† This theorem was given by Osgood, *Amer. Journal*, vol. XIX (1897), p. 182, in the case in which the terms of the series, and its sum, are continuous. The general theorem was given, for Riemann integration, by Arzelà, *Mem. di Bologna* (5), vol. VIII (1900).

$\int_a^x s(x)\,dx$, ***that the points at which the measure of non-uniform convergence of the series*** $\sum_{n=1} u_n(x)$ ***is infinite should form an enumerable set.***

The closed enumerable set of points at which the measure of non-uniform convergence of the series is infinite being denoted by G, let ξ be a point of (a, b) which does not belong to G. Since ξ is within an interval contiguous to G, in any interval $(\xi - \epsilon_1, \xi + \epsilon_2)$ interior to that contiguous interval, $| R_n(x) |$ has a finite upper boundary. Denoting by $U_n(x)$ the nth partial sum of the series $\sum_{n=1} \int_a^x u_n(x)\,dx$, and by $U(x)$ the continuous function to which $U_n(x)$ converges, as $n \sim \infty$, a value $\bar{n}$, of n, can be so determined that $| U(\xi) - U_n(\xi) | < \delta$, $| U(\xi + h) - U_n(\xi + h) | < \delta$, for $n \geqq \bar{n}$, where $\xi + h$ is a fixed point within the interval $(\xi - \epsilon_1, \xi + \epsilon_2)$, and δ is an arbitrarily chosen positive number.

We now have

$$\left| \frac{U(\xi + h) - U(\xi)}{h} - \frac{U_n(\xi + h) - U_n(\xi)}{h} \right| < \frac{2\delta}{|h|}.$$

Since the interval $(\xi, \xi + h)$ contains no points of G, it follows that, for all sufficiently large values of n,

$$\left| \int_\xi^{\xi+h} s_n(x)\,dx - \int_\xi^{\xi+h} s(x)\,dx \right| < \delta.$$

Therefore we have, provided n is not less than some fixed integer $\bar{n}_1$,

$$\left| \frac{U_n(\xi + h) - U_n(\xi)}{h} - \frac{S(\xi + h) - S(\xi)}{h} \right| < \frac{\delta}{|h|},$$

where $S(x)$ denotes $\int_a^x s(x)\,dx$.

From the two inequalities, we have

$$\left| \frac{U(\xi + h) - U(\xi)}{h} - \frac{S(\xi + h) - S(\xi)}{h} \right| < \frac{3\delta}{|h|};$$

and since δ is arbitrarily small, it follows that

$$\frac{U(\xi + h) - U(\xi)}{h} = \frac{S(\xi + h) - S(\xi)}{h}.$$

This holds for any point ξ that does not belong to G, and for any point $\xi + h$ in a neighbourhood of ξ that contains no points of G. It follows that any one of the four derivatives $D^+U(\xi)$, $D_+U(\xi)$, $D^-U(\xi)$, $D_-U(\xi)$, is equal to the corresponding derivative of $S(\xi)$. Since one of the four derivatives of the function $S(x)$, $U(x)$ is such that its value is the same for the two continuous functions, except at points belonging to an enumerable set, it follows (I, § 267) that the two functions differ by a constant; and since both vanish at the point a, they must be everywhere equal.

When the closed set G is not enumerable it contains a perfect component; and in that case the sum of the integrals of the terms of the series

is not necessarily equal to the integral of the sum, even when both exist and the condition of continuity of $\sum_{n=1}^{\infty}\int_a^x u_n(x)\,dx$ is satisfied.

It will be observed that, in accordance with the theorems which have been established, the term by term integration of a series may fail to give the integral of the sum, either (1), when the set G, of points of infinite measure of non-uniform convergence of the series, is finite or enumerable, but the condition that the sum of the series $\sum_1^{\infty}\int_a^x u_n(x)\,dx$ should be a continuous function of x is not satisfied; or (2), when G contains a perfect component.

216. If there exist, in the interval (a, b), points at which the series $\sum_{n=1} u_n(x)$ is not convergent, such points will be regarded as points of discontinuity of $s(x)$. Let it be assumed that these points form a non-dense set with an enumerable derivative, i.e. a reducible set; thus they are contained in a set G which is an enumerable closed set. Let it be further assumed that, in any interval (α, β) which contains, within it and at its ends, no point of G, the condition is satisfied that $|s_n(x)|$ is less than some fixed number, independent of n and x. Let it be also assumed that $\sum_{n=1}\int_a^x u_n(x)\,dx$ is convergent for all values of x in (a, b), and that its sum-function $U(x)$ is continuous in the closed interval (a, b). Let it be further assumed that $\int_a^x s(x)\,dx = S(x)$ is a continuous function of x, this will be the case when $s(x)$ is summable in (a, b), or more generally when it has a D-integral, or in particular, an HL-integral in (a, b). The enumerable set G contains all points of non-convergence of the given series and also every point at which the measure of non-uniform convergence of the series is infinite. With these assumptions, the proof of § 215 is applicable to establish the legitimacy of the integrability of the series $\Sigma u_n(x)$. We obtain accordingly the following theorem:

If the series $\sum_{n=1} u_n(x)$ converges to the function $s(x)$ at every point of the interval (a, b) which does not belong to a reducible set of points G, and if, in any interval (α, β) which contains within it, and at its ends, no points of G, $|s_n(x)|$ is bounded as a function of n and x; and if $\int_a^x u_n(x)\,dx$ exists as an L-integral, or as a D-integral, or an HL-integral, for every value of n, and the series $\sum_{n=1}^{\infty}\int_a^x u_n(x)\,dx$, for $a \leq x \leq b$, converges to a continuous function of x; then, if $\int_a^x s(x)\,dx$ exists as a continuous function of x,

$$\int_a^x s(x)\,dx = \sum_{n=1}^{\infty}\int_a^x u_n(x)\,dx.$$

The following theorem which also has reference to the case in which there are points at which the measure of non-uniform convergence is infinite is due to Vitali*:

If the points of a finite interval at which a convergent series has infinite measure of non-uniform convergence form a set G of which the measure is zero, then term by term integration is permissible, provided the sum-function is summable, and provided also the integrated series converges to a sum which is the integral of a summable function. The last condition is clearly also a necessary one when the first is satisfied.

In this case it is assumed that $U(x)$, the sum-function of $\sum\limits_{n=1} \int_a^x u_n(x)\,dx$, is an indefinite integral of a summable function, that is, that it is absolutely continuous in (a, b) (I, § 218). The closed set G being of measure zero, a finite set of intervals (Δ) of total measure $< \epsilon$ can be determined which include within them all the points of the set G. Since $U(x)$ is absolute continuous, the sum of its variations, each taken with its proper sign, over the intervals Δ is $< \delta_\epsilon$, where δ_ϵ is a number which converges to zero with ϵ. An interval (α, β), one of the intervals complementary to the finite set (Δ), contains no points of G, and therefore $U(\beta) - U(\alpha) = S(\beta) - S(\alpha)$; where $S(x)$ denotes $\int_a^x s(x)\,dx$. Now $U(b)$ is the sum of the variations of $U(x)$, taken over all the intervals (Δ) and all the complementary intervals (α, β), and the same remark applies to $S(b)$. It follows that $U(b)$ and $S(b)$ differ from one another by less than $\delta_\epsilon + \int_{(\Delta)} |s(x)|\,dx$. As ϵ converges to zero, so also do δ_ϵ and $m(\Delta)$; consequently $U(b)$ and $S(b)$ are equal. By considering the interval (a, x), where $a < x \leq b$, it follows also that $S(x) = U(x)$; and thus the theorem is established.

217. It is easily seen that, *in case all the terms of the series $\sum\limits_{n=1} u_n(x)$ are non-negative in an interval (a, b), finite or infinite, the term by term integrability of the series for (a, b) implies its complete integrability.*

For if e be any set of points in (a, b),

$$\int_{(e)} \{s(x) - s_n(x)\}\,dx \leq \int_a^b \{s(x) - s_n(x)\}\,dx,$$

since $s(x) - s_n(x) \geqq 0$. If the integral on the right-hand side converges to zero, as $n \sim \infty$, so also does the integral on the left-hand side.

The following theorem has reference to sequences which, in a given interval, are in general non-convergent:

In case the sequence $\{s_n(x)\}$ is not necessarily everywhere, or almost everywhere, convergent in the interval (a, b), and is such that $|s_n(x)| \leq \chi(x)$,

* *Rendiconti del Circ. Mat di Palermo*, vol. XXIII (1907), p. 155.

where $\chi(x)$ is summable in (a, b), then the upper and lower limits of $\int_a^x s_n(x)\,dx$, as $n \sim \infty$, are both integrals.

Let $\bar{F}(x) = \overline{\lim_{n\sim\infty}} \int_a^x s_n(x)\,dx$; and $\underline{F}(x) = \underline{\lim_{n\sim\infty}} \int_a^x s_n(x)\,dx$,

then $$\bar{F}(x_2) = \overline{\lim_{n\sim\infty}} \left\{\int_a^{x_1} s_n(x)\,dx + \int_{x_1}^{x_2} s_n(x)\,dx\right\}$$
$$\geqq \overline{\lim_{n\sim\infty}} \int_a^{x_1} s_n(x)\,dx - (x_2 - x_1)\int_a^b \chi(x)\,dx.$$

Thus $$\bar{F}(x_2) - \bar{F}(x_1) \geqq -(x_2 - x_1)A,$$
and similarly $$\underline{F}(x_2) - \underline{F}(x_1) \leqq (x_2 - x_1)A,$$
where A denotes $\int_a^b \chi(x)\,dx$. The sum of the values of the absolute variations of $\bar{F}(x)$, or of $\underline{F}(x)$, over any set of non-overlapping intervals, whose total measure is $< \epsilon$, is less than $A\epsilon$. It follows that both the functions are absolutely continuous in (a, b), and are therefore integrals of summable functions.

An extension of the theory of the integrability of convergent sequences to the case of non-convergent sequences which have an upper and a lower function, in relation to semi-integrals, has been developed* by W. H. Young.

218. When a convergent sequence of functions is defined in the infinite interval (a, ∞), and it is known that the sequence is integrable in every finite interval (a, b), it is desirable to possess a sufficient condition that the sequence should be integrable in the infinite interval.

The following sufficient conditions may be established:

(1) *Let the series $\sum_{n=1}^{\infty} u_n(x)$ have as its sum-function $s(x)$, summable in every finite interval, and let it be such that $\sum_{n=1} \int_a^C u_n(x)\,dx$ converges to $\int_a^C s(x)\,dx$, for every finite value of $C\ (> a)$, then if, corresponding to an arbitrarily chosen positive number ϵ, an integer n_ϵ, and a value of $C\ (> a)$ can be so chosen that $\left|\int_C^{C'} s_n(x)\,dx\right| < \epsilon$, for every value of $C'\ (> C)$, and for all values of $n \geqq n_\epsilon$, then $s(x)$ is integrable in (a, ∞), although not necessarily absolutely summable in (a, ∞), and $\sum_{n=1} \int^{\infty} u_n(x)\,dx$ converges to $\int_a^{\infty} s(x)\,dx$.*

(2) *On the assumption that the equation $\int_a^C s(x)\,dx = \sum_{n=1}^{\infty} \int_a^C u_n(x)\,dx$ holds for every value of $C\ (> a)$, then provided that $\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx$ is convergent,*

* See *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 286.

and that $\sum_{n=1} \int_a^C u_n(x)\,dx$ *converges to the value of* $\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx$, *when* C *is independently increased, it follows that* $\int_a^{\infty} s(x)\,dx$ *exists, and is equal to* $\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx$.

It should be observed that these theorems may be applied to cases in which the integrals $\int_a^C s_n(x)\,dx$, $\int_a^C s(x)\,dx$ exist only as non-absolutely convergent integrals such as D-integrals.

To prove (1) it is seen that $\lim_{n\sim\infty} \int_C^{C'} s_n(x)\,dx = \int_C^{C'} s(x)\,dx$; and assuming that n_ϵ and C can be so determined that $\left|\int_C^{C'} s_n(x)\,dx\right| < \epsilon$, for $n \geqq n_\epsilon$, and for all values of C' $(> C)$, it follows that $\left|\int_C^{C'} s(x)\,dx\right| \leq \epsilon$, for $C' > C$; and since ϵ is arbitrary, $\int_a^{\infty} s(x)\,dx$ exists. Assuming that all the integrals $\int_a^{\infty} s_n(x)\,dx$ exist, we have

$$\left|\int_a^{\infty} s(x)\,dx - \int_a^{\infty} s_n(x)\,dx\right| \leq \left|\int_a^{C} s(x)\,dx - \int_a^{C} s_n(x)\,dx\right| + \left|\int_C^{\infty} s(x)\,dx\right| + \left|\int_C^{\infty} s_n(x)\,dx\right|;$$

and by taking a sufficiently great value of n $(\geqq n_\epsilon)$, and a sufficiently large value of C, the expression on the right-hand side is $< 3\epsilon$. It thus appears that

$$\lim_{n\sim\infty} \int_a^{\infty} s_n(x)\,dx = \int_a^{\infty} s(x)\,dx.$$

To prove (2), we see that, if ϵ be fixed, C may be so chosen that

$$\left|\sum_{n=1}^{\infty} \int_C^{C'} u_n(x)\,dx\right| < \epsilon, \text{ for } C' > C,$$

and from this it follows that $\left|\int_C^{C'} s(x)\,dx\right| \leq \epsilon$, for $C' > C$. Since ϵ is arbitrary, $\int_a^{\infty} s(x)\,dx$ exists. Also since $\int_a^C s(x)\,dx = \lim_{n\sim\infty} \int_a^C s_n(x)\,dx$, we see that $\int_a^{\infty} s(x)\,dx$ is the limit to which $\sum_{n=1}^{\infty} \int_a^C s_n(x)\,dx$ converges as $C \sim \infty$, and this limit is by hypothesis $\sum_{n=1}^{\infty} \int_a^{\infty} u_n(x)\,dx$.

SEQUENCES OF FUNCTIONS THAT ARE INTEGRABLE (R)

219. Let $u_1(x)$, $u_2(x)$, ... $u_n(x)$, ... be functions each of which is bounded in the interval (a, b), and each of which is integrable (R) in that interval. Let it further be assumed that, in the whole interval, the series $u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$ converges to a function $s(x)$. Also let it be assumed that $s(x)$ is bounded in (a, b). It is proposed here to determine necessary and sufficient conditions that $s(x)$ may have an R-integral in (a, b).

Let E be a set of points, in (a, b), of measure zero; and let ϵ be an arbitrarily chosen positive number, and $\bar{n}$ an arbitrarily chosen integer. Let us suppose that, for each point x_1, of (a, b), which does not belong to a certain component E_ϵ, of E, an integer n_1 ($> \bar{n}$) can be determined, and also a neighbourhood $(x_1 - \delta, x_1 + \delta')$, such that the condition $| R_{n_1}(x) | < \epsilon$ is satisfied for every point x, in that neighbourhood. Then, provided this condition is satisfied for every value of ϵ, and E is such that each point of it belongs to E_ϵ, for some sufficiently small value of ϵ, the convergence of the sequence $\{s_n(x)\}$ to $s(x)$ is said to be *regular in* (a, b), except for the set E, of measure zero.

It will be observed that, for a fixed ϵ, the integer n_1 ($> \bar{n}$) depends in general upon the particular point x_1, which does not belong to E_ϵ. Moreover, since $\bar{n}$ is arbitrary, there exists, for a particular point x_1, an infinite number of values of n_1; the neighbourhood $(x_1 - \delta, x_1 + \delta')$ depending however in general upon the value of n_1 chosen.

In the particular case in which $u_n(x) \geqq 0$, for all values of n and x, so that the sequence $\{s_n(x)\}$ is monotone non-decreasing, when the condition $R_n(x) < \epsilon$ is satisfied for a particular value of n, it is also satisfied for all greater values. In the general case this does not hold; the condition is satisfied for an infinite set of greater values of n, but not necessarily for every such value.

It is easily seen that the set E_ϵ must, for each value of ϵ, be a non-dense closed set, although the set E is not necessarily non-dense, and may be everywhere dense in (a, b). For, if ξ be a limiting point of E_ϵ, then every neighbourhood of ξ contains points of E_ϵ, and it is impossible that the condition $| R_{n_1}(x) | < \epsilon$ can be satisfied for every point of such neighbourhood. Therefore ξ must belong to E_ϵ, and E_ϵ is consequently a closed set; and since its measure is zero, it must be non-dense in (a, b).

The following theorem will now be established:

The necessary and sufficient condition that the bounded function $s(x)$ may be integrable (R), is that the sequence of functions $\{s_n(x)\}$, all of which are integrable (R), shall converge to $s(x)$ regularly, except for a set of points E, of measure zero.

To prove that the condition stated is necessary, let it be assumed that $s(x)$ is integrable (R). Since $s(x)$, $s_1(x)$, $s_2(x)$, ... are all integrable (R), the set of points at which any one of these functions is discontinuous has measure zero, and it follows that the set of points at which one or more of these functions is discontinuous has its measure zero. It will be shewn that a point at which all these functions are continuous cannot belong to E_ϵ for any value of ϵ, and consequently cannot belong to E. Let ξ be a point at which all the functions are continuous, and let ϵ be a prescribed positive number. The integer n_1 $(> \bar{n})$ can be so chosen that $|s(\xi) - s_{n_1}(\xi)| < \frac{1}{3}\epsilon$; also δ can be so chosen that, for every point x, in the interval $(\xi - \delta, \xi + \delta)$, the inequalities $|s(\xi) - s(x)| < \frac{1}{3}\epsilon$, $|s_{n_1}(\xi) - s_{n_1}(x)| < \frac{1}{3}\epsilon$ are satisfied. From these three inequalities we deduce that the inequality

$$|s(x) - s_{n_1}(x)| < \epsilon$$

is satisfied for all points in the interval $(\xi - \delta, \xi + \delta)$; and therefore ξ does not belong to the set E_ϵ. Since ϵ is arbitrary, it follows that ξ does not belong to E. Hence every point of E must belong to the set of points at which one or more of the functions $s(x), s_1(x), s_2(x), \ldots s_n(x), \ldots$ is discontinuous; and therefore $m(E) = 0$.

To shew that the condition stated in the theorem is sufficient, let ϵ and $\bar{n}$ be fixed, then E_ϵ is a non-dense closed set of measure zero. A finite set of intervals, the sum of whose lengths is an arbitrarily chosen number η, can be so determined that every point of E_ϵ is within one of the intervals. The remainder of (a, b) consists of a finite set of intervals $\{\Delta\}$; and for any point x_1 in an interval Δ, a neighbourhood $(x_1 - \delta, x_1 + \delta')$ can be determined, and also an integer n $(> \bar{n})$, not necessarily the same for all such points x_1, such that $|R_n(x)| < \epsilon$ for all the points of $(x_1 - \delta, x_1 + \delta')$. This can be done for every point x_1 in the intervals (Δ), and we can consider the set of all such neighbourhoods $(x_1 - \delta, x_1 + \delta')$. To this set we may apply the Heine-Borel theorem; and consequently a finite set of the intervals $(x_1 - \delta, x_1 + \delta')$ exists such that every point of $\{\Delta\}$ is interior to one or more of the intervals of this finite set. In each one of the intervals of this finite set, the condition $|R_n(x)| < \epsilon$ is everywhere satisfied for some value of n $(> \bar{n})$. When the set of intervals of which the sum is η is excluded from (a, b), the remainder may be divided into a finite number of parts such that, in each part, the condition $|R_n(x)| < \epsilon$ is satisfied for a value of n belonging to a finite set $\bar{n} + p_1, n + p_2, \ldots \bar{n} + p_r$, of integers all greater than $\bar{n}$.

To shew that $s(x)$ is integrable (R) we now apply Riemann's test of integrability. Divide (a, b) into a number of parts $h_1, h_2, \ldots h_s$, so chosen that all the end-points of the excluded intervals, and also all the end-points of those finite parts for each of which $|R_n(x)| < \epsilon$, for a single value of n, are end-points of the parts $h_1, h_2, \ldots h_s$. For an interval h, in the excluded set, the product of h into the fluctuation of $s(x)$ is less than

$(M - m)\,h$, where M, m are the upper and lower boundaries of $s(x)$ in (a, b). For an interval h, for the whole of which $|R_{\bar{n}+p}| < \epsilon$, the fluctuation of $s(x)$ cannot exceed that of $s_{\bar{n}+p}(x)$ by more than 2ϵ. It follows that the sum of the products of each h into the corresponding fluctuation of $s(x)$ cannot exceed

$$(M - m)\,\eta + \sum_p \Sigma h\,\{2\epsilon + \text{fluctuation of } s_{\bar{n}+p}(x)\},$$

where, in the double summation, the first summation refers to all those of the h's which are in an interval for which p has one and the same value, and the second summation refers to the values $p_1, p_2, \ldots p_r$. Since $s_{n+p}(x)$ is integrable (R) in the interval to which it belongs, and for which p has a fixed value, it is seen that, when the number s is sufficiently increased, and the greatest of the h's is sufficiently small, $\sum_p \Sigma h \times$ fluctuation of $s_{\bar{n}+p}(x)$ becomes arbitrarily small. Since η and ϵ are arbitrarily small, it follows that Riemann's test of integrability of $s(x)$ is satisfied.

The general theorem having now been completely established, it appears, from the foregoing proof, that it may be stated as follows:

If $u_1(x) + u_2(x) + \ldots$ converges to a definite value $s(x)$, for all points of (a, b), and the functions $u_n(x)$ are all integrable (R) in (a, b), the necessary and sufficient conditions that $s(x)$ may be integrable (R) in (a, b) are (1), *that the upper boundary of $|s(x)|$ in (a, b) be finite, and* (2), *that, corresponding to two arbitrarily chosen positive numbers η, ϵ, and to any positive integer $\bar{n}$, a finite set of intervals whose sum is less than η can be excluded from (a, b), so that, in the remainder of (a, b), $|R_{\bar{n}+p}(x)| < \epsilon$, for every x, where p has one of a finite set of values which depend on x, but one such that the same p is applicable to all points x in a certain continuous interval.*

The condition (2), contained in this theorem, was obtained* first by Arzelà, and is expressed by him in the form, that a certain mode of convergence of the series, called *uniform convergence by segments in general* (convergenza uniforme a tratti in generale) holds good. This mode of convergence differs from that of uniform convergence by segments, considered in § 89, in that a finite set of intervals, of arbitrarily small sum, must be excluded from the domain, in order that the condition may be satisfied.

EXAMPLES

(1) Let $s_n(x) = nxe^{-nx^2}$, when n is odd, and $= 0$, when n is even. In this case the series is simply-uniformly convergent; the sum $s(x)$ is the continuous function 0. Then

$$\int_0^x s_n(x)\,dx = \tfrac{1}{2}(1 - e^{-nx^2}), \text{ or } 0,$$

* "Sulle serie di funzioni," Part II, *Mem. delle R. Accad. d. Sci. di Bologna* (5), vol. VIII (1900). A proof different from that in the text was given by Hobson, *Proc. Lond. Math. Soc.* (2), vol. I (1904), p. 382. It is shewn there that Arzelà's proof is invalid.

according as n is odd or even; thus

$$\lim_{n\sim\infty}\int_0^x s_n(x)\,dx$$

has no definite value, but $\displaystyle\int_0^x s(x)\,dx=0.$

The term by term integration fails in this case, because there is one point $x=0$, at which the measure of non-uniform convergence is indefinitely great, as may be seen from

$$R_n\left(\frac{1}{\sqrt{n}}\right)=-\sqrt{n}e^{-1}\quad(n\text{ odd}),$$

the limit $\displaystyle\lim_{n\sim\infty}\int_0^x s_n(x)\,dx$ not existing.

(2) Let $s_n(x)=2n^2xe^{-n^2x^2}$, then $s(x)=0$; at the point $x=0$, there is a point of indefinitely great measure of non-uniform convergence, since

$$s_n\left(\frac{1}{n}\right)=2ne^{-1}.$$

Here $$\int_{x_0}^x s_n(x)\,dx=e^{-n^2x_0^2}-e^{-n^2x^2},\quad x_0<0.$$

If x be different from zero, $\displaystyle\lim_{n\sim\infty}\int_{x_0}^x s_n(x)\,dx=0$, but at $x=0$ the limit is -1; thus, in any interval which contains the point 0, the function $\displaystyle\lim_{n\sim\infty}\int^x s_n(x)\,dx$ is discontinuous, and therefore cannot equal $\displaystyle\int_{x_0}^x s(x)\,dx$, which is zero.

(3) Let $$u_n(x)=\frac{x^{n-1}}{(n-1)!}+n^2xe^{-n^2x^2}-(n-1)^2xe^{-(n-1)^2x^2};$$
we find $s(0)=1$, and $s(x)=e^x$, for $|x|>0$.

We have $\displaystyle\int_0^x s(x)\,dx=e^x-1$. Also $\displaystyle\lim_{n\sim\infty}\int_0^x s_n(x)\,dx$ is discontinuous at the point $x=0$, which is a point at which the measure of non-uniform convergence is infinite; it converges to $e^x-\frac{1}{2}$ if $x>0$, and to zero if $x=0$.

(4) Let $$u_n(x)=\frac{k_n\phi_n'(x)}{1+\{\phi_n(x)\}^2}-\frac{k_{n+1}\phi_{n+1}'(x)}{1+\{\phi_{n+1}(x)\}^2},$$

where k_n is a function of n, and $\phi_n(x)$, $\phi_n'(x)$ are finite and continuous in the interval (a, b), and vanish for $x=a$. Further let it be assumed that $\phi_n(x)$, $\phi_n'(x)$ increase indefinitely with n, for every value of x except a, but so that $\lim_{n\sim\infty}u_n(x)$ is zero.

We have $$\int_a^x s_n(x)\,dx=-k_{n+1}\tan^{-1}\{\phi_{n+1}(x)\}+k_1\tan^{-1}\{\phi_1(x)\},$$
$$\int_a^x s(x)\,dx=k_1\tan^{-1}\{\phi_1(x)\};$$

the second integral and the limit of the first are not identical unless

$$k_{n+1}\tan^{-1}\{\phi_{n+1}(x)\}$$

has the limit zero. If $$\phi_n(x)=h_n(x-a)^2,$$
where h_n is positive and increases indefinitely with n, we have

$$\lim k_{n+1}\tan^{-1}\{\phi_{n+1}(x)\}=\tfrac{1}{2}\pi\lim k_{n+1}.$$

Hence, if $\lim k_{n+1}$ have a finite value, the two expressions have different finite values; if

it appears that a finite polynomial $P_n(x)$ can be so determined that $|\phi_n(x) - P_n(x)| < \epsilon_n$, at all points of Δ; it then follows that

$$\left|\int_{(E)}\{\phi_n(x) - P_n(x)\}\,dx\right| \leq \int_{(E)}|\phi_n(x) - P_n(x)|\,dx \leq \epsilon_n m(\Delta).$$

It is now seen that we have the following theorem:

If $f(x)$ be any function, summable in the p-dimensional cell Δ, a sequence $\{P_n(x)\}$ of finite polynomials can be so determined that $\int_{(E)} P_n(x)\,dx$ converges to $\int_{(E)} f(x)\,dx$, as $n \sim \infty$, for every measurable set of points E, contained in Δ, and uniformly with respect to all such sets.

The theorem of I, § 430, has been applied in various cases to extend properties of the integrals of continuous functions to the case of L-integrals in general. It now appears that such extensions can be made by starting from the simplest possible case, that of the integral of a finite polynomial.

THE OSCILLATIONS OF A SEQUENCE OF INTEGRALS

221. Some important properties will be given of a sequence of integrals $\int_a^x f_n(x)\,dx$, where the sequence $\{f_n(x)\}$, in general non-convergent, is defined in an interval (a, b). The theory has been fully investigated* by W. H. Young. We shall denote by $\bar{f}(x)$ and $\underline{f}(x)$ the upper and lower functions of the sequence $\{f_n(x)\}$, and by $\bar{F}(x)$, $\underline{F}(x)$ the upper and lower functions of the sequence $\{F_n(x)\}$, where $F_n(x)$ denotes the integral

$$\int_a^x f_n(x)\,dx.$$

It will be shewn that:

If $f_n(x)$ has a finite lower boundary with respect to (n, x), and $\{F_n(x)\}$ is such that at no point is $\bar{F}(x) = \underline{F}(x) = +\infty$, then

$$\underline{F}(x) \geqq \int_a^x \underline{f}(x)\,dx.$$

Similarly, if $f_n(x)$ has a finite upper boundary with respect to (n, x), and $\{F_n(x)\}$ satisfies the condition that at no point is $\bar{F}(x) = \underline{F}(x) = -\infty$, then

$$\bar{F}(x) \leqq \int_a^x \bar{f}(x)\,dx.$$

It will be sufficient to prove the first part of the theorem.

Let $w_n(x)$ be the function which, at each point x, has the value of the lower boundary of the sequence $f_n(x), f_{n+1}(x), \ldots$, at that point. Then $\{w_n(x)\}$ is a monotone non-diminishing sequence which converges to $\underline{f}(x)$. Since $w_n(x)$ has a finite lower boundary, and is $\leqq$ the summable function

* *Proc. Lond. Math. Soc.* (2), vol. IX (1910), p. 286: *ibid.* (2), vol. XI (1912), p. 43.

$f_n(x)$, it is summable, and $\int_a^x w_n(x)\,dx \leq \int_a^x f_n(x)\,dx$; and from this it follows that $\lim\limits_{n\sim\infty}\int_a^x w_n(x)\,dx \leq \underline{F}(x)$. Since $\underline{F}(x)$ is finite, and $\{w_n(x)\}$ monotone, it follows from the theorem of § 213 that $\{w_n(x)\}$ is an integrable sequence, and thus that $\underline{f}(x)$ is summable, and that

$$\lim_{n\sim\infty}\int_a^x w_n(x)\,dx = \int_a^x \underline{f}(x)\,dx;$$

therefore

$$\underline{F}(x) \geqq \int_a^x \underline{f}(x)\,dx.$$

It will next be proved that:

If $\underline{F}(x) \geqq \int_a^x \underline{f}(x)\,dx$, then $\overline{F}(x)$ and $\underline{F}(x)$ are upper semi-integrals; and if $\overline{F}(x) \leqq \int_a^x \overline{f}(x)\,dx$, then $\overline{F}(x)$ and $\underline{F}(x)$ are lower semi-integrals. In the first case it is assumed that $f_n(x)$ is bounded below, and in the second case that it is bounded above, with respect to (n, x).

An upper semi-integral has been defined in I, § 407, as the sum of an integral and a monotone non-diminishing function; a lower semi-integral is the sum of an integral and a monotone non-increasing function.

It will be sufficient to prove the first statement in the theorem.

Since

$$F_n(x+h) = F_n(x) + \int_x^{x+h} f_n(x)\,dx,$$

we have

$$\overline{\lim_{n\sim\infty}}\, F_n(x+h) \geqq \overline{\lim_{n\sim\infty}}\, F_n(x) + \underline{\lim_{n\sim\infty}}\int_x^{x+h} f_n(x)\,dx,$$

or

$$\underline{\lim_{n\sim\infty}}\int_x^{x+h} f_n(x)\,dx \leqq \overline{F}(x+h) - \overline{F}(x).$$

Similarly, it may be shewn that the lower limit on the left-hand side is $\leqq \underline{F}(x+h) - \underline{F}(x)$.

Employing the last theorem, we have

$$\int_x^{x+h} \underline{f}(x)\,dx \leqq \overline{F}(x+h) - \overline{F}(x)$$
$$\leqq \underline{F}(x+h) - \underline{F}(x).$$

We have thus

$$\int_a^{x+h} \underline{f}(x)\,dx - \overline{F}(x+h) \leqq \int_a^x \underline{f}(x)\,dx - \overline{F}(x),$$

and

$$\int_a^{x+h} \underline{f}(x)\,dx - \underline{F}(x+h) \leqq \int_a^x \underline{f}(x)\,dx - \underline{F}(x).$$

Hence

$$\int_a^x \underline{f}(x)\,dx - \overline{F}(x), \quad \int_a^x \underline{f}(x)\,dx - \underline{F}(x)$$

are both monotone non-increasing functions, and therefore $\overline{F}(x)$, $\underline{F}(x)$ are upper semi-integrals.

222. The following theorem* is independent of the supposition that $f(x)$ is a summable function:

If $\underline{\lim}_{m^*(e)\sim 0,\, n\sim\infty} \int_{(e)} f_n(x)\,dx \geqq 0$, *where e denotes any measurable set of points in the interval (a, b), then the sequence $\{F_n(x)\}$ is bounded below, and oscillates continuously and homogeneously below. Also if any subsequence $\{F_{n_p}(x)\}$ is convergent, its limiting function is an upper semi-integral, and also a lower semi-continuous function, and is consequently everywhere continuous on the left.*

We have $F_n(x+h) = F_n(x) + \int_x^{x+h} f_n(x)\,dx$; and as $n \sim \infty$ and $h \sim 0$, the second integral on the right has its lower double limit non-negative. Hence, if $\chi(x)$ denote the chasm function of the sequence $\{F_n(x)\}$, we have $\chi(x) \geqq \underline{F}(x)$. The functions $F_n(x)$ being all continuous, we have

$$\underline{F}(x) = \underline{\lim}_{n\sim\infty} \lim_{h\sim 0} F_n(x+h) \geqq \underline{\lim}_{n\sim\infty,\, h\sim 0} F_n(x+h) \geqq \chi(x).$$

It follows that $\underline{F}(x) = \chi(x)$, and thus the sequence oscillates continuously below; since the argument may be applied to any sub-sequence of $\{F_n(x)\}$, the continuous oscillation below is homogeneous.

To prove that $F_n(x)$ is bounded below in relation to (n, x), positive numbers ϵ, α, and an integer n_1 can be so determined that $\int_{(e)} f_n(x)\,dx > -\epsilon$, provided $m(e) < \alpha$, $n > n_1$.

If E be any measurable set of points in (a, b), the interval may be divided into r equal parts, each of length $< \alpha$; and thus the part of E in each of these sub-intervals has its measure $< \alpha$; it follows that

$$\int_{(E)} f_n(x)\,dx > -r\epsilon,$$

and in particular that $F_n(x) > -r\epsilon$, for $n > n_1$. Hence $F_n(x)$ is bounded below, for $n > n_1$, and therefore for all values of n.

Taking e to consist of a finite set of non-overlapping intervals (x_{s-1}, x_s), we have $\Sigma\{F_n(x_s) - F_n(x_{s-1})\} > -\epsilon$, for $n > n_1$, provided the measure of the set of intervals is $< \alpha$. It follows that, n being confined to have those values which it has in the sub-sequence $\{F_{n_p}(x)\}$ that converges to the unique function $F(x)$, we have

$$\Sigma\{F(x_s) - F(x_{s-1})\} \geqq -\epsilon.$$

This must also hold when the set of intervals is infinite, provided its measure is $< \alpha$.

To prove that the function $F(x)$ is of bounded variation, we observe that, if it be not of bounded variation, there must be at least one of the r parts of (a, b) each of which has measure $< \alpha$, in which the total variation

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. XI (1912), p. 51.

of $F(x)$ is infinite. This sub-interval (a', b') may be divided into a number of parts such that the sum of the absolute variations in those parts exceeds a number N, as large as we please. The sum of the negative variations over those parts is numerically less than ϵ; hence the sum of the positive variations exceeds $N - \epsilon$, which is impossible, if $N - \epsilon$ is sufficiently large, because the sum of the variations, each taken with its proper sign, is $F(b') - F(a')$. Therefore $F(x)$ is of bounded variation, and as in I, § 243, may be expressed by $P(x) - N(x)$, where $P(x)$ is the upper boundary of the positive variations over the meshes of all nets fitted on to (a, x), and $N(x)$ is the upper boundary of the numerical values of the negative variations. The functions $P(x)$, $N(x)$ are monotone non-diminishing; and it can be seen that $N(x)$, and consequently $-N(x)$, is an integral. For the sum of the variations of $N(x)$ over every finite or infinite set of non-overlapping intervals of which the measure is $< \alpha$ is $< \epsilon$; since, to each value of ϵ, there corresponds a value of α, it follows that $N(x)$ is absolutely continuous, and is therefore (see I, § 406) an integral. The function $F(x)$ being the sum of an integral and a bounded monotone increasing function, is an upper semi-integral. That $F(x)$ is lower semi-continuous follows from a theorem given in § 117, that the limit of a convergent sequence of continuous functions which oscillate continuously below is lower semi-continuous.

It follows from the theorem just established, and from the corresponding theorem for the case $\overline{\lim}_{m(e)\sim 0,\, n\sim\infty} \int_{(e)} f_n(x)\,dx \leq 0$, employing the theorem in § 123, that:

When $\lim_{m(e)\sim 0,\, n\sim\infty} \int_{(e)} f_n(x)\,dx = 0$, *the sequence* $\{F_n(x)\}$ *oscillates continuously and homogeneously, and there is in every sub-sequence of* $\{F_n(x)\}$, *a sub-sequence which converges uniformly to an integral.*

223. The following theorem, given* by W. H Young, is of use in the theory of series:

If $\{f_n(x)\}$ *is a sequence of non-negative functions, such that* $\int_a^x f_n(x)\,dx$ *forms a sequence* $\{F_n(x)\}$ *which oscillates boundedly, there is in every sub-sequence of* $\{F_n(x)\}$, *a sub-sequence which converges to a lower semi-continuous function which is an upper semi-integral.*

If the sequence $\{f_n(x)\}$ is bounded below, the theorem clearly also holds good. From the theorems given in § 221, it follows, since $\{f_n(x)\}$ is bounded below, with respect to (n, x), and $F_n(x)$ is bounded above, that $\overline{F}(x)$, $\underline{F}(x)$ are upper semi-integrals. Since this reasoning is applicable to any sub-sequence, it follows that all the upper functions and all the lower functions of the sequence $\{F_n(x)\}$ are upper semi-integrals.

* *Proc. Roy. Soc.* vol. LXXXVIII (1913), p. 571.

Since $f_n(x)$ is non-negative, it follows that

$$\lim_{n\sim\infty,\, \overline{m}(e)\sim 0} \int_{(e)} f_n(x)\, dx \geqq 0,$$

and we see, in accordance with the theorem given in § 222, that $\{F_n(x)\}$ oscillates continuously and homogeneously below. Hence all the lower functions of the sequence $\{F_n(x)\}$ are lower semi-continuous (see § 117). Since they are also upper semi-integrals, that is each is the sum of an integral and a monotone non-diminishing function, it follows that all these lower semi-continuous functions are therefore continuous on the left.

If all the upper or all the lower functions of a sequence are continuous on one side at least, the same side for all, then a sub-sequence of the functions exists which is convergent (see § 122). This sub-sequence satisfies the conditions of the theorem.

THE LIMIT OF AN INTEGRAL CONTAINING A PARAMETER

224. If E be a measurable set of points x, of any number of dimensions, and $f(x, y)$ is a function which is summable in E, for all values of the parameter y, contained in some finite, or infinite, linear interval, it is of importance to possess criteria for the convergence to a limit, of $\int_{(E)} f(x, y)\, dx$, as y converges to some value y_0, which may be finite or infinite. More generally there may be an exceptional set of values of y in the linear interval for which $f(x, y)$ is not summable. This exceptional set may be throughout disregarded, even if it be everywhere dense in the interval. Such convergence differs from the convergence of a sequence $\int_{(E)} f_n(x)\, dx$, as $n \sim \infty$, considered in §§ 201–213, only in the respect that the parameter y, approaches its limit y_0, or ∞, through a continuous (or at least unenumerable) set of values, whereas the parameter n is confined to have the values of the integer sequence. It will appear that the criteria obtained in §§ 201–213, have their analogues in the more general case here considered, in which the parameter has values in a continuous linear interval. It is sufficient to assume that y is confined to an interval $y_0 < y \leqq y + a$, on one side of the point y_0, or, in case y_0 is infinite, to the interval $A < y$. When y may have values both greater and less than y_0, the limits on the two sides of y_0 may then be considered separately.

Let E, in the first instance, have finite measure, and let it be assumed that, at each point x, of E, the limit $f(x, y_0 + 0)$, or $f(x, \infty)$, has a definite value. If, at points of a component of E, of measure zero, this condition is not satisfied, this exceptional set may be throughout disregarded. If ϵ be an arbitrarily chosen positive number, let e_h denote the set of points of E at which $| f(x, y) - f(x, y_0 + 0) | \leqq \epsilon$, for all the values

of y such that $y_0 < y \leq y + h$. In case y_0 is $+\infty$, e_A may be taken to denote the set of points for which $|f(x, y) - f(x, \infty)| \leq \epsilon$, provided $y \geqq A$. If $h > h'$, e_h is contained in $e_{h'}$; and if $A' > A$, e_A is contained in $e_{A'}$. Thus $m(e_h)$ is monotone non-diminishing as $h \sim 0$; and $m(e_A)$ is monotone non-diminishing as $A \sim \infty$. It can be shewn that $m(e_h)$ converges to $m(E)$, as $h \sim 0$; and that $m(e_A)$ converges to $m(E)$, as $A \sim \infty$. For if

$$\lim_{h\sim 0} m(E - e_h) = k\ (> 0),$$

a sequence $\{h_n\}$ of values of h converging to zero could be so determined that $\lim\limits_{n\sim\infty} m(E - e_{h_n}) = k$; there would then exist points ξ common to an infinite number of the sets $E - e_{h_n}$; and at such a point ξ, we should have $|f(\xi, y_0 + h_n) - f(x, y_0 + 0)| > \epsilon$ for an infinite set of values of n; and this is inconsistent with the existence of the limit $f(x, y_0 + 0)$. It thus follows that $\lim\limits_{h\sim 0} m(E - e_h) = 0$, or $m(E) = \lim\limits_{h\sim 0} m(e_h)$. In a similar manner, it is proved that $m(E) = \lim\limits_{A\sim\infty} m(e_A)$.

225. The following criterion can now be established:

If E be a measurable set of points, of any number of dimensions, of finite, or of infinite, measure, and if $f(x, y)$ be summable in E, for values of y in some interval $y_0 < y \leq y_0 + \alpha$; or in $y \geqq \alpha$, and if, for all (or almost all) values of x, the limit $f(x, y_0 + 0)$, or $f(x, \infty)$, exists, it is sufficient in order that $\int_{(E)} f(x, y_0 + 0)\, dx$, or $\int_{(E)} f(x, \infty)\, dx$, may exist and be equal to $\lim\limits_{y\sim y_0} \int_{(E)} f(x, y)\, dx$, or to $\lim\limits_{y\sim\infty} \int_{(E)} f(x, y)\, dx$, that a non-negative function $\phi(x)$, summable in E should exist, such that $|f(x, y)| \leq \phi(x)$, for all values of x, in E, and the values of y in the interval $y_0 < y \leq y_0 + \alpha$, or in $y \geqq \alpha$.

It is clear that $|f(x, y_0 + 0)| \leq \phi(x)$, or that $|f(x, \infty)| \leq \phi(x)$, and thus that $f(x, y_0 + 0)$, or $f(x, \infty)$, is summable in E. Let $m(E)$ be, in the first instance, finite, then we have

$$\left|\int_{(E)} \{f(x, y_0 + 0) - f(x, y)\}\, dx\right| \leq \int_{(e_h)} |f(x, y_0 + 0) - f(x, y)|\, dx$$
$$+ \int_{(E - e_h)} |f(x, y_0 + 0) - f(x, y)|\, dx < \epsilon m(e_h) + 2\int_{(E - e_h)} \phi(x)\, dx$$
$$< \epsilon m(E) + \epsilon,$$

provided h have a sufficiently small value.

Since ϵ is arbitrary, it follows that

$$\int_{(E)} f(x, y_0 + 0)\, dx = \lim_{y\sim y_0} \int_{(E)} f(x, y)\, dx.$$

The case in which y_0 is infinite can be treated in a precisely similar manner, so that

$$\int_{(E)} f(x, \infty)\, dx = \lim_{y\sim\infty} \int_{(E)} f(x, y)\, dx.$$

Next, let $m(E)$ be infinite; a part E_1 of E, such that $m(E_1)$ is finite, can be so determined that $\int_{(E-E_1)} \phi(x)\,dx$ is less than ϵ. We have then

$$\left|\int_{(E)} \{f(x, y_0+0) - f(x, y)\}\,dx\right| \leqq \left|\int_{(E_1)} \{f(x, y_0+0) - f(x, y)\}\,dx\right| + 2\epsilon \leqq 3\epsilon,$$

if y is sufficiently near to y_0. It then follows that

$$\int_{(E)} f(x, y_0+0)\,dx = \lim_{y \sim y_0} \int_{(E)} f(x, y)\,dx;$$

and similarly, it is seen that

$$\int_{(E)} f(x, \infty)\,dx = \lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx.$$

If in E there exists a set of points of measure zero, at which the limit $f(x, y_0+0)$, or $f(x, \infty)$, does not exist, it makes no difference in the application of the theorem, because the omission of this set of points from E does not affect the values of the integrals.

As in § 202, the criterion may be expressed as follows:

If $f(x, y)$ is absolutely summable in the measurable set E, of finite or infinite measure, for values of y in an interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$, and if $f(x, y)$ converges everywhere in E (or almost everywhere) to $f(x, y_0)$, or to $f(x, \infty)$, as the case may be; and if a non-negative function $\psi(x)$, summable in E, exists such that $|f(x, y) - f(x, y_0)|$, or $|f(x, y) - f(x, \infty)|$ is $\leqq \psi(x)$, then $\int_{(E)} f(x, y_0)\,dx$, or $\int_{(E)} f(x, \infty)\,dx$, exists, and is equal to

$$\lim_{y \sim y_0} \int_{(E)} f(x, y)\,dx, \text{ or to } \lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx.$$

For $|f(x, y_0)| \leqq |f(x, y) - f(x, y_0)| + |f(x, y)| \leqq \psi(x) + |f(x, y)|$, and therefore $f(x, y_0)$ is absolutely summable in E. Moreover

$$|f(x, y)| \leqq \psi(x) + |f(x, y_0)|,$$

which is a summable non-negative function, corresponding to $\phi(x)$.

In case E has finite measure, we obtain particular cases of the above criteria by taking $\phi(x)$, $\psi(x)$ constant, and equal to K. Thus we obtain the following:

If E be a set of points of finite measure, in any number of dimensions, and if $|f(x, y)| \leqq K$, for values of x in some interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$; and if for all (or almost all) values of x, the limit $f(x, y_0+0)$, or $f(x, \infty)$, exists, then $\int_{(E)} f(x, y_0+0)\,dx = \lim_{y \sim y_0} \int_{(E)} f(x, y)\,dx$, or

$$\int_{(E)} f(x, \infty)\,dx = \lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx.$$

If $f(x, y)$ be summable in the measurable set E, of finite measure, for all values of y in an interval $y_0 < y \leqq y_0 + a$, or $y \geqq a$; and if $f(x, y)$ converges everywhere (or almost everywhere) in E to $f(x, y_0)$, or to $f(x, \infty)$, and

$$|f(x, y) - f(x, y_0)|,$$

or $|f(x, y) - f(x, \infty)|$, is $\leqq K$, then $\int_{(E)} f(x, y_0)\,dx$, or $\int_{(E)} f(x, \infty)\,dx$, exists, and is equal to $\lim_{y \sim y_0} \int_{(E)} f(x, y)\,dx$, or to $\lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx$. This includes as a special case the condition that $f(x, y)$ should converge uniformly to $f(x, y_0)$, or to $f(x, \infty)$.

226. The following criterion can be deduced from that given in § 225:

If $f(x, y)$ be defined in the measurable set E, of finite, or of infinite, measure, for values of y in an interval $y_0 < y \leqq y_0 + a$, or $y \geqq a$, and if $f(x, y)$ be, for all x in E, monotone non-diminishing (or non-increasing) with respect to y, in the interval, and $|f(x, y_0 + a)|$, or $|f(x, a)|$, is summable in E, then $\int_{(E)} f(x, y_0 + 0)\,dx$ and $\lim_{y \sim y_0} \int_{(E)} f(x, y)\,dx$ are either both finite and equal, or else both are infinite. The same statement applies to $\int_{(E)} f(x, \infty)\,dx$ and $\lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx$.

The values of y considered may either be all those in the interval $y_0 < y \leqq y_0 + a$, or $y \geqq a$, or else they may be those corresponding to any set of points in the interval, of which y_0, or ∞, is a limiting point.

The proof will be given for the case $y_0 = \infty$; only a very slight modification is required to apply to the case in which y_0 is finite.

Since $f(x, \infty) = \{f(x, \infty) - f(x, a)\} + f(x, a)$ and $f(x, \infty) - f(x, a)$ is of fixed sign for all points x, in E, it follows that, when

$$\int_{(E)} \{f(x, \infty) - f(x, a)\}\,dx$$

is finite, so is $\int_{(E)} f(x, \infty)\,dx$, and when the first is infinite, so is the second. Since $|f(x, \infty)| \leqq |f(x, a)| + |f(x, \infty) - f(x, a)|$, it follows that $|f(x, \infty)|$ is summable in E if $|f(x, \infty) - f(x, a)|$ is summable in E; for by hypothesis $|f(x, a)|$ is summable in E. Since

$$|f(x, y) - f(x, \infty)| \leqq |f(x, a) - f(x, \infty)| \leqq |f(x, a)| + |f(x, \infty)|,$$

it follows that when $|f(x, \infty)|$ is summable in E, if y is in the interval $y > a$, $|f(x, y) - f(x, \infty)|$ is less than a non-negative function, summable in E. Thus the condition of the theorem of § 225 is satisfied, and consequently $\int_{(E)} f(x, \infty)\,dx = \lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx$. If $\lim_{y \sim \infty} \int_{(E)} f(x, y)\,dx = \infty$, then for all sufficiently large values of y, $\int_{(E)} f(x, y)\,dx$ is greater than an arbitrarily

Next, let $m(E)$ be infinite; a part E_1 of E, such that $m(E_1)$ is finite, can be so determined that $\int_{(E-E_1)} \phi(x)\,dx$ is less than ϵ. We have then

$$\left|\int_{(E)} \{f(x, y_0+0) - f(x,y)\}\,dx\right| \leqq \left|\int_{(E_1)} \{f(x, y_0+0) - f(x,y)\}\,dx\right| + 2\epsilon \leqq 3\epsilon,$$

if y is sufficiently near to y_0. It then follows that

$$\int_{(E)} f(x, y_0+0)\,dx = \lim_{y \sim y_0} \int_{(E)} f(x,y)\,dx;$$

and similarly, it is seen that

$$\int_{(E)} f(x, \infty)\,dx = \lim_{y\sim\infty} \int_{(E)} f(x,y)\,dx.$$

If in E there exists a set of points of measure zero, at which the limit $f(x, y_0+0)$, or $f(x,\infty)$, does not exist, it makes no difference in the application of the theorem, because the omission of this set of points from E does not affect the values of the integrals.

As in § 202, the criterion may be expressed as follows:

If $f(x,y)$ is absolutely summable in the measurable set E, of finite or infinite measure, for values of y in an interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$, and if $f(x,y)$ converges everywhere in E (or almost everywhere) to $f(x, y_0)$, or to $f(x,\infty)$, as the case may be; and if a non-negative function $\psi(x)$, summable in E, exists such that $|f(x,y) - f(x,y_0)|$, or $|f(x,y) - f(x,\infty)|$ is $\leqq \psi(x)$, then $\int_{(E)} f(x,y_0)\,dx$, or $\int_{(E)} f(x,\infty)\,dx$, exists, and is equal to

$$\lim_{y\sim y_0} \int_{(E)} f(x,y)\,dx, \textit{ or to } \lim_{y\sim\infty} \int_{(E)} f(x,y)\,dx.$$

For $\quad |f(x,y_0)| \leqq |f(x,y) - f(x,y_0)| + |f(x,y)| \leqq \psi(x) + |f(x,y)|$, and therefore $f(x,y_0)$ is absolutely summable in E. Moreover

$$|f(x,y)| \leqq \psi(x) + |f(x,y_0)|,$$

which is a summable non-negative function, corresponding to $\phi(x)$.

In case E has finite measure, we obtain particular cases of the above criteria by taking $\phi(x)$, $\psi(x)$ constant, and equal to K. Thus we obtain the following:

If E be a set of points of finite measure, in any number of dimensions, and if $|f(x,y)| \leqq K$, for values of x in some interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$; and if for all (or almost all) values of x, the limit $f(x, y_0+0)$, or $f(x,\infty)$, exists, then $\int_{(E)} f(x,y_0+0)\,dx = \lim_{y\sim y_0} \int_{(E)} f(x,y)\,dx$, or

$$\int_{(E)} f(x,\infty)\,dx = \lim_{y\sim\infty} \int_{(E)} f(x,y)\,dx.$$

If $f(x, y)$ be summable in the measurable set E, of finite measure, for all values of y in an interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$; and if $f(x, y)$ converges everywhere (or almost everywhere) in E to $f(x, y_0)$, or to $f(x, \infty)$, and

$$|f(x, y) - f(x, y_0)|,$$

or $|f(x, y) - f(x, \infty)|$, is $\leqq K$, then $\int_{(E)} f(x, y_0)\, dx$, or $\int_{(E)} f(x, \infty)\, dx$, exists, and is equal to $\lim_{y \sim y_0} \int_{(E)} f(x, y)\, dx$, or to $\lim_{y \sim \infty} \int_{(E)} f(x, y)\, dx$. This includes as a special case the condition that $f(x, y)$ should converge uniformly to $f(x, y_0)$, or to $f(x, \infty)$.

226. The following criterion can be deduced from that given in § 225:

If $f(x, y)$ be defined in the measurable set E, of finite, or of infinite, measure, for values of y in an interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$, and if $f(x, y)$ be, for all x in E, monotone non-diminishing (or non-increasing) with respect to y, in the interval, and $|f(x, y_0 + \alpha)|$, or $|f(x, \alpha)|$, is summable in E, then $\int_{(E)} f(x, y_0 + 0)\, dx$ and $\lim_{y \sim y_0} \int_{(E)} f(x, y)\, dx$ are either both finite and equal, or else both are infinite. The same statement applies to $\int_{(E)} f(x, \infty)\, dx$ and $\lim_{y \sim \infty} \int_{(E)} f(x, y)\, dx$.

The values of y considered may either be all those in the interval $y_0 < y \leqq y_0 + \alpha$, or $y \geqq \alpha$, or else they may be those corresponding to any set of points in the interval, of which y_0, or ∞, is a limiting point.

The proof will be given for the case $y_0 = \infty$; only a very slight modification is required to apply to the case in which y_0 is finite.

Since $f(x, \infty) = \{f(x, \infty) - f(x, \alpha)\} + f(x, \alpha)$ and $f(x, \infty) - f(x, \alpha)$ is of fixed sign for all points x, in E, it follows that, when

$$\int_{(E)} \{f(x, \infty) - f(x, \alpha)\}\, dx$$

is finite, so is $\int_{(E)} f(x, \infty)\, dx$, and when the first is infinite, so is the second. Since $|f(x, \infty)| \leqq |f(x, \alpha)| + |f(x, \infty) - f(x, \alpha)|$, it follows that $|f(x, \infty)|$ is summable in E if $|f(x, \infty) - f(x, \alpha)|$ is summable in E; for by hypothesis $|f(x, \alpha)|$ is summable in E. Since

$$|f(x, y) - f(x, \infty)| \leqq |f(x, \alpha) - f(x, \infty)| \leqq |f(x, \alpha)| + |f(x, \infty)|,$$

it follows that when $|f(x, \infty)|$ is summable in E, if y is in the interval $y > \alpha$, $|f(x, y) - f(x, \infty)|$ is less than a non-negative function, summable in E. Thus the condition of the theorem of § 225 is satisfied, and consequently $\int_{(E)} f(x, \infty)\, dx = \lim_{y \sim \infty} \int_{(E)} f(x, y)\, dx$. If $\lim_{y \sim \infty} \int_{(E)} f(x, y)\, dx = \infty$, then for all sufficiently large values of y, $\int_{(E)} f(x, y)\, dx$ is greater than an arbitrarily

chosen positive number N. In this case $f(x, y)$ is non-diminishing as y is increased, for all values of x in E; it therefore follows that $\int_{(E)} f(x, \infty)\, dx > N$. Since N is arbitrary, $\int_{(E)} f(x, \infty)\, dx$ is infinite, of the same sign as

$$\lim_{y \sim \infty} \int_{(E)} f(x, y)\, dx.$$

The case in which $f(x, y)$ is non-increasing can be treated in the same manner, the integral $\int_{(E)} f(x, \infty)\, dx$ then having the value $-\infty$.

227. From the criteria obtained that $\int_{(E)} f(x, y)\, dx$ may be continuous at a point, criteria are immediately deducible that the integral should be continuous in a finite, or infinite, interval of y. Thus we obtain the following criteria:

If, in an interval (α, β), of y, we have $|f(x, y)| \leq \phi(x)$, where $\phi(x)$ is a non-negative function, summable in the measurable set E, of finite, or infinite, measure, and if $f(x, y)$ be continuous with respect to y in (α, β), then $\int_{(E)} f(x, y)\, dx$ is continuous in any interval of y, interior to (α, β). If $\beta = \infty$, the integral is continuous in the interval (α', ∞), where $\alpha' > \alpha$.

In applying this theorem, $\phi(x)$ may be taken to be the maximum of $|f(x, y)|$ in the interval (α, β), of y.

If E have finite measure, and $|f(x, y)| \leq K$, in an interval (α, β), of y, and $f(x, y)$ be continuous in (α, β), with respect to y, then $\int_{(E)} f(x, y)\, dx$ is continuous in any interval interior to (α, β). If $\beta = \infty$, it is continuous in (α', ∞), where $\alpha' > \alpha$.

If $|f(x, y)|$ be summable in the measurable set E, of finite, or infinite measure, for all values of y in an interval (α, β), and $f(x, y)$ be for all values of x either monotone non-increasing, or monotone non-diminishing, and continuous with respect to y in the closed interval (α, β), then $\int_{(E)} f(x, y)\, dx$ is continuous in any interval interior to (α, β). If $\beta = \infty$, the integral is continuous in (α', ∞), where $\alpha' > \alpha$.

Theorems relating to cases in which $f(x, y)$ has discontinuities with respect to y have been given* by Hardy.

228. In the case of an integral $\int_a^\infty f(x, y)\, dx$, over the linear interval (a, ∞), the following criterion is of use:

If, in every finite interval (a, C), where $C > a$, the condition

$$\int_a^C f(x, y_0 + 0)\, dx = \lim_{y \sim y_0} \int_a^C f(x, y)\, dx$$

* *Quarterly Journal*, vol. XXXIV (1903), p. 28.

is satisfied, and if, corresponding to an arbitrarily fixed positive number ϵ, *a number* C ($> a$) *can be determined, and also a value* y_1, ($> y_0$), *of* y, *for which* $\left|\int_C^{C'} f(x, y)\,dx\right| < \epsilon$, *for every value of* C' ($> C$), *and for every value of* y *such that* $y_0 < y \leqq y_1$, *then* $\int_a^\infty f(x, y_0 + 0)\,dx$ *exists, and is equal to* $\lim\limits_{y \sim y_0}\int_a^\infty f(x, y)\,dx$. *This criterion holds good also when* $y_0 = \infty$, *in which case* $\left|\int_C^{C'} f(x, y)\,dx\right| < \epsilon$, *for every value of* C' ($> C$) *and for every value of* y *which is* $\geqq y_1$.

It will be observed that, in this theorem, no restriction is placed upon the nature of the integrals.

We have $\int_a^\infty \{f(x, y) - f(x, y_0 + 0)\}\,dx$

$$= \int_a^C \{f(x, y) - f(x, y_0 + 0)\}\,dx + \int_C^\infty f(x, y)\,dx - \int_C^\infty f(x, y_0 + 0)\,dx.$$

If C be sufficiently large, since, for $y_0 < y \leqq y_1$,

$$\int_C^\infty f(x, y)\,dx = \lim_{C'' \sim \infty}\int_C^{C''} f(x, y)\,dx,$$

we have

$$\left|\int_C^\infty f(x, y)\,dx\right| \leqq \epsilon.$$

Also $\int_C^{C'} f(x, y_0 + 0)\,dx = \lim\limits_{y \sim y_0}\int_C^{C'} f(x, y)\,dx$, hence $\left|\int_C^{C'} f(x, y_0 + 0)\,dx\right| \leqq \epsilon$, for all values of C', and thus $\left|\int_C^\infty f(x, y_0 + 0)\,dx\right| \leqq \epsilon$. Also, if y be sufficiently near to y_0, we have $\left|\int_a^C \{f(x, y) - f(x, y_0 + 0)\}\,dx\right| < \epsilon$. Hence, if y is sufficiently near to y_0, we have $\left|\int_a^\infty \{f(x, y) - f(x, y_0 + 0)\}\,dx\right| < 3\epsilon$; and thus the theorem is established. Only a slight modification is required for the case in which y_0 is infinite.

An alternative to the above criterion is the following:

If, in every finite interval (a, C), *where* $C > a$, *the condition*

$$\int_a^C f(x, y_0 + 0)\,dx = \lim_{y \sim y_0}\int_a^C f(x, y)\,dx,$$

is satisfied, and if $\lim\limits_{y \sim y_0}\int_a^\infty f(x, y)\,dx$ *exists, and also* $\lim\limits_{y \sim y_0}\int_a^C f(x, y)\,dx$ *converges to the value* $\lim\limits_{y \sim y_0}\int_a^\infty f(x, y)\,dx$, *when* C *is indefinitely increased, these conditions are sufficient to ensure that* $\int_a^\infty f(x, y_0 + 0)\,dx$ *exists, and is equal to* $\lim\limits_{y \sim y_0}\int_a^\infty f(x, y)\,dx$. *The case in which* $y_0 = \infty$ *is included.*

In order that $\int_a^\infty f(x, y)\,dx$ may be continuous on the right, at y_0, the additional condition must be satisfied that $f(x, y_0+0) = f(x, y_0)$, or more generally that $f(x, y_0+0) - f(x, y_0)$ should be an integrable null-function in an arbitrary interval of x.

229. In case the integrals $\int_a^b f(x, y)\,dx$, for values of y such that $y_0 < y \leqq y_0 + \alpha$, are not necessarily L-integrals, but may, for some or all such values of y exist as D-integrals, or as HL-integrals, we may apply the result of § 216 to obtain a set of sufficient conditions for the equality of $\int_a^b f(x, y_0+0)\,dx$ and $\lim\limits_{y\sim y_0}\int_a^b f(x, y)\,dx$. The case in which y_0 is infinite may be obtained by a slight modification of the statement of the following theorem:

If $f(x, y)$ converges to a definite limit $f(x, y_0+0)$, for all points x, of the interval (a, b) which do not belong to a closed enumerable set G, and the functions $f(x, y)$, for $y_0 < y \leqq y_0 + \alpha$, satisfy the conditions (1), *that, in any interval (α_1, β_1), contained in (a, b) and interior to an interval contiguous to G, $|f(x, y)|$ bounded with respect to (x, y), and* (2), *that $\int_a^b f(x, y)\,dx$ exists either as an L-integral, an HL-integral, or a D-integral, for each value of y such that $y_0 < y \leqq y + \alpha$; and* (3), *that $\lim\limits_{y\sim y_0}\int_a^x f(x, y)\,dx$, for $a \leqq x \leqq b$ is convergent and represents a continuous function of x; and* (4), *$\int_a^b f(x, y_0+0)\,dx$ exists as an L-integral, an HL-integral, or a D-integral; then the equality $\int_a^b f(x, y_0+0)\,dx = \lim\limits_{y\sim y_0}\int_a^b f(x, y)\,dx$ holds good.*

In order to deduce this theorem from that of § 216, it is sufficient to choose a sequence of values of y converging to y_0.

EXAMPLES

(1) If $y>0$, we have $\int_0^\infty \frac{\sin yx}{x}\,dx = \frac{1}{2}\pi$, but when $y=0$, $\int_0^\infty \frac{\sin yx}{x}\,dx$ vanishes; and thus $\int_0^\infty \frac{\sin yx}{x}\,dx$ is discontinuous at $y=0$.

In any interval $(0, C)$ of x, and $0<y\leqq\alpha$, $\left|\frac{\sin yx}{x}\right|$ is bounded, and thus the condition $\lim\limits_{y\sim 0}\int_0^C \frac{\sin yx}{x}\,dx = 0$ is satisfied. But $\int_C^{C'} \frac{\sin yx}{x}\,dx = \int_{Cy}^{C'y} \frac{\sin\theta}{\theta}\,d\theta$; and however C be fixed, a value y_1 of y can be so chosen that $Cy_1 = \beta < \frac{\pi}{2}$; taking C' such that $C'y_1 = \frac{\pi}{2}$, we have

$$\int_C^{C'} \frac{\sin y_1 x}{x}\,dx = \int_\beta^{\frac{1}{2}\pi} \frac{\sin\theta}{\theta}\,d\theta > \frac{2}{\pi}\cos\beta.$$

Thus it is impossible to choose C so that $\left|\int_C^{C'} \frac{\sin yx}{x}\,dx\right| < \epsilon$, for $C' > C$ and for every value of y in an interval $(0, \alpha)$; and thus the condition in the theorem of § 228 is not satisfied.

Since $\lim_{y\sim 0}\int_0^C \frac{\sin yx}{x}\,dx = 0$, the condition in the second theorem of § 228, that this limit must converge, as $C\sim\infty$, to the value $\lim_{y\sim y_0}\int_0^\infty \frac{\sin yx}{x}\,dx = \frac{\pi}{2}$ is not satisfied.

(2) The equality $\lim_{y\sim y_0}\int_a^b \phi(x) f(x, y)\,dx = \int_a^b \phi(x) f(x, y_0+0)\,dx$ holds if $|f(x, y)|$ is bounded for all values of x and y such that $a \leqq x \leqq b$, $y_0 < y \leqq y_0 + a$; and provided also $\phi(x)$ is summable in (a, b), and has infinite discontinuities only at points of a closed enumerable set. For $\phi(x) f(x, y)$ converges to $\phi(x) f(x, y_0+0)$, in accordance with the condition of § 229, boundedly in any interval interior to an interval contiguous to the exceptional set; and the theorem is therefore applicable. The result may be extended to the case in which $b=\infty$, provided $\phi(x)$ be absolutely summable in (a, ∞); then under the same conditions the equality holds. For $\left|\int_C^{C'} f(x, y)\phi(x)\,dx\right| < K\int_C^{C'} |\phi(x)|\,dx$, where K is the upper boundary of $|f(x, y)|$; and therefore, assuming the existence of $\int_a^\infty |\phi(x)|\,dx$, we have $\left|\int_C^{C'} f(x,y)\phi(x)\,dx\right| < \epsilon$, provided C is sufficiently great. It follows that $\int_a^\infty f(x, y_0+0)\phi(x)\,dx$ exists, and is equal to $\lim_{y\sim y_0}\int_a^\infty f(x, y)\phi(x)\,dx$.

(3) Consider $\int_a^b e^{-yx}\phi(x)\,dx$, where b may be finite or infinite. It follows from Ex. 2, that provided $\phi(x)$ is absolutely summable in (a, b), and has at most a set of points of infinite discontinuity which form a reducible set, $\lim_{y\sim 0}\int_a^b e^{-yx}\phi(x)\,dx = \int_a^b \phi(x)\,dx$.

The theorem holds, however, whenever $\int_a^b e^{-yx}\phi(x)\,dx$ has a definite value for all values of y such that $0 \leqq y \leqq \alpha$, where α is some positive number. If $\psi(x)$ denote the continuous function $\int_a^x \phi(x)\,dx$, we have

$$\int_a^b e^{-yx}\phi(x)\,dx = e^{-by}\psi(b) + y\int_a^b e^{-yx}\psi(x)\,dx,$$

b being taken to be finite. Since $|\psi(x)|$ has a finite upper limit U, in (a, b), we have

$$\left|\int_a^b e^{-yx}\psi(x)\,dx\right| < Ue^{-ay}(b-a), \text{ if } a \geqq 0;$$

therefore
$$\lim_{y\sim 0}\int_a^b e^{-yx}\phi(x)\,dx = \psi(b) = \int_a^b \phi(x)\,dx.$$

In case $b=\infty$, we have

$$\int_a^\infty e^{-yx}\phi(x)\,dx = y\int_a^\infty e^{-yx}\psi(x)\,dx = y\int_a^{\frac{1}{\sqrt{y}}} e^{-yx}\psi(x)\,dx + y\int_{\frac{1}{\sqrt{y}}}^\infty e^{-yx}\psi(x)\,dx,$$

hence, applying the first mean value theorem, we have

$$\int_a^\infty e^{-yx}\phi(x)\,dx = \psi(x_1)\left(e^{-ay} - e^{-\sqrt{y}}\right) + \psi(x_2)\,e^{-\sqrt{y}};$$

where x_1 is some number between a and $1/\sqrt{y}$, and x_2 some number greater than $1/\sqrt{y}$. When y converges to the limit zero, the first term on the right-hand side converges to zero, and the second to the limit $\psi(\infty)$, or $\int_a^\infty \phi(x)\,dx$. It is sufficient if $\phi(x)$ have a D-integral in the infinite interval (a, ∞).

(4) The integral $\int_0^\infty f(x)\cos xy\,dx$ is a continuous function of y, in any finite interval of y, interior to $(0, \infty)$, provided either (1), $|f(x)|$ is summable in $(0, \infty)$, or (2), $f(x)$ is summable in every finite interval, $f(x) \sim 0$, as $x \sim \infty$, and is of bounded variation in some interval (A, ∞).

For, in case (1), $|f(x)\cos xy| \leqq |f(x)|$, which is summable in $(0, \infty)$, and thus the result follows from the theorem in § 225.

In case (2), if $f(x)$ be monotone decreasing in (A', ∞), we have

$$\int_{A'}^{A''} f(x)\cos xy\,dx = f(A')\int_{A'}^{A'''}\cos xy\,dx = \frac{f(A')}{y}(\sin A'''x - \sin A'x)$$

or $\left|\int_{A'}^{A''} f(x)\cos xy\,dx\right| < \frac{2f(A')}{|y|}$, provided $A < A' < A''$, where A''' is in the interval (A', A''). It follows that $\left|\int_{A'}^{\infty} f(x)\cos xy\,dx\right| \leqq \frac{2f(A')}{|y|} < \epsilon$, provided y is in an interval interior to $(0, \infty)$, and A' is taken sufficiently large.

It is clear that, if $f(x)$ is the difference of two such monotone functions, that is, of bounded variation in (A, ∞), and $\lim\limits_{x \sim \infty} f(x) = 0$, the same result holds good. Denoting the integral by $I(y)$, we have

$$\left| I(y+h) - I(y) - \int_0^{A'} f(x)\{\cos x(y+h) - \cos xy\,dx\}\right| < 2\epsilon,$$

and since $\int_0^{A'} f(x)\cos xy\,dx$ is continuous because

$$|f(x)\cos xy| \leqq |f(x)| \text{ and } \int_0^{A'} |f(x)|\,dx$$

exists, we have $|I(y+h) - I(y)| < 3\epsilon$, provided $|h|$ is small enough. Thus the condition of continuity of $I(y)$ is satisfied.

(5) If* $\int_a^A f(x, y)\,\phi(x)\,dx$ is continuous with respect to y in an interval (a, β), for each finite value of $A\ (>a)$, then $\int_a^\infty f(x, y)\,\phi(x)\,dx$ exists, and is continuous with respect to y in (a, β), if either of the following sets of conditions are satisfied:

(i), $\int_a^\infty \phi(x)\,dx$ exists; $f(x, y)$ is monotone decreasing with respect to x, and $\geqq 0$, for each va ue of y, in (a, β); and $|f(a, y)|$ is less than a number K, independent of y.

(ii), $\int_a^\infty \phi(x)\,dx$ oscillates between finite limits; $f(x, y)$ is, for each value of y in (a, β), positive and monotone with respect to x; and $f(x, y)$ converges to zero, as $x \sim \infty$, uniformly with respect to y.

We have, by Bonnet's form of the second mean value theorem,

$$\int_A^{A'} f(x, y)\,\phi(x)\,dx = f(A, y)\int_A^{A''}\phi(x)\,dx,$$

where $A < A'' \leqq A'$. In case (1), $|f(A, y)| < K$, and A may be so chosen that

$$\left|\int_A^{A''}\phi(x)\,dx\right| < \epsilon.$$

* See Bromwich's *Theory of Infinite Series*, pp. 434–436, where these theorems are given in a slightly different form.

Therefore $\left| \int_A^{A'} f(x, y)\, \phi(x)\, dx \right|$ is less than an assigned positive number, if A be properly chosen, for all values of A' ($>A$), and for all values of y, in (α, β). That $\int_a^\infty f(x, y)\, \phi(x)\, dx$ exists, and is continuous with respect to y, now follows (see § 229).

In case (ii), A may be so chosen that $|f(A, y)| < \epsilon$, for all the values of y, and $\left| \int_A^{A''} \phi(x)\, dx \right|$ is less than a fixed number, independent of A''; thus the same result follows.

230. The following theorem is of use in connection with the theory of Fourier's series:

If $g(t)$ is a summable function, defined in the cell or interval (a, b), and if $f(t)$ is a summable function, defined in the cell or interval $(a + \alpha, b + \beta)$, then $\int_a^b f(x + t)\, g(t)\, dt$ exists and is a continuous function of x in the cell or interval (α, β), provided either (1), $g(t)$ is bounded in (a, b), or (2), $|f(t)|^p$, $|g(t)|^q$ are summable in the cells or intervals $(a + \alpha, b + \beta)$, (a, b) respectively, for some positive values of p and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

In case the variables x and t are in r dimensions, the integral denotes

$$\int_{(a^{(1)}, a^{(2)}, \dots a^{(r)})}^{(b^{(1)}, b^{(2)}, \dots b^{(r)})} f(x^{(1)} + t^{(1)}, x^{(2)} + t^{(2)}, \dots, x^{(r)} + t^{(r)}),$$
$$g(t^{(1)}, t^{(2)}, \dots, t^{(r)})\, d(t^{(1)}, t^{(2)}, \dots t^{(r)}).$$

A precisely similar result holds good for an integral $\int_a^b f(x - t)\, g(t)\, dt$. This theorem was established* by W. H. Young, for the linear case, but the proof given below suffices in the case of functions of a variable of any number of dimensions.

If $P(t)$ denotes a finite polynomial in t, we have

$$\int_a^b f(x + t)\, g(t)\, dt = \int_a^b \{f(x + t) - P(x + t)\}\, g(t)\, dt + Q(x),$$

where $Q(x)$ is the finite polynomial

$$\int_a^b P(x + t)\, g(t)\, dt.$$

Considering first the case (1), the polynomial $P(t)$ may be so chosen that $\int_{a+\alpha}^{b+\beta} |f(t) - P(t)|\, dt < \eta$, where η is an assigned positive number (see I, § 430). We have then $\left| \int_a^b \{f(x + t) - P(x + t)\}\, g(t)\, dt \right| < \bar{g}\eta$, where $\bar{g}$ is the upper boundary of $|g(t)|$ in (a, b). It follows that, if ξ be any fixed value of x in (α, β), and x' any point in a certain neighbourhood of ξ, the difference of the values of $\int_a^b f(x + t)\, g(t)\, dt$ for ξ and x' is numerically

* *Proc. Roy. Soc.* (A), vol. LXXXV (1911), pp. 404–408.

less than $\eta\,(1 + 2\bar{g})$, or than ϵ, if η be chosen to be $\leqq \epsilon/(1 + 2\bar{g})$ Therefore $\int_a^b f(x+t)\,g(t)\,dt$ is continuous with respect to x at the point ξ.

Next, in case (2), we see that (see I, § 435)

$$\int_a^b \{f(x+t) - P(x+t)\}\,g(t)\,dt$$

is numerically not greater than

$$\left[\int_a^b |f(x+t) - P(x+t)|^p\,dt\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q\,dt\right]^{\frac{1}{q}}.$$

By the theorem given in § 173, the polynomial $P(t)$ can be so chosen that

$$\int_{a+\alpha}^{b+\beta} |f(t) - P(t)|^p\,dt < \eta;$$

then
$$\int_a^b \{f(x+t) - P(x+t)\}\,g(t)\,dt$$

is numerically less than $\quad \eta^{\frac{1}{p}}\left[\int_a^b |g(t)|^q\,dt\right]^{\frac{1}{q}},$

or than $\frac{1}{2}\epsilon$, if η be properly chosen. Since $Q(x)$ is a continuous function of x, in a certain neighbourhood of the point ξ the fluctuation of $Q(x)$ is $< \frac{1}{2}\epsilon$; hence the fluctuation of $\int_a^b f(x+t)\,g(t)\,dt$, in that neighbourhood is $< \epsilon$, which is the condition of continuity of the function of x.

THE DIFFERENTIATION OF SERIES

231. If $s(x)$ denote the sum-function of an infinite series

$$u_1(x) + u_2(x) + \dots,$$

and it be assumed that, either at a particular point, or in a continuous linear interval of x, all the terms $u_1(x)$, $u_2(x)$, ... are continuous and differentiable, it is a subject for investigation under what conditions $s(x)$ possesses a differential coefficient which is the limiting sum of the infinite series $u_1'(x) + u_2'(x) + \dots$, of which the terms are the differential coefficients of the original series. It may happen that (1), $s(x)$ possesses no differential coefficient, or (2), that the series $u_1'(x) + u_2'(x) + \dots$ is not convergent, or both (1) and (2) may be the case, or (3) it may happen that $s'(x)$ exists and the series of differential coefficients is also convergent, but that its limiting sum is not $s'(x)$.

Writing $s(x) = s_n(x) + R_n(x)$, we have, at any point of convergence of the series, $\lim_{n\sim\infty} R_n(x) = 0$; further we have

$$\frac{s(x+h) - s(x)}{h} = \frac{s_n(x+h) - s_n(x)}{h} + \frac{R_n(x+h) - R_n(x)}{h}.$$

On the hypothesis that all the terms of the series have finite differential coefficients at the point x, we have $\lim\limits_{h\sim 0} \dfrac{s_n(x+h)-s_n(x)}{h} = s_n'(x)$. If $R_n'(x)$ exists, at the point x, and converges, as $n \sim \infty$, to the value zero, we have

$$s'(x) = \lim_{n\sim\infty} s_n'(x) = \lim_{n\sim\infty} \{u_1'(x) + u_2'(x) + \ldots + u_n'(x)\}.$$

In case $R_n'(x)$ either does exist, or if it exists but does not converge to zero, as $n \sim \infty$, the term by term differentiation of the series is inapplicable.

Let it be assumed that, in a given interval (a, b), the terms of the convergent series $u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$ are differentiable, and that their differential coefficients are integrable in (a, b), in accordance with Lebesgue's definition, or more generally in accordance with that of Denjoy. Let it be further assumed that, for each value of n,

$$\int_a^x u_n'(x)\,dx = u_n(x) - u_n(a);$$

this condition is certainly satisfied if $u_n'(x)$ is finite at every point, and summable in (a, b), and $u_n(x)$ is of bounded variation (see I, § 553); or more generally, if $u_n(x)$ is of bounded variation, and $u_n'(x)$ is infinite only at points of a reducible set, and is summable in (a, b). In case $u_n'(x)$ is everywhere finite in (a, b), $\int_a^x u_n'(x)\,dx$ always exists as a D-integral, and the condition $\int_a^x u_n'(x)\,dx = u_n(x) - u_n(a)$ is certainly satisfied (see I, § 471). Let it also be assumed that the series $u_1'(x) + u_2'(x) + \ldots$ is convergent everywhere in (a, b); then, denoting the sum-function of this series by $\phi(x)$, we may apply the theorems given in §§ 214–218 to obtain sufficient conditions that $\phi(x)$ possesses an integral $\int_a^x \phi(x)\,dx$, where $a \leqq x \leqq b$, and that the series $\{u_1(x) - u_1(a)\} + \{u_2(x) - u_2(a)\} + \ldots$ converges to the value $\int_a^x \phi(x)\,dx$. If the condition that $\sum\limits_{n=1} \{u_n(x) - u_n(a)\}$ converges to the integral $\int_a^x \phi(x)\,dx$, is satisfied, we have $s(x) - s(a) = \int_a^x \phi(x)\,dx$; from which it follows that, almost everywhere in (a, b), and certainly at every point of continuity of $\phi(x)$, the differential coefficient $s'(x)$ exists, and has the value $\phi(x)$; or $s'(x) = \sum\limits_{n=1}^{\infty} u_n'(x)$.

Accordingly, *it is sufficient for the validity of term by term differentiation of the series* $u_1(x) + u_2(x) + \ldots$, *almost everywhere in* (a, b), *that:*

(1) $\int_a^x u_n'(x)\,dx$ *exists as an L-integral, or a D-integral, and has the value* $u_n(x) - u_n(a)$.

(2) *The sum-function* $\phi(x)$, *of the series* $\sum_{n=1}^{\infty} u_n'(x)$ *has an integral* $\int_a^x \phi(x)\,dx$, *in* (a, b), *to which the series* $\{u_1(x) - u_1(a)\} + \{u_2(x) - u_2(a)\} + \dots$ *converges.*

Condition (1) *is always satisfied if* $u_n'(x)$ *is everywhere finite in* (a, b).

The simplest sufficient condition for the validity of term by term differentiation of a series is the following:

If the series $\sum_{n=1} u_n(x)$ *converges everywhere in the finite interval* (a, b), *and the terms of the series* $\sum_{n=1} u_n'(x)$ *be all continuous in* (a, b), *and this latter series is uniformly convergent in* (a, b), *then* $s'(x)$ *exists, and is the sum of the series* $\sum_{n=1}^{\infty} u_n'(x)$, *at all points of* (a, b).

For, if the series of continuous functions $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly, its sum-function $\phi(x)$ is continuous, and has an R-integral $\int_a^x \phi(x)\,dx$, to which the series $\sum_{n=1}^{\infty} \{u_n(x) - u_n(a)\}$ converges (see § 214 (1)).

232. The following theorem gives less stringent sufficient conditions for the validity of term by term differentiation of a series:

If $\sum_{n=1} u_n(x)$ *be everywhere convergent in the interval* (a, b), *and the differential coefficients all have finite values everywhere in the interval, and* $u_n'(x)$ *be summable, and the series* $\sum_{n=1}^{\infty} u_n'(x)$ *be everywhere convergent in* (a, b), *then, almost everywhere in the interval, and certainly at every point of continuity of* $\sum_{n=1}^{\infty} u_n'(x)$, *the relation* $\frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n'(x)$ *holds, provided either* (1), $\sum_{n=1} u_n'(x)$ *converges uniformly in the interval, or* (2), $\left|\sum_{n=1}^{\infty} u_n'(x)\right|$ *is, for every value of* n *and* x, *less than the value of some summable function* $\psi(x)$, *or* (3), *if* $\sum_{n=1}^{\infty} u_n'(x)$ *is continuous in* (a, b), *and the set of points in whose neighbourhood* $\left|\sum_{n=1}^{m} u_n'(x)\right|$ *is not bounded for all values of* m, *is enumerable.*

Since $u_n'(x)$ is summable, and everywhere finite, the L-integral $\int_a^x u_n'(x)\,dx$ exists and has the value $u_n(x) - u_n(a)$. If either of the conditions (1), (2), (3), (4), of the theorem is satisfied, it follows from the theorems established in §§ 214–216, that $\sum_{n=1} \int_a^x u_n'(x)\,dx$ converges to $\int_a^x \phi(x)\,dx$, where $\phi(x) = \sum_{n=1}^{\infty} u_n'(x)$; thus both of the conditions in § 231 are satisfied.

The condition of the above theorem, that $u_n'(x)$ be everywhere finite, may be so far relaxed, that it may have infinite values at points of a reducible set. If then $u_n'(x)$ be still summable over the part of (a, b) which remains when the reducible set is removed, in accordance with I, § 413, $\int_a^x u_n'(x) = u(x) - u(a)$.

In this case, the theorem of § 216 is applicable to prove that, under certain conditions $\sum_{n=1}^{\infty} \int_a^x u_n'(x)\,dx$ converges to $\int_a^x \sum_{n=1}^{\infty} u_n'(x)\,dx$.

We have accordingly the following theorem:

If $\sum_{n=1} u_n(x)$ be everywhere convergent in (a, b), and has a continuous sum, $u_n'(x)$ be finite except at points belonging to a reducible set, and be summable in (a, b), and if further $\sum_{n=1} u_n'(x)$ converges to a function $\phi(x)$, at every point which does not belong to a reducible set G, and so that $\left|\sum_{n=1}^{m} u_n'(x)\right|$ is bounded, as a function of m and x, in every interval that contains no point of G as interior or end-point, then term by term differentiation holds good almost everywhere in the interval.

233. The following theorem is due* to Fubini:

If all the functions of the convergent series $\sum_{n=1} u_n(x)$ are monotone non-diminishing, or all are monotone non-increasing, and the series converges in (a, b) to $s(x)$, then $s'(x)$ exists and is the sum-function of $\sum_{n=1} u_n'(x)$, almost everywhere in (a, b).

Let $u_n(x)$ be monotone non-diminishing; it has almost everywhere in (a, b), a differential coefficient $u_n'(x) \geq 0$. Moreover $u_n'(x)$ is summable over the set of points at which it exists, and $\int_a^x u_n'(x)\,dx = u_n(x) - u_n(a)$. In accordance with theorem (10) of § 214, since $\sum_{n=1}^{\infty} \int_a^b u_n'(x)\,dx$ is convergent, the series $\sum_{n=1} u_n'(x)$ converges almost everywhere to a function $\phi(x)$, summable in (a, b), and $\sum_{n=1} \int_a^x u_n'(x)\,dx$ converges uniformly to $\int_a^x \phi(x)\,dx$; therefore $s(x) - s(a) = \int_a^x \phi(x)\,dx$, from which it follows that $s'(x)$ exists almost everywhere, and has the value $\phi(x)$, to which $\sum_{n=1} u_n'(x)$ converges almost everywhere.

A theorem, similar to this, is the following†:

If $\sum_{n=1} u_n(x)$ is a convergent series such that $u_n'(x)$ is, for each value of n,

* *Rend. Acc. Lincei*, (5) vol. XXIV (1915), p. 204, where a direct proof of the theorem is given. Another proof has been given by A. Rajchman and S. Sales, *Fundamenta Mat.* vol. IV, pp. 211–13.

† See W. H. Young, *Camb. Phil. Trans.* vol. XXI (1910), p. 408.

finite at every point that does not belong to a reducible set of points, and if $\sum_{n=1} u_n'(x)$ *converges almost everywhere to a function which is itself a summable differential coefficient, and finite, except at points belonging to a reducible set, then* $\frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n'(x)$ *almost everywhere.*

For we have, as in the last case, $\int_a^x u_n'(x)\,dx = u_n(x) - u_n(a)$; and $\sum_{n=1} u_n'(x)$ converges almost everywhere to a function $\phi(x)$, equal to a function $\psi'(x)$ almost everywhere. Also, in virtue of the condition that $\psi'(x)$ is summable, and finite, except at points belonging to a reducible set, we have $\int_a^x \phi(x)\,dx = \psi(x) - \psi(a)$.

234. The condition of the validity of term by term differentiation of the convergent series $\Sigma u(x)$, at a particular point a of the domain of x, is identical with the condition that the two repeated limits of

$$\frac{s(a+h, y) - s(a, y)}{h}$$

for $h = 0, y = 0$, should exist, and have one and the same value. By applying the theorems of I, §§ 305, 306, which contain the necessary and sufficient conditions for the existence and equality of repeated limits of a function at a point, we obtain the following theorems:

If the series $\Sigma u_n(x)$ *everywhere converge in a sufficiently small neighbourhood of a point* a, *and the differential coefficients* $u_n'(a)$ *exist, and are finite, then the necessary and sufficient conditions that* $\frac{d}{dx}s(x)$ *at* $x = a$, *may exist and be equal to* $\sum_1^{\infty} u_n'(a)$ *are* (1), *that* $\Sigma u_n'(a)$ *be convergent, and* (2), *that,* ϵ *being an arbitrarily chosen positive number, and* n_0 *an arbitrarily chosen positive integer, a number* η, *positive and* > 0 *can be found, and also a positive integer* $n > n_0$, *such that the condition* $\left|\frac{R_n(a+h) - R_n(a)}{h}\right| < \epsilon$ *is satisfied for this value of* n, *and for every value of* h *such that* $0 < |h| < \eta$, *and for which* $a + h$ *is interior to the given neighbourhood of* a.

If the series $\Sigma u_n(x)$ *everywhere converge in a sufficiently small neighbourhood of a point* a, *and the differential coefficients* $u_n'(a)$ *exist, and are finite, then the necessary and sufficient condition that* $\frac{d}{dx}s(x)$ *at* $x = a$, *may exist and be equal to* $\sum_1^{\infty} u_n'(a)$ *is that, corresponding to any arbitrarily chosen positive number* ϵ, *an integer* n_0 *exists, such that corresponding to each integer* $n > n_0$, *a positive number* η, *in general dependent on* n, *can be found, such that the condition* $\left|\frac{R_n(a+h) - R_n(a)}{h}\right| < \epsilon$ *is satisfied for every value of*

h such that $0 < |h| < \eta$, *and for which* $a + h$ *is interior to the given neighbourhood of* a.

It is clear from I, § 305, that *the uniform convergence of* $\dfrac{s_n(a+h) - s_n(a)}{h}$ *to the limit* $\dfrac{s(a+h) - s(a)}{h}$, *for all values of* h, *except* 0, *in a fixed interval* $(-\delta, \delta')$ *for* h, *is a sufficient condition that* $s'(a)$ *exists, and that the series* $\Sigma u_n'(a)$ *converges to* $s'(a)$.

235. The following theorem* is sometimes more convenient than the theorems of § 234, for the purpose of ascertaining whether a function defined by a convergent series of functions is differentiable or not.

If the series $\Sigma u_n(x)$ *converge in* (a, b), *and the differential coefficients* $u_n'(a)$ *exist, and are finite, then the necessary and sufficient conditions that* $\dfrac{d}{dx} s(x)$ *may exist at* $x = a$, *and be the sum of the series* $\Sigma u_n'(a)$, *are* (1), *that the series* $\Sigma u_n'(a)$ *be convergent, and* (2), *that, corresponding to an arbitrarily fixed positive number* ϵ, *and an arbitrarily fixed integer* m', *a positive number* δ *can be determined such that, for each value of* h *numerically less than* δ, *and for which* $a + h$ *is in* (a, b), *an integer* $m\ (> m')$, *in general varying with* h, *can be found, for which the three numbers*

$$\sum_{n=1}^{m}\left\{\frac{u_n(a+h) - u_n(a)}{h} - u_n'(a)\right\},\quad \frac{R_m(a+h)}{h},\quad \frac{R_m(a)}{h}$$

are all numerically less than ϵ.

The convenience in application of this theorem arises from the fact that it provides a test in which only a single value of h is employed. To prove that the conditions stated in the theorem are sufficient, we have

$$\frac{s(a+h) - s(a)}{h} - \sum_{n=1}^{\infty} u_n'(a) = \sum_{1}^{m}\left\{\frac{u_n(a+h) - u_n(a)}{h} - u_n'(a)\right\} + \frac{R_m(a+h)}{h} - \frac{R_m(a)}{h} - R_m',$$

where R_m' denotes the remainder, after m terms, of the series $\Sigma u_n'(a)$. The number m' can be so chosen that $|R_n'| < \epsilon$, for $n \geqq m'$, since the series $\Sigma u_n'(a)$ is convergent. If m be chosen $> m'$, and such that the second condition in the theorem is satisfied, we see that

$$\left|\frac{s(a+h) - s(a)}{h} - \sum_{n=1}^{\infty} u_n'(a)\right| < 4\epsilon,$$

provided $|h| < \delta$; and therefore $\lim\limits_{h\sim 0} \dfrac{s(a+h) - s(a)}{h}$ is $\sum\limits_{n=1}^{\infty} u_n'(a)$. Therefore the conditions are sufficient.

To shew that the conditions stated are necessary; it is clear that (1) must be satisfied, and therefore that m' can be determined so that

* Dini, *Grundlagen*, p. 152.

$|R_m'| < \frac{1}{4}\epsilon$, if $m \geqq m'$. Moreover, a positive number δ can be determined such that $\frac{s(a+h)-s(a)}{h} - \sum_{n=1}^{\infty} u_n'(a)$ is numerically less than $\frac{1}{4}\epsilon$, if $|h| < \delta$. Also since $\Sigma u_n(x)$ is convergent, for each value of h, a corresponding value of m ($\geqq m'$) exists, such that $\frac{R_m(a+h)}{h}$, $\frac{R_m(a)}{h}$ are each numerically $< \frac{1}{4}\epsilon$. It then follows that, for these values of h and m, the condition

$$\sum_1^m \left\{\frac{u_n(a+h)-u_n(a)}{h} - u_n'(a)\right\} < \epsilon$$

is satisfied. Therefore the conditions in the theorem are necessary.

EXAMPLES

(1) Let $u_n(x) = \frac{1}{n}\sin nx$; the series $\Sigma u_n(x)$ converges everywhere in any interval, but the series $\Sigma \cos nx$ does not converge. The term by term differentiation of the given series is therefore inapplicable.

(2) Let $u_n(x) = \frac{x^n}{n} - \frac{x^{n+1}}{n+1}$; the series $\Sigma u_n(x)$ converges to the sum-function $s(x) = x$, in the interval $(0, 1)$. The series $\Sigma(x^{n-1} - x^n)$ converges to $s'(x) = 1$, for all values of x in the interval $(0, 1)$, except for $x = 1$, when it converges to 0, which is not equal to $s'(0)$. The series $\Sigma(x^{n-1} - x^n)$ has the point $x = 1$ for a point of non-uniform convergence, and thus the convergence is not uniform in the interval $(0, 1)$.

(3) The series $\sum_{n=1} b^n \cos(a^n x)$, where $0 < b < 1$, converges uniformly in any interval. The series $-\Sigma(ab)^n \sin(a^n x)$, for $ab > 1$, is not convergent. It will be shewn later that the function defined by the given series is not differentiable for any value of x, provided ab exceeds a certain value.

INVERSION OF THE ORDER OF REPEATED INTEGRALS

236. It is an important case of the problem of the inversion of the order of repeated limits to investigate sufficient criteria for the equality of the repeated integrals

$$\int_a^\alpha dx \int_b^\beta f(x, y)\, dy, \quad \int_b^\beta dy \int_a^\alpha f(x, y)\, dx,$$

where $f(x, y)$ is a function of two variables, defined in the cell $(a, b; \alpha, \beta)$. It will be assumed that $f(x, y)$, whether it be bounded or not, is measurable in the cell.

The plane set of points at which $f(x, y) > A$, is, for each value of A, a measurable plane set E_A. It has been shewn in I, § 427, that the section of E_A by an ordinate y, corresponding to an abscissa x, is linearly measurable, for almost all values of x; hence the set of points on the ordinate y, at which $f(x, y) > A$, is, for almost all values of x, linearly measurable. Assigning to A the values of an enumerable set of numbers, everywhere dense in the indefinite interval $(-\infty, \infty)$, we see, taking account of a

theorem given in I, § 383, that $f(x, y)$ is, for almost every value of x, linearly measurable with respect to y.

In order that the repeated integrals $\int_a^\alpha dx \int_b^\beta f(x, y)\, dy$ may have a meaning, it is sufficient that $\int_b^\beta f(x, y)\, dy$ should have a definite value $\phi(x)$, either as an L-integral, or as a non-absolutely convergent integral, such as a D-integral, for almost all values of x, and that $\int_a^\alpha \phi(x)\, dx$ should also exist; where, in the integration, those points of (a, α) at which $\phi(x)$ is not definite, forming a set of measure zero, are left out of account. A similar statement will apply to $\int_a^\alpha f(x, y)\, dy$. It is not absolutely necessary for the existence of $\int_a^\alpha dx \int_b^\beta f(x, y)\, dy$ that $\int_b^\beta f(x, y)\, dy$, or $\phi(x)$, should have a definite value, almost everywhere in the interval (a, α). If, in accordance with any definition, $\phi(x)$ has an upper value $\bar{\phi}(x)$, and a lower value $\underline{\phi}(x)$, the repeated integral may exist where $\int_a^b \{\bar{\phi}(x) - \underline{\phi}(x)\}\, dx = 0$. This possibility will however not be here further considered; it will be assumed throughout that $\int_b^\beta f(x, y)\, dy$ exists almost everywhere in the interval (a, α), and that $\int_a^\alpha f(x, y)\, dx$ exists almost everywhere in the interval (b, β).

237. In case it is known that $f(x, y)$ is summable in the cell $(a, b;\ \alpha, \beta)$ we have the theorem established in I, § 429:

If $f(x, y)$ be a function, bounded or unbounded, that is summable in the cell $(a, b;\ \alpha, \beta)$, the repeated integrals

$$\int_a^\alpha dx \int_b^\beta f(x, y)\, dy, \quad \int_b^\beta dy \int_a^\alpha f(x, y)\, dx$$

are equal to one another, and have the same value as the integral of $f(x, y)$ over the cell.

It is of importance to possess a criterion which does not depend upon a knowledge that the function is summable over the cell, in view of the fact that, in general, an integral over the cell can only be evaluated by means of one of the corresponding repeated integrals; and it is in general not known, apart from such valuation, whether a given unbounded measurable function is summable, or not. For this purpose, the second theorem in I, § 429, may be employed:

If one of the repeated integrals $\int_a^\alpha dx \int_b^\beta |f(x, y)|\, dy$, $\int_b^\beta dy \int_a^\alpha |f(x, y)|\, dx$, exists as a finite number, then $f(x, y)$ is summable over the cell $(a, b;\ \alpha, \beta)$, and therefore $\int_a^\alpha dx \int_b^\beta f(x, y)\, dy = \int_b^\beta dy \int_a^\alpha f(x, y)\, dx$.

In particular, we have the result that:

If $f(x, y) \geqq 0$, in the cell $(a, b; \alpha, \beta)$, and if one of the repeated integrals of $f(x, y)$ exists as a finite number, then the other exists, and the two have the same value.

The following test may often* be conveniently applied:

If $\phi(x, y)$ be $\geqq 0$, and unbounded, and one of the repeated integrals of $\phi(x, y)$ is finite; and $\psi(x, y)$ be a bounded measurable function, the repeated integrals

$$\int_a^\alpha dx \int_b^\beta \phi(x, y)\,\psi(x, y)\,dy, \quad \int_b^\beta dy \int_a^\alpha \phi(x, y)\,\psi(x, y)\,dx$$

are both finite, and are equal to one another.

For the function $\phi(x, y)\,|\,\psi(x, y)\,|$ is summable over the cell, since $\phi(x, y)$ is summable and $|\,\psi(x, y)\,|$ is bounded; therefore $\phi(x, y)\,\psi(x, y)$ is summable over the cell, and the result then follows from the first theorem.

238. If $f(x, y)$, although measurable, is not summable in the cell, the repeated integrals may exist, and they may have different values. An example of this possibility has been given in I, p. 578, for the case

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},$$

the cell being $(0, 0; 1, 1)$.

For the case in which $f(x, y)$ is not summable, or is not known to be summable, over the cell, the following general theorem is applicable:

If (1), $\left|\int_a^x f(x, y)\,dx\right| \leqq \phi(y)$, $\left|\int_b^y f(x, y)\,dy\right| \leqq \psi(x)$, *for all values of (x, y) in the cell $(a, b; \alpha, \beta)$; where $\phi(y)$ is some non-negative function, summable in the interval (b, β), and $\psi(x)$ is some non-negative function, summable in the interval (a, α); and if*, (2), *the points of infinite discontinuity of $f(x, y)$ are distributed on a limited number of arcs of continuous curves representing monotone functions, then*

$$\int_a^\alpha dx \int_b^\beta f(x, y)\,dy = \int_a^\beta dy \int_a^\alpha f(x, y)\,dx.$$

In applying the theorem, $\phi(y)$ will be taken to be the maximum of $\left|\int_a^x f(x, y)\,dx\right|$, for a fixed y, for all values of x in (a, α). A similar remark applies to $\psi(y)$. The functions $\phi(x)$, $\psi(y)$ may be infinite, or indeterminate, for sets of values of x, y which have linear measure zero; and still they may be summable in (a, α), (b, β) respectively, when these sets are left out of account.

* See W. H. Young, *Camb. Phil. Trans.* vol. XXI (1910), p. 364, where the theorem is given in slightly different form.

The particular case of this theorem, which is a generalization of one given* by Jordan, that arises when $\phi(y)$ and $\psi(x)$ are both constants, so that $\int_a^x f(x, y)\,dx$, $\int_b^y f(x, y)\,dy$ are both bounded functions of (x, y) in the cell, was given† by W. H. Young.

Let each point of each curve, belonging to the finite set, be enclosed in a rectangle with centre at the point, and sides parallel to the axes, and of lengths ϵ_n, ϵ_n'; where $\{\epsilon_n\}, \{\epsilon_n'\}$ are two sequences of diminishing positive numbers which converge to zero. Employing the Heine-Borel theorem, there exists a finite set of these rectangles which contains all the points of the curves. In this manner a finite set Δ_n, of cells, is obtained, such that every point of infinite discontinuity is interior to one of them. On any straight line parallel to one of the axes, there are at most r segments, in which the straight line intersects Δ_n; where r is the number of the curves.

Let $f_n(x, y) = 0$ at all points in any of the rectangles of Δ_n, and let $f_n(x, y) = f(x, y)$, at all remaining points of the cell. The function $f_n(x, y)$ is summable in the cell $(a, b; \alpha, \beta)$, and therefore

$$\int_a^\alpha dx \int_b^\beta f_n(x, y)\,dy = \int_a^\beta dy \int_a^\alpha f_n(x, y)\,dx = \int_{(a, b)}^{(\alpha, \beta)} f_n(x, y)\,d(x, y).$$

Denoting $\int_b^\beta f_n(x, y)\,dy$ by $\chi_n(x)$, and $\int_b^\beta f(x, y)\,dy$ by $\chi(x)$, it will be shewn that $\int_a^\alpha \chi(x)\,dx$ exists and is equal to $\lim_{n\sim\infty} \int_a^\alpha \chi_n(x)\,dx$.

That $\int_a^\alpha \chi(x)\,dx$ exists, follows from the condition (1), of the theorem, since $|\chi(x)| \leqq \psi(x)$, which is summable in (a, α). The difference of the two functions $\chi(x)$, $\chi_n(x)$ is the sum of at most r integrals $\int f(x, y)\,dy$ each taken over a segment in which the ordinate, corresponding to the abscissa x, intersects the cells Δ_n; it follows that $|\chi(x) - \chi_n(x)| \leqq 2r\psi(x)$. Therefore, employing the theorem of § 202, we have

$$\int_a^\alpha \chi(x)\,dx = \lim_{n\sim\infty} \int_a^\alpha \chi_n(x)\,dx;$$

or
$$\int_a^\alpha dx \int_b^\beta f(x, y)\,dy = \lim_{n\sim\infty} \int_a^\alpha dx \int_b^\beta f_n(x, y)\,dy.$$

In a precisely similar manner, it can be shewn that

$$\int_b^\beta dy \int_a^\alpha f(x, y)\,dx = \lim_{n\sim\infty} \int_b^\alpha dy \int_a^\alpha f_n(x, y)\,dx;$$

and the two limits on the right-hand side being the same, the theorem has been established.

* *Cours d'Analyse*, vol. II, p. 67.

† *Camb. Phil. Trans.* vol. XXI (1910), p. 365.

It can be shewn that:

In the theorem, if condition (1) *be replaced by* (1)′, *that, corresponding to any fixed positive number* ϵ, *positive numbers* h_1, k_1 *exist, such that*

$$\left|\int_x^{x+h} f(x,y)\,dx\right| < \epsilon, \quad \left|\int_y^{y+k} f(x,y)\,dy\right| < \epsilon$$

for $|h| \leqq h_1$, $|k| \leqq k_1$, *and for every value of* (x, y) *in the cell* $(a, b;\ \alpha, \beta)$, *the theorem holds good; the condition* (2) *being unaltered.*

The condition (1)′ is more stringent than the condition (1); accordingly if (1)′ be adopted, the theorem becomes less general. For, if the condition (1)′ is satisfied, we have, since

$$\int_a^x f(x,y)\,dx = \left\{\int_a^{h_1} + \int_{h_1}^{2h_1} + \ldots + \int_{(s-1)h_1}^{(s-1)h_1+h'}\right\} f(x,y)\,dx,$$

where s is the least integer such that $sh_1 \geqq x$, and $h' \leqq h_1$,

$$\left|\int_a^x f(x,y)\,dx\right| < s\epsilon.$$

Now s cannot exceed the smallest integer $\bar{s}$, such that $\bar{s}h \geqq \alpha$; hence $\left|\int_a^x f(x,y)\,dx\right| \leqq \bar{s}\epsilon$, and thus $\int_a^x f(x,y)\,dx$ is bounded, for all points (x, y) in the cell. Similarly it is seen that $\int_b^y f(x,y)\,dy$ is bounded. It follows that the condition (1) of the theorem is satisfied.

When the conditions (1) and (2) of the theorem are satisfied, it does not follow that $f(x, y)$ is summable in the cell, but it follows that it has a non-absolutely convergent double integral of the kind defined in I, § 368, p. 494, subject to the extension that $\int_{(D_n)} f(x,y)\,d(x,y)$ may exist only as an L-integral, and not necessarily as an R-integral. Such a non-absolutely convergent integral defined as $\lim_{n\sim\infty}\int_{(D_n)} f(x,y)\,d(x,y)$, for a finite set of rectangles D_n, which contain none of the points of infinite discontinuity of $f(x, y)$, may be termed a *restricted Jordan double integral.*

The converse does not hold good, that (1) and (2) follow from the existence of the restricted Jordan integral.

Investigations of conditions of equality of the repeated integrals were given by de la Vallée Poussin*, and by Hobson†. The results there obtained have now been in the main superseded, owing to the later development of the theory of Lebesgue integration.

* See *Annales de la soc. sc. de Bruxelles*, vol. XVI (B) (1892); *Liouville's Journal* (4), vol. VIII (1892); *ibid.* (5), vol. V, p. 191.

† *Proc. Lond. Math. Soc.* (2), vol. IV (1906), p. 148.

If it be assumed only that the restricted Jordan integral

$$\int_{(a,\,b)}^{(\alpha,\,\beta)} f(x, y)\, d(x, y)$$

exists; taking $f_n(x, y)$ as before to be zero in the finite set Δ_n of cells which include all the points of infinite discontinuity of $f(x, y)$, we have

$$\int_{(a,\,b)}^{(\alpha,\,\beta)} f(x, y)\, d(x, y) = \lim_{n\sim\infty} \int_{(a,\,b)}^{(\alpha,\,\beta)} f_n(x, y)\, d(x, y) = \lim_{n\sim\infty} \int_a^\alpha dx \int_b^\beta f_n(x, y)\, dy,$$

and therefore $$\int_{(a,\,b)}^{(\alpha,\,\beta)} f(x, y)\, d(x, y) = \int_a^\alpha dx \int_b^\beta f(x, y)\, dy,$$

provided $\lim_{n\sim\infty} \int_a^\alpha dx \int_{\Delta_n(x)} f(x, y)\, dy = 0$; where $\Delta_n(x)$ denotes that finite set of intervals which forms the section of Δ_n by the ordinate corresponding to the abscissa x. This condition will be satisfied when the conditions (1) and (2) of the theorem of § 238 are satisfied; but it may be satisfied when (1) and (2) are not satisfied.

In order to obtain criteria for the equality of the repeated integrals of $f(x, y)$ taken over any measurable bounded set of points E, we may take a cell which contains E, and assume $f(x, y)$ to be defined over the whole cell by taking its values to be zero at all points of the cell which belong to the complement of E relatively to the cell. The preceding theory will then be applicable to this case.

EXAMPLES

(1) For the function defined in I, § 365, Ex. 1, only one of the repeated integrals exists, in accordance with the definition there employed; neither does the R-double integral exist. The Lebesgue double integral exists, and has the value 1. For the set of points at which $f(x, y) = 1$ has measure zero; and therefore the function has the same L-integral as that function which, at every point (x, y), has the value $2y$. The other repeated integral necessarily exists, in accordance with Lebesgue's definition, as may be easily verified; and both the repeated integrals have the value 1.

(2) For the function defined in I, § 365, Ex. 4, both the repeated integrals exist, in accordance with the definition there employed, and they have the value c; the double R-integral, however, does not exist. But the double L-integral exists and has the value c; for the points at which $f(x, y) = c'$, although they are everywhere dense, form a set of plane measure zero.

(3) Let $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, and let the domain be the cell $(0, 0;\ 1, 1)$. The function is not summable over the cell; neither does the restricted Jordan integral exist. For if the rectangle $(0, 0;\ h, k)$ be excluded from the domain, the double integral over the remainder of the domain is $\int_0^1 dx \int_k^1 \dfrac{x^2 - y^2}{(x^2 + y^2)^2}\, dy + \int_h^1 dx \int_0^k \dfrac{x^2 - y^2}{(x^2 + y^2)^2}\, dy$,

or $$\int_0^1 \left(\frac{1}{x^2+1} - \frac{k}{x^2+k^2}\right) dx + \int_h^1 \frac{k}{x^2+k^2}\, dx,$$

which is equal to $\frac{1}{4}\pi - \tan^{-1}\dfrac{h}{k}$; and this has no definite limit, as h and k converge independently to zero.

It follows that one at least of the conditions (1), (2) of the theorem of § 238 cannot be satisfied; and it is clearly the condition (1) which should be examined.

We have $\left|\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right| = \frac{x}{x^2+y^2}$, and the maximum value of this is, for a fixed value of y, $\frac{1}{2y}$ which is not summable in $(0, 1)$; thus the condition (1) is not satisfied.

(4) It has been shewn in the Example in I, § 368, that the double integral of $\frac{1}{x}\sin\frac{1}{x}$ over $(0, 0; a, b)$ does not exist. In this case the restricted Jordan integral exists; for

$$\int_0^b dy \int_\epsilon^a \frac{1}{x}\sin\frac{1}{x}\,dx = b\int_\epsilon^a \frac{1}{x}\sin\frac{1}{x}\,dx,$$

and this has a definite limit, as $\epsilon\sim 0$. The condition $\lim\limits_{n\sim\infty}\int_0^b dy\int_{\Delta_n(y)} \frac{1}{x}\sin\frac{1}{x}\,dx = 0$ is satisfied, for $\Delta_n(y)$ is independent of y, and consists of the interval $(0, \epsilon)$. The repeated integrals accordingly exist, and are equal to the restricted Jordan integral. However the condition (1) of the theorem in § 238 is not satisfied, for $\left|\int_0^y \frac{1}{x}\sin\frac{1}{x}\,dy\right| \equiv \left|\frac{y}{x}\sin\frac{1}{x}\right|$ has the maximum $b\left|\frac{1}{x}\sin\frac{1}{x}\right|$ which is not summable in the interval $(0, a)$.

(5) Let $f(x, y) = (x-y)^{-\frac{2}{3}}$, in the domain $(a, 0; b, c)$, when $c > a$. In this case the function is non-negative; in order to shew that the repeated integrals exist, it is only necessary to verify that one of them is finite.

(6) Let the function* $\psi(x)$ be defined for the domain $(0, 1)$ by the rule that, for every rational value of x of the form $\frac{2m+1}{2^n}$, $(n \geqq 0)$, $\psi(x) = \frac{1}{2^n}$; and that, for every other value of x, $\psi(x) = 0$. Let $\psi(x, y) = \left|\frac{1}{y}\sin\frac{1}{y}\right|\psi(x)$ in the cell $(0, 0; 1, 1)$. Since $\psi(x, y)$ is non-negative, it is sufficient to shew that one of the repeated integrals is finite, in order to prove that $\psi(x, y)$ is summable in the cell, and consequently that the repeated integrals are equal. Since $\psi(x)\left|\frac{1}{y}\sin\frac{1}{y}\right|$ is zero for almost all values of x, when y is fixed, we have

$$\int_0^1 \left|\frac{1}{y}\sin\frac{1}{y}\right|\psi(x)\,dx = 0,$$

and therefore $\int_0^1 dy\int_0^1 \psi(x, y)\,dx = 0$; therefore the other repeated integral is zero.

THE INVERSION OF REPEATED INTEGRALS OVER AN INFINITE DOMAIN

239. Let the measurable function $f(x, y)$ be defined for the infinite domain $(a, b;\ \infty,\ \infty)$; criteria will be obtained which are sufficient to ensure the equality of the two repeated integrals

$$\int_a^\infty dx\int_b^\infty f(x, y)\,dy,\quad \int_b^\infty dy\int_a^\infty f(x, y)\,dx;$$

which are equivalent respectively to

$$\lim_{\alpha\sim\infty}\int_a^\alpha dx\,.\lim_{\beta\sim\infty}\int_b^\beta f(x, y)\,dy,\quad \lim_{\beta\sim\infty}\int_b^\beta dy\,.\lim_{\alpha\sim\infty}\int_a^\alpha f(x, y)\,dx.$$

* See Stolz, *Grundzüge*, vol. III, p. 149. The function $\psi(x)$ was first given by Du Bois-Reymond, *Crelle's Journal*, vol. XCVI, p. 278.

Let it be, for the present, assumed that $f(x, y)$ is a non-negative function, summable in every finite cell $(a, b;\ \alpha, \beta)$. Denoting the integral of $f(x, y)$ over the finite cell by $F(\alpha, \beta)$; we have

$$F(\alpha, \beta) = \int_{(a, b)}^{(\alpha, \beta)} f(x, y)\, d(x, y) = \int_a^\alpha dx \int_b^\beta f(x, y)\, dy = \int_b^\beta dy \int_a^\alpha f(x, y)\, dx.$$

In case the double limit of $F(\alpha, \beta)$, as $\alpha \sim \infty$, $\beta \sim \infty$, exists, as a finite number, $f(x, y)$ is summable over the domain $(a, b;\ \infty, \infty)$ (see I, § 437). Moreover, since $F(\alpha, \beta)$ is monotone non-diminishing, as α increases, and also as β increases, it is sufficient for the existence of the double limit that either of the repeated limits $\lim\limits_{\alpha\sim\infty} \lim\limits_{\beta\sim\infty} F(\alpha, \beta)$, $\lim\limits_{\beta\sim\infty} \lim\limits_{\alpha\sim\infty} F(\alpha, \beta)$ should exist.

Let us consider $\lim\limits_{\alpha\sim\infty} \lim\limits_{\beta\sim\infty} \int_a^\alpha dx \int_b^\beta f(x, y)\, dy$; it will be shewn that this is equal to $\lim\limits_{\alpha\sim\infty} \int_a^\alpha dx \,.\, \lim\limits_{\beta\sim\infty} \int_b^\beta f(x, y)\, dy$ which is $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$.

Let $\chi(x, \beta)$ denote $\int_b^\beta f(x, y)\, dy$, and let $\chi(x, \infty)$ denote $\int_b^\infty f(x, y)\, dy$. Since $\chi(x, \beta)$ is a monotone non-diminishing function of y, the theorem of § 226 is applicable, and shews that

$$\int_a^\alpha \chi(x, \infty)\, dx, \quad \lim_{\beta\sim\infty} \int_a^\alpha \chi(x, \beta)\, dx$$

are either both finite and equal, or both $+\infty$.

Thus we have

$$\int_a^\alpha dx \int_b^\infty f(x, y)\, dy = \lim_{\beta\sim\infty} \int_a^\alpha dx \int_b^\beta f(x, y)\, dy$$

if either of these expressions has a finite value; otherwise both are infinite.

We now have

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \lim_{\alpha\sim\infty} \lim_{\beta\sim\infty} \int_a^\alpha dx \int_b^\beta f(x, y)\, dy$$

if the repeated limit on the right-hand side has a finite value; otherwise both sides are infinite.

It thus appears that

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \lim_{\substack{\alpha\sim\infty \\ \beta\sim\infty}} \int_{(a, b)}^{(\alpha, \beta)} f(x, y)\, d(x, y)$$

when either of the expressions is known to be finite: otherwise both sides are $+\infty$.

Similarly $$\int_b^\infty dy \int_a^\infty f(x, y)\, dx = \lim_{\substack{\alpha\sim\infty \\ \beta\sim\infty}} \int_{(a, b)}^{(\alpha, \beta)} f(x, y)\, d(x, y),$$

both expressions being finite, or both infinite.

The following theorem has now been established:

If $f(x, y)$ be a non-negative measurable function, defined in the domain $(a, b; \infty, \infty)$, the three expressions

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy, \quad \int_b^\infty dy \int_a^\infty f(x, y)\, dy, \quad \int_{(a,b)}^{(\infty,\infty)} f(x, y)\, d(x, y)$$

are all finite and equal, or else all infinite.

240. If $f(x, y)$ be no longer non-negative, the above theorem may be applied to $|f(x, y)|$. If then either of the repeated integrals

$$\int_a^\infty dx \int_b^\infty |f(x, y)|\, dy, \quad \int_b^\infty dy \int_a^\infty |f(x, y)|\, dx$$

is known to be finite, the other one is finite, and $|f(x, y)|$ is summable over the domain $(a, b; \infty, \infty)$. If $f(x, y)$ be expressed by $f^+(x, y) - f^-(x, y)$, where $f^+(x, y)$, $f^-(x, y)$ are both non-negative functions, one at least of which is zero at each point, we have

$$|f(x, y)| = f^+(x, y) + f^-(x, y).$$

Since $f^+(x, y) \leq |f(x, y)|$, $f^-(x, y) \leq |f(x, y)|$, it follows that, if $|f(x, y)|$ is summable in the domain, so also are $f^+(x, y)$, $f^-(x, y)$, and therefore $f(x, y)$ is summable.

Hence the repeated integrals of each of these functions are finite and equal.

Since

$$\int_b^\infty f(x, y)\, dy = \lim_{\beta \sim \infty} \int_b^\beta \{f^+(x, y) - f^-(x, y)\}\, dy$$

$$= \lim_{\beta \sim \infty} \int_b^\beta f^+(x, y)\, dy - \lim_{\beta \sim \infty} \int_b^\beta f^-(x, y)\, dy$$

when the limits on the right-hand side exist, we have

$$\int_b^\infty f(x, y)\, dy = \int_b^\infty f^+(x, y)\, dy - \int_b^\infty f^-(x, y)\, dy,$$

and hence

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_a^\infty dx \int_b^\infty f^+(x, y)\, dy - \int_a^\infty dx \int_b^\infty f^-(x, y)\, dy,$$

when the integrals on the right-hand side exist.

We obtain, in the same manner, the corresponding result when the order of integration is inverted.

If then one of the repeated integrals of $|f(x, y)|$ over the domain is finite, $\int_{(a,b)}^{(\infty,\infty)} f(x, y)\, d(x, y)$ is finite, and since it is the difference of the integrals of $f^+(x, y)$, $f^-(x, y)$, it is equal to

$$\int_a^\infty dx \int_b^\infty f^+(x, y)\, dy - \int_a^\infty dx \int_b^\infty f^-(x, y)\, dy$$

which has been shewn to be equal to $\int_a^\infty dx \int_b^\infty f(x, y)\, dy$; similarly the integral of $f(x, y)$ is equal to $\int_b^\infty dy \int_a^\infty f(x, y)\, dx$. The following theorem has now been established:

If one of the repeated integrals $\int_a^\infty dx \int_b^\infty |f(x, y)|\, dy$, $\int_b^\infty dy \int_a^\infty |f(x, y)|\, dx$ *is known to be finite, then the repeated integrals of* $f(x, y)$ *and the integral of* $f(x, y)$ *over the domain* $(a, b; \infty, \infty)$ *are all finite and equal.*

In order to extend the results to the case in which $(-\infty, -\infty; \infty, \infty)$ is the domain of integration, it is only necessary to consider that an integral of $f(x, y)$, or a repeated integral, is the sum of the integrals, or repeated integrals, of the four functions $f(x, y)$, $f(-x, y)$, $f(x, -y)$, $f(-x, -y)$ over the domain $(0, 0; \infty, \infty)$.

If $f(x, y)$ be defined over a measurable set E, of infinite measure, we may suppose $f(x, y)$ to be defined over the whole domain $(-\infty, -\infty; \infty, \infty)$ by taking $f(x, y) = 0$, at every point that does not belong to E. The above theorems are then applicable to any measurable domain, of infinite measure.

We thus obtain an extension of Fubini's theorem given in I, § 429, and applicable to integrals over a domain of finite measure:

If $|f(x, y)|$ *be summable over a measurable domain* E, *of infinite measure, then* $\int_{(E)} f(x, y)\, d(x, y)$ *is equal to either of the repeated integrals of* $f(x, y)$ *taken over* E.

241. When the sufficient conditions that have been obtained are inapplicable, further criteria will be required. The integrals which are employed are not necessarily L-integrals, but may be non-absolutely convergent.

Let us consider, in the first instance, the case in which the domain of integration is $(a, b; \alpha, \infty)$. Let it be assumed that, for every finite value of β, the condition

$$\int_a^\alpha dx \int_b^\beta f(x, y)\, dy = \int_b^\beta dy \int_a^\alpha f(x, y)\, dx$$

is satisfied.

If $\int_b^\infty dy \int_0^\alpha f(x, y)\, dx$ exists, as a definite number, it is equal to

$$\lim_{\beta \sim \infty} \int_a^\alpha dx \int_b^\beta f(x, y)\, dy.$$

Denoting $$\int_b^\beta f(x, y)\, dy \text{ by } \chi(x, \beta),$$

if $$\lim_{\beta \sim \infty} \int_a^\alpha \chi(x, \beta)\, dx = \int_a^\alpha \chi(x, \infty)\, dx,$$

we have

$$\int_b^\infty dy \int_a^a f(x, y)\, dx = \lim_{\beta \sim \infty} \int_a^a dx \int_b^\beta f(x, y)\, dy = \int_a^a dx \int_b^\infty f(x, y)\, dy.$$

We have thus the following theorem:

On the supposition that the two repeated integrals of $f(x, y)$ over the domain $(a, b;\ a, \beta)$ exist, and have equal values, it is sufficient for the equality of the repeated integrals over $(a, b;\ a, \infty)$,

(1), *that* $\int_b^\infty dy \int_a^a f(x, y)\, dx$ *shall have a definite value, and*

(2), *that* $\int_a^a dx \int_\beta^\infty f(x, y)\, dy$ *shall converge to zero, as* $\beta \sim \infty$.

Sufficient conditions may be obtained that condition (2) of this theorem is satisfied. The condition is that $\lim_{\beta \sim \infty} \int_a^a \chi(x, \beta)\, dx = \int_0^a \chi(x, \infty)\, dx$, where

$$\chi(x, \beta) = \int_b^\beta f(x, y)\, dy, \text{ and } \chi(x, \infty) = \int_b^\infty f(x, y)\, dy.$$

Referring to the results in §§ 225–229, it is seen to be sufficient, in order that (2) may be satisfied, that one of the following conditions should be satisfied:

(2)′ *If* $\int_b^\beta f(x, y)\, dy$ *is a monotone function of* β *for all values of* x *in* (a, a). *This condition is satisfied, in particular, if* $f(x, y) \geqq 0$.

(2)″ *If* $\left|\int_b^\beta f(x, y)\, dy\right|$ *has a maximum* $\phi(x)$, *for all values of* β *in* (b, ∞), *and* $\phi(x)$ *is summable in the interval* (a, a). *This condition is satisfied, in particular if* $\left|\int_b^\beta f(x, y)\right| dy$ *is a bounded function of* (x, β). *The condition may be satisfied when there is an exceptional set of points* x, *of measure zero, at which* $\int_b^\infty f(x, y)\, dy$ *is oscillatory.*

(2)‴ *If* $\int_b^\beta f(x, y)\, dy$ *converges uniformly to* $\int_b^\infty f(x, y)\, dy$ *in the interval* (a, a), *of* x.

(2)$^{\text{IV}}$ *If* $\lim_{\beta \sim \infty} \int_a^a dx \int_b^\beta |f(x, y)|\, dy = \int_0^a dx \int_b^\infty |f(x, y)|\, dy$.

242. Next, let the measurable function $f(x, y)$ be as before defined in the domain $(a, b;\ \infty, \infty)$. It will be assumed that the repeated integrals $\int_a^a dx \int_b^\beta f(x, y)\, dy$, $\int_b^\beta dy \int_a^a f(x, y)\, dx$ exist, and are equal, for all finite values of a and β; let their value be denoted by $\phi(a, \beta)$. We have now

$$\lim_{\beta \sim \infty} \phi(a, \beta) = \int_b^\infty dy \int_a^a f(x, y)\, dx = \lim_{\beta \sim \infty} \int_a^a dx \int_b^\beta f(x, y)\, dy,$$

it being assumed that this limit has a definite value, for each value of a.

If now $$\lim_{\beta\sim\infty}\int_a^\alpha dx\int_b^\beta f(x,y)\,dy=\int_a^\alpha dx\int_b^\infty f(x,y)\,dy,$$

we have $$\lim_{\alpha\sim\infty}\lim_{\beta\sim\infty}\phi(\alpha,\beta)=\int_a^\infty dx\int_b^\infty f(x,y)\,dy;$$

the condition that this may be the case can be expressed in the form

$$\lim_{\beta\sim\infty}\int_a^\alpha dx\int_\beta^\infty f(x,y)\,dy=0.$$

Similarly, if $$\lim_{\alpha\sim\infty}\int_b^\beta dy\int_\alpha^\infty f(x,y)\,dx=0,$$

we have $$\lim_{\beta\sim\infty}\lim_{\alpha\sim\infty}\phi(\alpha,\beta)=\int_b^\infty dy\int_a^\infty f(x,y)\,dx,$$

it being assumed that $\lim\limits_{\alpha\sim\infty}\phi(\alpha,\beta)$ has a definite value for each value of β.

If the further condition is satisfied that

$$\lim_{\alpha\sim\infty}\lim_{\beta\sim\infty}\phi(\alpha,\beta)=\lim_{\beta\sim\infty}\lim_{\alpha\sim\infty}\phi(\alpha,\beta),$$

then $$\int_a^\infty dx\int_b^\infty f(x,y)\,dy=\int_b^\infty dy\int_a^\infty f(x,y)\,dx.$$

The following theorem has accordingly been established:

It being assumed that the repeated integrals of $f(x,y)$ in the domain $(a,b;\ \alpha,\beta)$ exist, and are equal, for every pair of finite values of α,β, it is sufficient for the existence and equality of the two repeated integrals of $f(x,y)$ over the infinite domain $(a,b;\ \infty,\infty)$ that the following conditions be satisfied.

(1) *That* $\int_a^\infty dx\int_b^\beta f(x,y)\,dy$, $\int_b^\infty dy\int_a^\alpha f(x,y)\,dx$ *have definite values for finite values of β and α, respectively.*

(2) *That* $$\lim_{\beta\sim\infty}\int_a^\alpha dx\int_\beta^\infty f(x,y)\,dy=0,$$

and $$\lim_{\alpha\sim\infty}\int_0^\beta dy\int_\alpha^\infty f(x,y)\,dx=0.$$

(3) *That* $$\lim_{\alpha\sim\infty}\lim_{\beta\sim\infty}\int_a^\alpha dx\int_b^\beta f(x,y)\,dy=\lim_{\beta\sim\infty}\lim_{\alpha\sim\infty}\int_a^\alpha dx\int_b^\beta f(x,y)\,dy.$$

This condition is satisfied, in particular, if $\int_a^\alpha dx\int_b^\beta f(x,y)\,dy$ *have a double limit as $\alpha\sim\infty$, $\beta\sim\infty$.*

Alternatively, the condition may be applied to $\int_b^\beta dy\int_a^\alpha f(x,y)\,dx$.

Sufficient conditions to be satisfied by $\int_\beta^\infty f(x,y)\,dy$, $\int_\alpha^\infty f(x,y)\,dx$ have already been given in § 242, that the condition (2) may be satisfied.

The condition (3) may be expressed in a somewhat different form by making use of (2); thus we may replace (3) by

(3)′ *That* $\lim\limits_{\beta\sim\infty}\int_a^\infty dx\int_\beta^\infty f(x,y)\,dy=0$, *or else* $\lim\limits_{\alpha\sim\infty}\int_b^\infty dy\int_\alpha^\infty f(x,y)\,dx=0$.

(3)′ is equivalent to the condition that, if ϵ be arbitrarily chosen, a number β_ϵ exists such that

$$\left|\int_a^\infty dx \int_\beta^\infty f(x, y)\, dy\right| < \epsilon, \text{ for } \beta > \beta_\epsilon,$$

and hence that, corresponding to each such value of β, a number λ_β can be chosen so large that

$$\left|\int_a^\lambda dx \int_\beta^\infty f(x, y)\, dy\right| < \epsilon, \text{ for } \beta > \beta_\epsilon,$$

and for $\lambda > \lambda_\beta$. The condition, in this form, might have been deduced from the theorem in I, § 305.

243. The following theorem, due* to de la Vallée Poussin, much more restricted in its scope, may be deduced from the theorem of § 242.

It is sufficient for the existence and equality of the repeated integrals, with infinite limits, (1), *that the repeated integrals between finite limits always exist, and are equal; and* (2), *that* $\int_a^\infty f(x, y)\, dx$ *be uniformly convergent in an arbitrary interval of* y*; and* (3), *that* $\int_b^\infty f(x, y)\, dy$ *satisfies the similar condition; and* (4), *that* $\int_a^\infty dx \int_b^\beta f(x, y)\, dy$ *converges uniformly in the unlimited interval of* β.

If $\int_b^\infty f(x, y)\, dy$ be uniformly convergent in the interval (a, α) of x, then, for a fixed positive number η, β_η can be so determined that $\left|\int_\beta^\infty f(x, y)\, dy\right| < \eta$ for $\beta \geqq \beta_\eta$, and for every value of x in the interval (a, α); it then follows that $\left|\int_a^\alpha dx \int_\beta^\infty f(x, y)\, dy\right| < \eta\,(\alpha - a)$; for a fixed α, η can be chosen equal to $\epsilon/(\alpha - a)$, and thus $\left|\int_a^\alpha dx \int_\beta^\infty f(x, y)\, dy\right| < \epsilon$, for $\beta \geqq \beta_\eta$, hence the condition $\lim_{\beta\sim\infty} \int_a^\alpha dx \int_\beta^\infty f(x, y)\, dy = 0$ is satisfied. Similarly it can be shewn that the other part of condition (2), of § 242, is satisfied.

The condition (4) of the present theorem may be stated in the form that $|\phi(\alpha, \beta) - \lim_{\alpha\sim\infty} \phi(\alpha, \beta)| < \epsilon$ for every value of β, and for all values of α not less than a fixed value α_ϵ. Since, on account of (2),

$$\lim_{\alpha\sim\infty} \phi(\alpha, \beta) = \int_b^\beta dy \int_a^\infty f(x, y)\, dx,$$

it is seen that $\left|\int_b^\beta dy \int_\alpha^\infty f(x, y)\, dx\right| < \epsilon$ for every value of β, and for $\alpha \geqq \alpha_\epsilon$.

* *Liouville's Journal* (4), vol. VIII (1892), p. 464.

Thus $$\left|\int_b^\infty dy \int_a^\infty f(x, y)\, dx\right| < \epsilon, \text{ for } a \geqq a_\epsilon,$$

hence $$\lim_{a\sim\infty} \int_b^\infty dy \int_a^\infty f(x, y)\, dx = 0,$$

which is one of the conditions (3)′ of the theorem of § 242. The theorem has been established, since it has been shewn that, if its conditions are satisfied, so also are those of the theorem of § 242

EXAMPLES

(1) It will be found that $\int_1^\infty dx \int_1^\infty \frac{x^2 - y^2}{(x^2+y^2)^2} dy = -\frac{1}{4}\pi$, and that $\int_1^\infty dy \int_1^\infty \frac{x^2 - y^2}{(x^2+y^2)^2} dx = \frac{1}{4}\pi$. In this case the first condition of the theorem in § 242 is satisfied. For

$$\int_1^a dx \int_1^\beta \frac{x^2 - y^2}{(x^2+y^2)^2} dy = -\int_1^\beta dy \int_1^a \frac{x^2 - y^2}{(x^2+y^2)^2} dx = \tan^{-1}\frac{a}{\beta} - \frac{\pi}{4},$$

hence $$\int_1^\infty dx \int_1^\beta \frac{x^2 - y^2}{(x^2+y^2)^2} dy = \frac{\pi}{4}, \quad \int_1^\infty dy \int_1^\beta \frac{x^2 - y^2}{(x^2+y^2)^2} = -\frac{\pi}{4}.$$

The second condition is however not satisfied; for

$$\lim_{a\sim\infty} \int_1^\infty dy \int_a^\infty \frac{x^2 - y^2}{(x^2+y^2)^2} dx = \lim_{a\sim\infty} \int_1^\infty \frac{a}{y^2+a^2} dy = \lim_{a\sim\infty}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{a}\right) = \frac{\pi}{2}.$$

(2) Let* $\phi(z) = \frac{pz^p}{1+z^{2p}}$, where $p > 1$; then $\phi'(z) = \frac{p^2 z^{p-1}(1-z^{2p})}{(1+z^{2p})^2}$. The repeated integral $\int_0^c dy \int_0^\infty \phi'(xy)\, dx = 0$, but $\int_0^\infty dx \int_0^c \phi'(xy)\, dy = \frac{1}{2}\pi$. In this case

$$\int_0^c dy \int_a^\infty \phi'(xy)\, dx = -\tan^{-1}(c^p a^p),$$

which does not converge to 0, as $a \sim \infty$. Thus the condition of the theorem in § 241 is violated.

(3) We have $\int_0^\infty dy \int_0^a \cos xy\, dx = \frac{\pi}{2}$, for $a > 0$, but $\int_0^a dx \int_0^\infty \cos xy\, dy$ does not exist.

(4) It may be shewn that $\int_0^\infty dx \int_0^c e^{-xy} dy = \int_0^c dy \int_0^\infty e^{-xy} dx$. For $e^{-xy} \geqq 0$, and one of the repeated integrals exists.

(5) Let† $V = \left(\frac{1}{x} + \frac{1}{y}\right) \sin \pi x \sin \pi y$; in this case we find

$$\int_1^h dx \int_1^k \frac{\partial^2 V}{\partial x \partial y} dy = \int_1^h dy \int_1^k \frac{\partial^2 V}{\partial x \partial y} = \left(\frac{1}{h} + \frac{1}{k}\right) \sin \pi h \sin \pi k.$$

The repeated integral $\int_1^\infty dx \int_1^\infty \frac{\partial^2 V}{\partial x \partial y}$ does not exist, for $\int_1^\infty \frac{\partial^2 V}{\partial x \partial y} dy$, or $\left[\frac{\partial V}{\partial x}\right]_1^\infty$ has no definite value, for any value of x. The double limit $\lim_{\substack{h\sim\infty\\k\sim\infty}} \int_1^h dy \int_1^k \frac{\partial^2 V}{\partial x \partial y} dx$ exists, and is equal to zero.

* See Stolz, *Grundzüge*, vol. III, pp. 8, 182, where the example is ascribed to Du Bois-Reymond.

† Bromwich, *Proc. Lond. Math. Soc.* (2), vol. I, p. 182.

(6) Let $V=\left(\frac{1}{x}+\frac{1}{y}\right)\left(1-\frac{1}{y}\right)\sin \pi x$, and $f(x, y)=\frac{\partial^2 V}{\partial x \partial y}$. We find, in this case,

$$\int_1^\infty dx \int_1^\infty f(x, y)\, dy = 0,$$

but the other repeated integral does not exist, since $\int_1^\infty f(x, y)\, dx$ has no definite value.

The double limit $\lim\limits_{\substack{h\sim\infty \\ k\sim\infty}} \int_1^h dx \int_1^k f(x, y)\, dy = 0$.

(7) Let $f(x, y)=\frac{\partial^2 V}{\partial x \partial y}$, $V=\frac{xy}{1+x^2+y^2}$. In this case, the two repeated integrals

$$\int_0^\infty dx \int_0^\infty f(x, y)\, dy, \quad \int_0^\infty dy \int_0^\infty f(x, y)\, dx$$

exist, and are both zero.

The conditions of the theorem of § 242 are satisfied. We find that

$$\int_0^a dx \int_\beta^\infty f(x, y)\, dy = \int_0^a \frac{\beta(1+\beta^2-x^2)}{(1+\beta^2+x^2)^2}\, dx;$$

since $\frac{\beta(1+\beta^2-x^2)}{(1+\beta^2+x^2)^2}$ is a bounded function of (x, β), the limit, when $\beta\sim\infty$, of the integral is accordingly zero. The function $\phi(a, \beta)=\frac{a\beta}{1+a^2+\beta^2}$; and its repeated limits as $a\sim\infty$, $\beta\sim\infty$ are both zero, although the double limit does not exist.

(8) It can be proved that the order of integration in $\int_0^\infty \sin y\, dy \int_0^\infty e^{-yx^2}\, dx$ can be reversed. Since the function $e^{-yx^2}\sin y$ is bounded, its repeated integrals over a finite rectangle $(0, 0;\ a, \beta)$ are equal. Also $\int_0^\beta e^{-yx^2}\sin y\, dy = \frac{1}{x^4+1}\{1-e^{-\beta x^2}(\cos\beta + x^2\sin\beta)\}$; and the expression on the right-hand side is numerically less than $\frac{2+x^2}{x^4+1}$ for all values of $\beta\ (\geqq 0)$, and this is a summable function of x in the interval $(0, \infty)$; therefore

$$\int_0^\infty dx \int_0^\infty e^{-yx^2}\sin y\, dy = \lim_{\beta\sim\infty}\int_0^\infty dx \int_0^\beta e^{-yx^2}\sin y\, dy,$$

and thus one of the conditions (1) of the theorem of § 242 is satisfied. Again

$$\int_0^a e^{-yx^2}\sin y\, dx = \sin y \int_0^a e^{-yx^2} dx < \frac{\sin y}{\sqrt{y}}\int_0^\infty e^{-t^2}\, dt;$$

hence $\int_0^a e^{-yx^2}\sin y\, dx$ is bounded for all values of $y\ (\geqq 0)$, and therefore

$$\int_0^\infty dy \int_0^\infty e^{-yx^2}\sin y\, dx = \lim_{a\sim\infty}\int_0^\infty dy \int_0^a e^{-yx^2}\sin y\, dx,$$

thus the second of the conditions (1) of the theorem of § 242 is satisfied.

We have also $\int_0^\infty dx \int_\beta^\infty e^{-yx^2}\sin y\, dy = \int_0^\infty \frac{e^{-\beta x^2}}{1+x^4}(\cos\beta + x^2\sin\beta)$; if we divide the integral on the right-hand side into two parts from 0 to 1, and from 1 to ∞, the first of these integrals has the limit 0, as $\beta\sim\infty$; the second is numerically less than $\int_1^\infty e^{-\beta x^2}\, dx$, or is less than $\frac{1}{\sqrt{\beta}}\int_0^\infty e^{-x^2}dx$, and thus converges to zero, as $\beta\sim\infty$. Therefore the second condition of the theorem is satisfied.

(9*) Let $f(x, y) = x^{n-1}e^{-x}(e^{-y} - e^{-xy})/y$, the field of integration being $(0, 0; \infty, \infty)$. In $(1, 0; \infty, \infty)$ we have $f(x, y) \geqq 0$, and in $(0, 0; 1, \infty)$, $f(x, y) \leqq 0$; thus the repeated integrals

$$\int_1^\infty dx \int_0^\infty f(x, y)\, dy, \quad \int_0^1 dx \int_0^\infty f(x, y)\, dy$$

may conveniently be considered separately. Since in each case the integrand is of fixed sign, we need only shew that one of the repeated integrals exists, both when the range of x is $(0, 1)$, and when it is $(1, \infty)$. It can thus be shewn that the repeated integrals are equal.

DIFFERENTIATION OF AN INTEGRAL WITH RESPECT TO A PARAMETER

244. Let $f(x, y)$ be a function of x defined in the interval or cell (a, b), and for each value of y in the interval $(y_0, y_0 + \alpha)$. This function $f(x, y)$, defined in the $p + 1$ dimensional cell $[x \text{ in } (a, b),\ y_0 \leqq y \leqq y_0 + x]$, will be assumed to be summable in (a, b) for almost all the values of y. It is a problem of importance to find sufficient conditions that

$$\frac{\partial}{\partial y}\int_a^b f(x, y)\, dx = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx, \text{ for } y = y_0.$$

This rule, first employed by Leibniz, is spoken of as differentiation under the sign of integration, and is an important example of the employment of the process of changing the order of repeated limits; a process, the validity of which is always subject to conditions, the sufficiency of which is a subject of investigation. In this rule the differentiation at y_0 is on one side; thus $\frac{\partial f(x, y)}{\partial y}$, at y_0, denotes the derivative on the right. If the function be defined for an interval $(y_0 - \alpha, y_0 + \alpha)$, of y, the derivative on the left may be treated in a similar manner, and when sufficient conditions on both sides of y_0 are satisfied, $\frac{\partial f(x, y)}{\partial y}$, at y_0, may be regarded as the differential coefficient in the ordinary sense.

Conditions sufficient to ensure that Leibniz's rule is applicable have been investigated by Jordan†, Harnack‡, de la Vallée Poussin§, G. H. Hardy||, and W. H. Young¶, and others. The problem has also been considered, of obtaining the differential coefficient when Leibniz's rule is not applicable.

245. Two methods may be employed to determine the requisite sufficient conditions. The first method, which will be here developed, depends upon the convergence of integrals of the incrementary ratio

* W. H. Young, *Camb. Phil. Trans* vol. XXI (1910), p. 375.

† *Cours d'Analyse*, vol. II, p. 155 (2nd ed.).

‡ *Elemente der Diff u. Integralrechnung.*

§ *Liouville's Journal*, (4) vol. VIII (1892), p. 421, and *Ann. de la soc. sc. de Bruxelles*, vol. XVI (B) (1891–2).

|| *Quarterly Journal of Math.* vol. XXXII (1901), p. 66.

¶ *Trans. Camb. Phil. Soc.* vol. XXI (1910), p. 397.

$\dfrac{f(x, y_0 + h) - f(x, y_0)}{h}$, as h converges to zero. Denoting the value of $\int_a^b f(x, y)\,dx$ by u_y, we have $\dfrac{u_{y_0+h} - u_{y_0}}{h} = \int_a^b \dfrac{f(x, y_0 + h) - f(x, y_0)}{h}\,dx$, where $h \leq \alpha$. If then $\lim_{h\sim 0} \int_a^b \dfrac{f(x, y_0 + h) - f(x, y_0)}{h}\,dx$ has a definite value, as the continuous variable h converges to zero, $\left(\dfrac{\partial u}{\partial y}\right)_{y=y_0}$ exists, and has the same value.

If further

$$\lim_{h\sim 0} \int_a^b \frac{f(x, y_0 + h) - f(x, y_0)}{h}\,dx = \int_a^b \lim_{h\sim 0} \frac{f(x, y_0 + h) - f(x, y_0)}{h}\,dx,$$

and $\dfrac{\partial f(x, y)}{\partial y}$ exists for almost all values of x, we then have

$$\left(\frac{\partial u}{\partial y}\right)_{y=y_0} = \int_a^b \left(\frac{\partial f(x, y)}{\partial y}\right)_{y=y_0} dx.$$

Assuming that $\dfrac{\partial f(x, y)}{\partial y}$ exists at all points y interior to the interval $(y_0, y_0 + h)$, and that $f(x, y)$ is continuous with respect to y in the closed interval $(y_0, y_0 + h)$, we have, employing the mean value theorem of I, § 262, $\dfrac{f(x, y_0 + h) - f(x, y_0)}{h} = \dfrac{\partial f(x, y_0 + \theta h)}{\partial y}$; where θ is such that $0 < \theta < 1$. We thus have $\dfrac{u_{y_0+h} - u_{y_0}}{h} = \int_a^b \dfrac{\partial f(x, y_0 + \theta h)}{\partial y}\,dx$; the number θ depending upon h and x.

If it be further assumed that $\dfrac{\partial f(x, y)}{\partial y}$ exists for all the values of x, and that it converges to $\left\{\dfrac{\partial f(x, y)}{\partial y}\right\}_{y=y_0}$, uniformly for all values of x in (a, b); we have, provided h is sufficiently small,

$$\frac{\partial f(x, y_0 + \theta h)}{\partial y} = \left\{\frac{\partial f(x, y)}{\partial y}\right\}_{y=y_0} + \beta(x),$$

where $|\beta(x)| < \epsilon$, for all the values of x.

Under these conditions, we have, since $\left|\int_a^b \beta(x)\,dx\right|$ is less than the arbitrary small number $\epsilon m(\Delta)$, where Δ denotes the cell or interval (a, b),

$$\left(\frac{\partial u}{\partial y}\right)_{y=y_0} = \lim_{h\sim 0} \frac{u_{y_0+h} - u_{y_0}}{h} = \int_a^b \left\{\frac{\partial f(x, y)}{\partial y}\right\}_{y=y_0} dx.$$

The condition of uniform convergence of $\dfrac{\partial f(x, y)}{\partial y}$ is satisfied, in particular, if $\dfrac{\partial f(x, y)}{\partial y}$ is continuous with respect to (x, y) in the $(p + 1)$-dimensional domain $[x \text{ in } (a, b);\ y_0 \leq y < y_0 + h]$.

Accordingly, the following theorem has been established:

If $f(x, y)$ be defined in the finite $(p+1)$ dimensional domain

$$[x \text{ in } (a, b);\ y_0 \leq y \leq y_0 + \alpha],$$

and $\int_a^b f(x, y)\, dy$ exists as an L-integral, for every value of y, it is a sufficient condition for the existence of the differential coefficient at y_0, and for the validity of the differentiation of $\int_a^b f(x, y)\, dy$, at y_0, on one side, under the integral sign, that $\frac{\partial f(x, y)}{\partial y}$ should be a continuous function of (x, y) in the whole domain; or more generally, it is sufficient that $\frac{\partial f(x, y)}{\partial y}$ should converge to $\left\{\frac{\partial f(x, y)}{\partial y}\right\}_{y=y_0}$ uniformly for all values of x in (a, b).

246. Less stringent conditions for the validity of the rule for the differentiation of the integral under the sign of integration may be obtained by employing the sufficient conditions given in §§ 225–227, for the convergence of $\int_a^b \frac{f(x, y_0+h) - f(x, y_0)}{h}\, dx$ to $\int_a^b \lim_{h \sim 0} \left\{\frac{f(x, y_0+h) - f(x, y_0)}{h}\right\} dx$.

In accordance with the theorem of § 225, it is sufficient that $\left\{\frac{\partial f(x, y)}{\partial y}\right\}_{y=y_0}$ should exist for all values of x, and that a function $\phi(x)$, summable in the finite, or infinite cell, or interval (a, b), should exist, and be such that $\left|\frac{f(x, y_0+h) - f(x, y_0)}{h}\right| \leq \phi(x)$, for all values of x in (a, b), and all values of h such that $0 < h \leq \alpha$. This condition is satisfied, in particular, when (a, b) is finite, if $\phi(x)$ has the constant value K. In case $\frac{\partial f(x, y)}{\partial y}$ exists in the whole domain $[x \text{ in } (a, b);\ y_0 \leq y \leq y_0 + \alpha]$, the above condition is satisfied if $\left|\frac{\partial f(x, y)}{\partial y}\right| \leq \phi(x)$, in the whole domain; in accordance with the theorem of I, § 280.

Thus the following theorem has been established:

If $\frac{\partial f(x, y)}{\partial y}$ exist in the finite, or infinite, domain

$$[x \text{ in } (a, b);\ y_0 \leq y < y_0 + \alpha],$$

and be such that $\left|\frac{\partial f(x, y)}{\partial y}\right| \leq \phi(x)$, where $\phi(x)$ is summable in the cell, or interval, (a, b); then $\int_a^b \frac{\partial f(x, y_0)}{\partial y_0}\, dx = \frac{\partial}{\partial y_0} \int_a^b f(x, y_0)\, dx$. When (a, b) is finite, we may have in particular $\left|\frac{\partial f(x, y)}{\partial y}\right| \leq K$, a constant.

In case $\frac{\partial f(x, y)}{\partial y}$ is a bounded function in the $(p+1)$-dimensional domain, finite or infinite, and $\psi(x)$ is absolutely summable in (a, b), we have

$$\left|\frac{\partial f(x, y)}{\partial y} \psi(x)\right| \leq K \mid \psi(x) \mid ,$$

and therefore:

If $\left|\frac{\partial f(x, y)}{\partial y}\right|$ *be bounded in* $[x$ *in* $(a, b),\ y_0 \leqq y \leqq y_0 + \alpha]$, *and* $\psi(x)$ *be absolutely summable in* (a, b), *then*

$$\int_a^b \frac{\partial f(x, y)}{\partial y} \psi(x)\, dx = \frac{\partial}{\partial y}\int_a^b f(x, y)\, \psi(x)\, dx.$$

If the theorem of § 225 be employed, it is seen to be sufficient for the application of the rule for differentiation under the sign of integration that $\frac{f(x, y_0 + h) - f(x, y_0)}{h}$ should be monotone with respect to h, for each value of x. But a simpler condition is obtained by assuming that $\frac{\partial f(x, y)}{\partial y}$ exists and is a monotone function of y in the interval $y_0 < y \leqq y_0 + \alpha$, for each value of x in (a, b). For $\frac{f(x, y_0 + h) - f(x, y)}{h}$ lies in the interval bounded by $\frac{\partial f(x, y_0)}{\partial y}$ and $\frac{\partial f(x, y_0 + h)}{\partial y}$, and therefore

$$\int_a^b \frac{f(x, y_0 + h) - f(x, y_0)}{h}\, dx$$

lies in the interval bounded by $\int_a^b \frac{\partial f(x, y_0)}{\partial y}\, dx$ and $\int_a^b \frac{\partial f(x, y_0 + h)}{\partial y}\, dx$.

Since $\frac{\partial f(x, y_0 + h)}{\partial y}$ is monotone with respect to h, employing the theorem of § 225, we have

$$\lim_{h \sim 0} \int_a^b \frac{\partial f(x, y_0 + h)}{\partial y}\, dx = \int_a^b \frac{\partial f(x, y_0)}{\partial y}\, dx;$$

for $\lim\limits_{\sim 0} \frac{\partial f(x, y_0 + h)}{\partial y} = \frac{\partial f(x, y_0)}{\partial y}$, on account of the fact that $\frac{\partial f(x, y)}{\partial y}$ is monotone with respect to y, and therefore in accordance with the theorem in I, § 283, $\frac{\partial f(x, y)}{\partial y}$ is continuous at y_0, since it cannot have a discontinuity of the second kind.

It now follows that

$$\lim_{h \sim 0} \int_a^b \frac{f(x, y_0 + h) - f(x, y_0)}{h}\, dx = \int_a^b \frac{\partial f(x, y_0)}{\partial y}\, dx.$$

The following theorem has now been established:

If $\dfrac{\partial f(x, y)}{\partial y}$ *is monotone with respect to* y, *in the interval* $y_0 \leqq y \leqq y_0 + \alpha$, *for each value of* x, *in the finite, or infinite, cell* (a, b), *then*

$$\int_a^b \frac{\partial}{\partial y} f(x, y)\, dx = \frac{\partial}{\partial y}\int_a^b f(x, y)\, dx,$$

for $y_0 \leqq y < y_0 + \alpha$, *it being assumed that* $\int_a^b f(x, y)\, dx$ *exists for each such value of* y.

247. The second method of obtaining sufficient conditions for the validity of the differentiation of an integral under the integral sign depends upon the condition for the equality of two repeated integrals.

Let it be assumed that, for almost all values of x, the relation

$$f(x, y_0 + h) - f(x, y_0) = \int_{y_0}^{y_0+h} \frac{\partial f(x, y)}{\partial y}\, dy$$

holds good. This is equivalent to the assumption that $f(x, y)$ is an indefinite integral in y, for almost all values of x.

We have then

$$\frac{u_{y_0+h} - u_{y_0}}{h} \equiv \int_a^b \frac{f(x, y_0 + h) - f(x, y_0)}{h}\, dx = \frac{1}{h}\int_a^b dx \int_{y_0}^{y_0+h} \frac{\partial f(x, y)}{\partial y}\, dy;$$

if now the order of integration in the repeated integral may be reversed, we have

$$\frac{u_{y_0+h} - u_{y_0}}{h} = \frac{1}{h}\int_{y_0}^{y_0+h} dy \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx,$$

and then, in case $\int_a^b \frac{\partial f(x, y)}{\partial y}\, dx$ be continuous with respect to y at the point y_0, we have

$$\left(\frac{\partial u}{\partial y}\right)_{y=y_0} = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx.$$

We thus obtain the following theorem:

If (1), $f(x, y)$ *is an indefinite L-integral in* y, *for almost all values of* x, *and* (2), *the repeated integrals of* $\dfrac{\partial f(x, y)}{\partial y}$ *over the domain*

$$[x \text{ in } (a, b);\ y_0 \leqq y \leqq y + \alpha]$$

have equal values, and (3), $\int_a^b \frac{\partial f(x, y)}{\partial y}\, dx$ *exists and is continuous with respect to* y, *at* y_0, *then*

$$\left\{\frac{\partial}{\partial y}\int_a^b f(x, y)\, dx\right\}_{y=y_0} = \int_a^b \frac{\partial f(x, y_0)}{\partial y}\, dx.$$

The condition (2) is satisfied in particular, when (a, b) is finite, if $\dfrac{\partial f(x, y)}{\partial y}$ is summable over the domain $[x \text{ in } (a, b);\ y_0 \leqq y \leqq y_0 + \alpha]$.

In case the function $f(x, y)$ is monotone with respect to y, $\frac{\partial f(x, y)}{\partial y}$ is of fixed sign, and in this case the condition (2) is necessarily satisfied (see § 237) whether (a, b) be finite or infinite; we have accordingly the following theorem:

If in the finite, or infinite, domain $[x$ in (a, b); $y_0 \leq y \leq y_0 + \alpha]$, $f(x, y)$ is monotone with respect to y, and is an indefinite L-integral in y, for almost every value of x, then, if also $\int_a^b \frac{\partial f(x, y)}{\partial y} dx$ is continuous with respect to y at y_0,

$$\left\{\frac{\partial}{\partial y}\int_a^b f(x, y)\, dx\right\}_{y=y_0} = \int_a^b \frac{\partial f(x, y_0)}{\partial y} dx.$$

248. The case in which (a, b) is a linear interval, and $b = \infty$, may be specially considered; in this case the condition that $\frac{\partial f(x, y)}{\partial y}$ should be summable in the domain $(a \leq x < \infty\,;\, y_0 \leq y \leq y_0 + \alpha)$ is not sufficient to ensure that the order of the repeated integral may be reversed; it may in fact happen that $\int_a^\infty \frac{\partial f(x, y)}{\partial y} dy$ does not exist.

We have

$$u_{y_0+h} - u_{y_0} = \lim_{X\sim\infty}\int_a^X \{f(x, y_0 + h) - f(x, y_0)\}\, dx$$
$$= \lim_{X\sim\infty}\int_a^X dx \int_{y_0}^{y_0+h} \frac{\partial f(x, y)}{\partial y} dy;$$

it being assumed as before that, in the finite domain

$$(a \leq x \leq X;\, y_0 \leq y \leq y_0 + \alpha),$$

$\frac{\partial f(x, y)}{\partial y}$ exists and is an L-integral with respect to y, whatever value X may have. If $\frac{\partial f(x, y)}{\partial y}$ is summable in the finite domain, we have

$$u_{y_0+h} - u_{y_0} = \lim_{X\sim\infty}\int_{y_0}^{y_0+h} dy \int_a^X \frac{\partial f(x, y)}{\partial y} dx \qquad \text{......(A)}.$$

If now
$$\lim_{X\sim\infty}\int_{y_0}^{y_0+h} dy \int_X^\infty \frac{\partial f(x, y)}{\partial y} dx = 0 \qquad \text{......(B)},$$

and further, if $\int_a^\infty \frac{\partial f(x, y)}{\partial y} dy$ be continuous with respect to y at y_0, we have

$$\left(\frac{\partial u}{\partial y}\right)_{y_0} = \int_a^\infty \frac{\partial f(x, y_0)}{\partial y} dx.$$

In case $\int_a^\infty \frac{\partial f(x, y)}{\partial y} dx$ does not exist, or the equation (B) be otherwise not valid, the equation (A) still holds, and it may in certain cases be applied to determine $\left(\frac{\partial u}{\partial y}\right)_{y=y_0}$.

Let us assume* that $\int_a^X \frac{\partial f(x,y)}{\partial y}\,dx$ can be divided into two components, so that $\int_a^X \frac{\partial f(x,y)}{\partial y}\,dx = \phi(X,y) + \int_a^X \psi(x,y)\,dx$, where $\phi(X,y)$ is such that $\lim_{X\sim\infty}\int_{y_0}^{y_0+h} \phi(X,y)\,dy = 0$, and where $\psi(x,y)$ is such that

$$\int_a^\infty dx \int_{y_0}^{y_0+h} \psi(x,y)\,dy = \int_{y_0}^{y_0+h} dy \int_a^\infty \psi(x,y)\,dx.$$

We find then, provided $\int_a^\infty \psi(x,y)\,dx$ is a continuous function of y and y_0, that $\left(\frac{\partial u}{\partial y}\right)_{y=y_0} = \int_a^\infty \psi(x,y_0)\,dx$.

249. In ordinary cases, a special case of the criteria of § 241 may be applied to establish the validity of the inversion involved in the use of the equation

$$\int_a^\infty dx \int_{y_0}^{y_0+h} \frac{\partial f(x,y)}{\partial y}\,dy = \int_{y_0}^{y_0+h} dy \int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx;$$

and then, provided $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx$ is continuous at y_0, with respect to y, we have

$$\left(\frac{\partial u}{\partial y}\right)_{y=y_0} = \int_a^\infty \left\{\frac{\partial f(x,y)}{\partial y}\right\}_{y=y_0} dx.$$

It is thus established that:

A sufficient condition for the differentiability of $\int_a^\infty f(x,y)\,dx$ at y_0, under the sign of integration, is that $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx$ shall converge uniformly for all values of y in the interval $(y_0, y_0+\alpha)$, and shall be a continuous function of y at y_0.

It may be observed that:

The condition that $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx$ shall be a continuous function of y, at y_0, may be replaced by the condition that $\int_a^X \frac{\partial f(x,y)}{\partial y}\,dx$ be continuous, whatever value X $(> a)$ may have, it being assumed that the condition of uniform convergence of $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx$ is satisfied.

For $\int_a^\infty \frac{\partial f(x,y)}{\partial y}\,dx = \int_a^X \frac{\partial f(x,y)}{\partial y}\,dx + \eta(y)$; where $|\eta(y)| < \epsilon$, provided X is sufficiently large.

* De la Vallée Poussin, *Ann. de la soc. sc. de Bruxelles*, vol. XVI (B) (1892), p. 150.

Hence, we have

$$\int_a^\infty \frac{\partial f(x, y_0+h)}{\partial y}\,dx - \int_a^\infty \frac{\partial f(x, y_0)}{\partial y}\,dx = \int_a^X \left\{\frac{\partial f(x, y_0+h)}{\partial y} - \frac{\partial f(x, y_0)}{\partial y}\right\} dx + \zeta,$$

where $|\zeta| < 2\epsilon$. From this it follows that, for all sufficiently small values of h, $\int_a^\infty \frac{\partial f(x, y_0+h)}{\partial y}\,dx - \int_a^\infty \frac{\partial f(x, y_0)}{\partial y}\,dx$ is numerically less than 3ϵ; and since ϵ is arbitrary $\int_a^\infty \frac{\partial f(x, y_0)}{\partial y}\,dx$ is continuous at y_0.

EXAMPLES

(1*) Let
$$f(x, y) = \sin\left(4\tan^{-1}\frac{y}{x}\right) - \frac{4xy}{x^2+y^2}\cos\left(4\tan^{-1}\frac{y}{x}\right);$$
then
$$\int_0^X f(x, y)\,dx = X\sin\left(4\tan^{-1}\frac{y}{X}\right).$$
We find that
$$\frac{\partial}{\partial y}\int_0^X f(x,y)\,dx = \frac{4X^2}{X^2+y^2}\cos\left(4\tan^{-1}\frac{y}{X}\right);$$
therefore, at the point $y=0$, $\frac{\partial}{\partial y}\int_0^X f(x, y)\,dx = 4$. The value of $\int_0^X \frac{\partial f(x, y)}{\partial y}\,dx$ is found to be $\frac{4X^2}{X^2+y^2}\cos\left(4\tan^{-1}\frac{y}{X}\right)$, when $y>0$, and it is zero when $y=0$. Since this integral is not continuous at $y=y_0$, the conditions of § 247 for differentiation under the sign of integration are not satisfied at $y=0$; in fact we have $\int_0^X \frac{\partial f(x, 0)}{\partial y}\,dx = 0$. The function $f(x, y)$ is discontinuous at the point $(0, 0)$.

(2) Consider the integral $\int_0^\infty \frac{\sin xy}{x}\,dx$, where $y>0$. This integral is not differentiable under the sign of integration for any value of y; for $\int_0^\infty \cos xy\,dx$ does not exist.

(3†) The integral $\int_0^X (x-y)^{\frac{1}{3}}\,dx$ may be differentiated under the sign of integration, for every value of y. For it has been shewn in § 238, Ex. (5), that $(x-y)^{-\frac{2}{3}}$ has an L-integral in the domain $(0, 0; X, h)$. Also $\int_0^X (x-y)^{-\frac{2}{3}}\,dx$ exists and is a continuous function of y; therefore the conditions of the theorem of § 247 are both satisfied.

(4) Consider the integral $u = \int_0^\infty \frac{\cos xy}{1+x^2}\,dx$, where $y>0$. The integral $\int_0^\infty \frac{x\sin xy}{1+x^2}\,dx$ converges uniformly for all values of y greater than a positive number y_0. For, integrating by parts, we find

$$\int_X^{X'} \frac{x\sin xy}{1+x^2}\,dx = \left[-\frac{x\cos xy}{(1+x^2)\,y}\right]_X^{X'} + \frac{1}{y}\int_X^{X'} \frac{1-x^2}{(1+x^2)^2}\cos xy\,dx;$$

hence, if $X'>X>1$, the absolute value of the integral on the left-hand side is less than

$$\frac{2X}{(1+X^2)\,y_0} + \frac{1}{y_0}\left(\frac{\pi}{2} - \tan^{-1}X\right)$$

which is $<\epsilon$, if X be chosen sufficiently large. It is clear that $\int_0^X \frac{x\sin xy}{1+x^2}\,dx$ is, for each value of X, a continuous function of $y\ (>0)$, for the integrand is bounded in the rectangle $(0, y; X, y+h)$, and thus the theorem of § 225 is applicable. It thus appears that the conditions of the theorem of § 249 are satisfied.

* Harnack's *Diff. and Int. Calc.*, Cathcart's translation, p. 266.

† Hardy, *Quarterly Journal of Math.* vol. XXXII (1901), p. 67

(5) Let $f(x, a)$ have the values $(a-x)^{\frac{1}{2}}-x, 0, (a-x)^{\frac{1}{2}}+x$ according as $x \lesseqgtr a$, where $-1<a<1$; then

$$\frac{d}{da}\int_{-1}^{1} f(x, a)\,dx = \int_{-1}^{1}\frac{\partial f(x, a)}{\partial a}\,dx - 2a.$$

(6*) The integral $\int_a^b f(y \pm x)\,\psi(x)\,dx$ is differentiable under the sign of integration, where y is in an interval $(-A, A)$ either if (1), $\psi(x)$ is summable in (a, b), and $f(t)$ has a differential coefficient that is bounded in $(a-A, b+A)$, or (2), if both $f(t)$ is an integral in $(a-A, b+A)$, and $\psi(x)$ is an integral in (a, b).

The case (1) is a particular case of the second theorem of § 246.

To prove (2), let $F(t) = \int^t f(t)\,dt$, then

$$\int_a^b f(y \pm x)\,\psi(x)\,dx = \Big[F(y \pm x)\,\psi(x)\Big]_a^b - \int_a^b F(y \pm x)\,\psi'(x)\,dx;$$

the integral on the right hand falls under case (1), and may therefore be differentiated. Thus we have

$$\frac{d}{dy}\int_a^b f(y \pm x)\,\psi(x)\,dx = \Big[f(y \pm x)\,\psi(x)\Big]_a^b - \int_a^b f(y \pm x)\,\psi'(x)\,dx = \int_a^b f'(y \pm x)\,\psi(x)\,dx.$$

250. Let (a, b) now denote a linear integral, and let $f(x, y)$ be defined in the interval $(a-\epsilon, b+\epsilon)$, for all values of y in some linear interval; and let $\int_a^b f(x, y)\,dx$ be denoted by $u(y, a, b)$.

We have $$\frac{u(y, a, b+k) - u(y, a, b)}{k} = \frac{1}{k}\int_b^{b+k} f(x, y)\,dx,$$

and thus $\frac{\partial u}{\partial b} = \lim_{h \sim 0}\frac{1}{k}\int_b^{b+k} f(x, y)\,dx$, provided the limit on the right-hand side exists. If, for a particular value of y, $f(x, y)$ is continuous with respect to x, at $x = b$, the limit on the right-hand side is equal to $f(b, y)$.

Again, if, for a particular value of y, $f(x, y)$ is, in a neighbourhood of the point $x = b$, the finite differential coefficient of a function $F(x)$ of x, we have (see I, § 471) $\int_b^x f(x, y)\,dx = F(x) - F(b)$, the integral being in general a D-integral; and thus $\frac{\partial u}{\partial b} = F'(b) = f(b, y)$.

The following theorem has now been established:

The integral $\int_a^b f(x, y)\,dx$ has, for a particular value of y, a differential coefficient with respect to b, of which the value is $f(b, y)$, if either (1), *$f(x, y)$ be continuous with respect to x at $x = b$, or more generally* (2), *if, in a neighbourhood of $x = b$, $f(x, y)$ is everywhere the finite differential coefficient of a function of x.*

The sufficient condition that the differential coefficient of the integral with respect to a is $-f(a, y)$ is precisely similar.

* See W. H. Young, *Proc. Roy. Soc.* (A), vol. XCIII (1917), p. 280.

251. Sufficient conditions have now been obtained that $u(y, a, b)$ should have, at a particular point (y, a_0, b_0), partial differential coefficients, of which the values are respectively $\int_{a_0}^{b_0} \frac{\partial f(f, y)}{\partial y} dx$, $-f(a_0, y)$, $f(b_0, y)$; we proceed to determine sufficient conditions that $u(y, a, b)$, regarded as a function of (y, a, b), should have a total differential at the particular point (y_0, a_0, b_0).

If $\phi(x, y, z)$ be a function of the three variables x, y, z, it can be shewn, as in I, § 309, where the case of a function of two variables is dealt with, that, it is sufficient, in order that $\phi(x_0 + h, y_0 + k, z_0 + l) - \phi(x_0, y_0, z_0)$ should be expressible in the form $h\frac{\partial\phi}{\partial x_0} + k\frac{\partial\phi}{\partial y_0} + l\frac{\partial\phi}{\partial z_0} + h\rho + k\sigma + l\tau$, where ρ, σ, τ converge to zero as h, k, l do so in any manner, that $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ have definite values at (x_0, y_0, z_0), and that one of these partial differential coefficients, say $\frac{\partial\phi}{\partial z}$, exists everywhere in some three-dimensional neighbourhood of (x_0, y_0, z_0), and is continuous at (x_0, y_0, z_0), and also that another of them, say $\frac{\partial\phi}{\partial y}$, exists in a two-dimensional neighbourhood of (x_0, y_0), for $z = z_0$, and is continuous at (x_0, y_0).

Applying this result to the function $u(y, a, b)$, we obtain the following theorems* which are found by replacing in different orders the three variables x, y, z by y, a, b.

If $f(x, y)$ be continuous with respect to (x, y) at the points (a_0, y_0), (b_0, y_0), and if $\int_{a_0}^{b_0} f(x, y)\, dx$ have a partial differential coefficient with respect to y, at the point y_0, then $\int_a^b f(x, y)\, dx$ has a total differential at the point (y_0, a_0, b_0) with respect to (y, a, b).

If $f(x, y)$ is, in neighbourhoods of the points a_0, b_0, for $y = y_0$, a finite differential coefficient of some summable function, with respect to x, and is continuous with respect to x at the point (b_0, y_0); and if further $u(y, a, b)$ has a differential coefficient with respect to y which is continuous with respect to (y, a, b) at the point (y_0, a_0, b_0), then $u(y, a, b)$ has a total differential with respect to (y, a, b) at the point (y_0, a_0, b_0).

If y, a, b are all differentiable functions of a single variable t, and the conditions of either of the above theorems are satisfied, we have

$$\frac{d}{dt} u(y_0, a_0, b_0) = f(b_0, y_0)\frac{db_0}{dt} - f(a_0, y_0)\frac{da_0}{dt} + \frac{\partial u}{\partial y_0}\frac{dy_0}{dt}.$$

* See W. H. Young, *Trans. Camb. Phil. Soc.* vol. XXI (1910), p. 402.

GENERALIZED INTEGRALS

252. In I, § 389, the generalized upper and lower integrals of a function have been defined in a manner dependent upon the division of the measurable set E, the field of integration, into a finite, or enumerably infinite, set of measurable parts. The relation of this definition, which is due to W. H. Young, with the definition of Lebesgue will here be investigated.

The following preliminary theorem is required:

If H be a closed set of points, in p-dimensions, and $\phi(x)$ be a function, defined in H, and of which U and L ($\geqq 0$) are the upper and lower boundaries in H, the necessary and sufficient condition that the $(p+1)$-dimensional set of points (x, y), defined by $[x$ in H, $0 \leqq y \leqq \phi(x)]$ should be closed, is that $\phi(x)$ should be upper semi-continuous in the closed set H. Also the necessary and sufficient condition that the set $[x$ in H, $\phi(x) \leqq y \leqq U]$ should be closed, is that $\phi(x)$ should be lower semi-continuous in H.

In the first part of the theorem, it is clear that the necessary and sufficient condition is that the set of points $(x, \phi(x))$ should have no limiting point $(\xi, \bar{y})$, such that $\bar{y} > \phi(\xi)$. This is equivalent to the condition that, ϵ being an arbitrarily chosen positive number, a p-dimensional neighbourhood of ξ can be so determined that $\phi(x) < \phi(\xi) + \epsilon$, for all points x in that neighbourhood; and this for every point ξ, of H. This is equivalent to the condition that $\phi(x)$ be upper semi-continuous in H.

The second part of the theorem can be established, in a similar manner, from the consideration that the necessary and sufficient condition is that the set of points $(x, \phi(x))$ should have no limiting point (ξ, y) such that $y < \phi(\xi)$.

It is clear that the condition $L \geqq 0$ may be removed, for, if $L < 0$, we can consider the function $\phi(x) - L$, for which the lower boundary is zero.

253. The following theorem will be established:

If $f(x)$ be a function defined for all points x, in a measurable set E, of p-dimensions, and if E be divided into two measurable parts E_1 and E_2, the functions $f_1(x)$, $f_2(x)$ being such that $f_1(x) = f(x)$, over E_1, and $f_1(x) = 0$, over E_2; $f_2(x) = f(x)$, over E_2, and $f_2(x) = 0$, over E_1; then

$$\int_{(E_1)} f_1(x)\,dx + \int_{(E_2)} f_2(x)\,dx = \int_{(E)} f(x)\,dx.$$

Also the corresponding result holds for the upper generalized integrals.

The set E_1 can be divided into a finite, or enumerably infinite, number of measurable parts $e^{(1)}$, such that $\Sigma\, l^{(1)} m(e^{(1)}) > \int_{(E_1)} f_1(x)\,dx - \frac{1}{2}\epsilon$; moreover E_2 can be similarly divided into parts $e^{(2)}$, such that

$$\Sigma\, l^{(2)} m(e^{(2)}) > \int_{(E_2)} f_2(x)\,dx - \tfrac{1}{2}\epsilon;$$

where $l^{(1)}$ is the lower boundary of $f_1(x)$ in $e^{(1)}$, and $l^{(2)}$ is the lower boundary of $f_2(x)$ in $e^{(2)}$. Since $e^{(1)}$ and $e^{(2)}$ taken together form a set of sub-divisions of E into measurable parts, we see that $\Sigma l^{(1)} m(e^{(1)}) + \Sigma l^{(2)} m(e^{(2)})$ forms a corresponding sum for the function $f(x)$ in the set E, and this is consequently $\leqq \underline{\int}_{(E)} f(x)\,dx$; it follows that

$$\underline{\int}_{(E)} f(x)\,dx > \underline{\int}_{(E_1)} f_1(x)\,dx + \underline{\int}_{(E_2)} f_2(x)\,dx - \epsilon;$$

and since ϵ is arbitrary, we have

$$\underline{\int}_{(E)} f(x)\,dx \geqq \underline{\int}_{(E_1)} f_1(x)\,dx + \underline{\int}_{(E_2)} f_2(x)\,dx.$$

In order to prove that this relation cannot be an inequality, let it be assumed that, if possible, E is divided into a set of measurable parts e, such that

$$\Sigma lm(e) > \underline{\int}_{(E_1)} f_1(x)\,dx + \underline{\int}_{(E_2)} f_2(x)\,dx.$$

These sets e will in general fall into three classes, those, $g^{(1)}$ which contain only points of E_1; those, $g^{(2)}$ which contain only points of E_2; and those h, which contain points both of E_1 and of E_2. Thus $\Sigma lm(e)$ consists of three sets of terms; let us consider a term $lm(h)$. The set h can be divided into two measurable parts $h^{(1)}$, and $h^{(2)}$, where $h^{(1)}$ consists entirely of points of E_1, and $h^{(2)}$ consists entirely of points of E_2. The term $lm(h)$ will then, in this further sub-division, be replaced by $l^{(1)}m(h^{(1)}) + l^{(2)}m(h^{(2)})$, where $l^{(1)} \geqq l$, and $l^{(2)} \geqq l$; and thus the term $lm(h)$ may be increased, but cannot be diminished. We thus obtain a set of sub-divisions of E_1 into measurable parts $g^{(1)}$, $h^{(1)}$, and also a set of sub-divisions of E_2 into a set of measurable sets $g^{(2)}$, $h^{(2)}$. It follows that

$$\Sigma lm(g^{(1)}) + \Sigma l^{(1)} m(h^{(1)}) \leqq \underline{\int}_{(E_1)} f_1(x)\,dx,$$

and
$$\Sigma lm(g^{(2)}) + \Sigma l^{(2)} m(h^{(2)}) \leqq \underline{\int}_{(E_2)} f_2(x)\,dx;$$

adding together the expressions on each side, and remembering that

$$\Sigma lm(h) \leqq \Sigma l^{(1)} m(h^{(1)}) + \Sigma l^{(2)} m(h^{(2)}),$$

we have
$$\Sigma lm(e) \leqq \underline{\int}_{(E_1)} f_1(x)\,dx + \underline{\int}_{(E_2)} f_2(x)\,dx,$$

which is contrary to the assumption made above. It now follows that

$$\underline{\int}_{(E)} f(x)\,dx = \underline{\int}_{(E)} f_1(x)\,dx + \underline{\int}_{(E)} f_2(x)\,dx.$$

If we employ u, the upper boundary of the function in a set e, instead of l, a similar proof will establish the fact that

$$\overline{\int}_{(E)} f(x)\,dx = \overline{\int}_{(E)} f_1(x)\,dx + \int_{(E)} f_2(x)\,dx.$$

The following corollary follows from the above theorem:

If $f(x)$ be a function, defined in a measurable set E, of any number of dimensions, and if E be contained in another measurable set F, and the function $g(x)$ be defined by $g(x) = f(x)$, in E, and $g(x) = 0$, in $F - E$, then

$$\overline{\int}_{(E)} f(x)\,dx = \overline{\int}_{(F)} g(x)\,dx, \text{ and } \underline{\int}_{(E)} f(x)\,dx = \underline{\int}_{(F)} g(x)\,dx.$$

It can be easily seen that:

If $f_1(x) \geqq f_2(x)$, then $\overline{\int}_{(E)} f_1(x)\,dx \geqq \overline{\int}_{(E)} f_2(x)\,dx$, *and*

$$\underline{\int}_{(E)} f_1(x)\,dx \geqq \underline{\int}_{(E)} f_2(x)\,dx.$$

For $\overline{\int}_{(E)} f_1(x)\,dx$ is the lower boundary of all the sums $\Sigma u^{(1)} m(e)$, where $u^{(1)}$ is the upper boundary of $f_1(x)$ in e; this sum is $\geqq \Sigma u^{(2)} m(e)$, where $u^{(2)}$ is the upper boundary of $f_2(x)$ in (e). It follows that the lower boundary in the first case is not less than the lower boundary in the second case. The second part of the theorem follows similarly from the fact that, in any set e, $l^{(1)} \geqq l^{(2)}$.

254. It will now be proved that:

If $f(x)$ be a non-negative function, defined in a measurable set E, of finite measure, the upper, and the lower, generalized integrals of $f(x)$ over E are equal, respectively, to the exterior and interior measures of the $(p+1)$-dimensional set of points (x, y), defined by $[x \text{ in } E,\ 0 \leqq y \leqq f(x)]$.

Let H be a closed set, contained in E, and let $\phi(x)$ be an upper semi-continuous function, defined in E, and such that $0 \leqq \phi(x) \leqq f(x)$. It has been shewn that the set $[x \text{ in } H,\ 0 \leqq y \leqq \phi(x)]$ is a closed set, and it is contained in the set $[x \text{ in } E,\ 0 \leqq y \leqq f(x)]$.

It will first be shewn that the lower generalized integral of $\phi(x)$, over H, is equal to the measure of the closed set $[x \text{ in } H,\ 0 \leqq y \leqq \phi(x)]$, which will be denoted by H_ϕ. It is impossible that $\underline{\int}_{(H)} \phi(x)\,dx > m(H_\phi)$; for if this inequality held good, H could be divided into a finite, or enumerably infinite, set of measurable parts e_ι, such that $\Sigma l_{e_\iota} m(e_\iota) > m(H_\phi)$; where l_{e_ι} denotes the lower boundary of $\phi(x)$ in e_ι. A finite number n, of these sets e_ι, could be so determined that $\sum_{\iota=1}^{\iota=n} l_{e_\iota} m(e_\iota) > m(H_\phi)$; further, in each of the sets e_ι $(\iota = 1, 2, \ldots n)$, a closed component f_ι could be so determined that $\sum_{\iota=1}^{\iota=n} l_{e_\iota} m(f_\iota) > m(H_\phi)$; where l_{e_ι} might have a greater value than before, when it is taken to refer to f_ι instead of e_ι, but could not have a lesser value. The set of points (x, y) such that x is in (f_ι), when $\iota = 1, 2, \ldots n$;

and $0 \leq y \leq l_{e_r}$, is a closed set, contained in H_ϕ, and it would have a measure greater than that of H_ϕ, which is impossible. It has thus been proved that $\int_{(H)} \phi(x)\,dx \leq m(H_\phi)$; and it will now be shewn that this relation cannot be an inequality. For, let it be assumed, if possible, that

$$m(H_\phi) > \int_{(H)} \phi(x)\,dx;$$

and let the net $(a_0, a_1, \dots a_n)$ be fitted on to the linear interval bounded by the lower and the upper boundaries of $\phi(x)$ in H; let e_r denote the measurable set of points x, such that $a_{r-1} \leq \phi(x) < a_r$, for $r = 1, 2, 3, n-1$; and let e_n denote the set for which $a_{r-1} \leq \phi(x) \leq a_n$. We may take the net such that the breadths of all its meshes are less than η. Then $m(H_\phi)$ lies between $\sum_{r=1}^{r=n} a_{r-1} m(e_r)$ and $\sum_{r=1}^{r=n} a_r m(e_r)$, which are the measures of sets contained in, and containing H_ϕ, respectively; and these measures differ from one another by less than $\eta m(H)$. Choosing η sufficiently small, we now see that $\sum_{r=1}^{r=n} a_{r-1} m(e_r) > \int_{(H)} \phi(x)\,dx$; thus we have, since a_{r-1} is the lower boundary of $\phi(x)$ in the set e_r, a finite set of measurable parts of H, such that $\Sigma l_{e_r} m(e_r)$ is greater than $\underline{\int}_{(H)} \phi(x)\,dx$, which is impossible.

It has now been proved that $m(H_\phi) = \int_{(H)} \phi(x)\,dx$.

The case will first be considered in which $f(x)$ is bounded in E. From the theorem of § 253, we have

$$\underline{\int}_{(E)} \phi(x)\,dx = \int_{(H)} \phi(x)\,dx + \int_{(E-H)} \phi(x)\,dx$$
$$= m(H_\phi) + \int_{(E-H)} \phi(x)\,dx;$$

the second lower integral on the right-hand side is less than $Um(E-H)$, where U is the upper boundary of $f(x)$ in E; and this will be arbitrarily small, since H may be so chosen that $m(E-H)$ is arbitrarily small. The interior measure of the set $[x \text{ in } E,\ 0 \leq y \leq f(x)]$ is the upper boundary of the measures of all closed sets interior to it, and is consequently the upper boundary of $m(H_\phi)$, as $m(H)$ converges to $m(E)$, and for all upper semi-continuous functions $\phi(x)$, such that $0 \leq \phi(x) \leq f(x)$. For any closed set contained in $[x \text{ in } E,\ 0 \leq y \leq f(x)]$ has for its section by a y-ordinate a closed set of which the upper and lower boundaries are $\phi(x)$, $\psi(x)$, where $\phi(x)$ is upper semi-continuous; and the measure of such a set is clearly not greater than that of the set $[x \text{ in } E,\ 0 \leq y \leq \phi(x)]$. It thus appears that the interior measure of the set $[x \text{ in } E,\ 0 \leq y \leq f(x)]$ is the upper boundary of $\underline{\int}_{(E)} \phi(x)\,dx$, for all functions $\phi(x)$ which are upper semi-continuous,

and such that $0 \leq \phi(x) \leq f(x)$. Let $\phi(x)$ be so determined that the interior measure of the set $[x \text{ in } E, 0 \leq y \leq f(x)]$ exceeds $\int_{(E)} \phi(x)\,dx$ by less than ϵ; since $\underline{\int}_{(E)} f(x)\,dx \geq \int_{(E)} \phi(x)\,dx$, it follows that the interior measure of $[x \text{ in } E, 0 \leq y \leq f(x)] \leq \int_{(E)} f(x)\,dx + \epsilon$, thus, since ϵ is arbitrary, it is $\leq \int_{(E)} f(x)\,dx$. It is impossible that this relation can be an inequality, for, as before, if it were so, a finite set e_ι of parts of E could be so determined that $\sum_{\iota=1}^{\iota=n} l_\iota m(e_\iota)$ would be greater than the interior measure of the set $[x$ in $E, 0 \leq y \leq f(x)]$; and by taking suitable closed parts g_ι of the sets e_ι, we should have $\sum_{\iota=n}^{\iota=l} l_\iota m(g_\iota) >$ the interior measure of $[x \text{ in } E, 0 \leq y < f(x)]$; and thus this last set of points would contain a closed set of measure greater than the interior measure of the set itself, which is impossible. Therefore $\int_{(E)} f(x)\,dx$ is equal to the interior measure of the set $[x \text{ in } E, 0 \leq y \leq f(x)]$.

Let us next consider the function $U - f(x)$, then $\underline{\int}_{(E)} \{U - f(x)\}\,dx$ is the interior measure of the set $[x \text{ in } E, 0 \leq y \leq U - f(x)]$, which is equal to the excess of $Um(E)$ over the exterior measure of the set

$$[x \text{ in } E, 0 \leq y \leq f(x)].$$

Also $\int_{(E)} \{U - f(x)\}\,dx$ is the upper boundary of sums $\Sigma (U - u_\iota)\, m(e_\iota)$, or $Um(E) - \Sigma u_\iota m(e_\iota)$; and is thus equal to $Um(E) - \overline{\int}_{(E)} f(x)\,dx$. It now follows that $\int_{(E)} f(x)\,dx$ is the exterior measure of the set

$$[x \text{ in } E, 0 \leq y \leq f(x)].$$

The theorem has now been established for the case in which $f(x)$ is bounded in E; we proceed to the case in which it is unbounded.

If $\overline{\int}_{(E)} f(x)\,dx$ has a finite value, a mode of division of E into measurable sets e_ι can be so determined that $\sum_{\iota=1}^{\infty} u_\iota m(e_\iota) - \int_{(E)} f(x)\,dx = \theta\epsilon$, where θ is such that $0 \leq \theta < 1$; ϵ being an arbitrarily chosen positive number.

The numbers u_ι can be divided into two sets, those, $u_\iota^{(1)}$ which are $\leq N_0$, and those, $u_\iota^{(2)}$ which are $> N_0$; where N_0 is an arbitrarily prescribed positive number.

Thus $$\sum_{\iota=1}^{\infty} u_\iota m(e_\iota) = \sum_{\iota=1}^{\infty} u_\iota^{(1)} m(e_\iota) + \sum_{\iota'=1}^{\infty} u_{\iota'}^{(2)} m(e_{\iota'});$$

and the sum $\sum_{\iota'=1}^{\infty} u_{\iota'}^{(2)} m(e_{\iota'})$, being convergent, can be expressed by

$$\sum_{\iota'=1}^{\iota'=s} u_{\iota'}^{(2)} m(e_{\iota'}) + \theta'\epsilon,$$

where θ' is such that $0 \leq \theta' < 1$. Let N be greatest of the s numbers $u_{\iota'}^{(2)}$, where $\iota' = 1, 2, 3, \dots s$. Amalgamating the two sums

$$\sum_{\iota=1}^{\infty} u_\iota^{(1)} m(e_\iota), \quad \sum_{\iota'=1}^{\iota'=s} u_{\iota'}^{(2)} m(e_{\iota'}),$$

we have a single sum $\sum_{\iota=1}^{\infty} u_\iota m(e_\iota)$, where all the numbers u_ι are $\leq N$. We have now, in this new notation

$$\sum_{\iota=1}^{\infty} u_\iota m(e_\iota) + \theta'\epsilon - \overline{\int}_{(E)} f(x)\,dx = \theta\epsilon.$$

Let F denote the set $\sum_{\iota=1}^{\infty} e_\iota$; this set F is a measurable component of E. Consider $\overline{\int}_{(F)} f^{(N)}(x)\,dx$, where $f^{(N)}(x)$ is the function defined by $f^{(N)}(x) = f(x)$, when $f(x) \leq N$; and $f^{(N)}(x) = N$, when $f(x) > N$. If $\sum_{\iota=1}^{\infty} u_\iota m(e_\iota) - \overline{\int}_{(F)} f^{(N)}(x)\,dx > \epsilon$, the set F may be divided into a set of measurable parts $\bar{e}_\iota$, such that $\sum_{\iota=1}^{\infty} u_\iota m(\bar{e}_\iota) - \overline{\int}_{(F)} f^{(N)}(x)\,dx = \theta''\epsilon$; where $0 \leq \theta'' < 1$. The set E has been divided into parts u_ι $(\iota = 1, 2, 3, \dots)$ and $u_{\iota'}^{(2)}$ $(\iota' = s+1, s+2, \dots)$; thus

$$\sum_{\iota=1}^{\infty} \bar{u}_\iota m(e_\iota) + \theta'\epsilon \geq \overline{\int}_{(E)} f(x)\,dx,$$

or $$\overline{\int}_{(F)} f^{(N)}(x)\,dx \geq \overline{\int}_{(E)} f(x)\,dx - 2\epsilon.$$

Now, by the corollary in § 244, the upper generalized integral of $f^{(N)}(x)$ over F is equal to that, over E, of the function which has the values $f^{(N)}(x)$ in F, and the value zero over $E - F$, and this cannot exceed the upper generalized integral of $f^{(N)}(x)$ over E.

Therefore $$\overline{\int}_{(E)} f^{(N)}(x)\,dx \geq \overline{\int}_{(E)} f(x)\,dx - 2\epsilon.$$

As $\epsilon \sim 0$, $N \sim \infty$, we have

$$\lim_{N\sim\infty} \overline{\int}_{(E)} f^{(N)}(x)\,dx \geq \overline{\int}_{(E)} f(x)\,dx.$$

It is impossible that this relation can be an inequality, for, if it were so, N could be chosen so large that

$$\overline{\int}_{(E)} f^{(N)}(x)\,dx > \overline{\int}_{(E)} f(x)\,dx,$$

which is impossible, since $f^{(N)}(x) \leq f(x)$ (see § 244). Therefore

$$\overline{\int}_{(E)} f(x)\,dx = \lim_{N \sim \infty} \overline{\int}_{(E)} f^{(N)}(x)\,dx.$$

Now $\overline{\int}_{(E)} f^{(N)}(x)\,dx$ is the exterior measure of the set

$$[x \text{ in } E;\ 0 \leq y \leq f^{(N)}(x)].$$

The exterior measure of the set $[x \text{ in } E;\ 0 \leq y \leq f(x)]$ being defined as the limit of the exterior measure of $[x \text{ in } E;\ 0 \leq y \leq f^{(N)}(x)]$, as $N \sim \infty$, it has now been shewn that $\overline{\int}_{(E)} f(x)\,dx$ is the exterior measure of the set

$$[x \text{ in } E;\ 0 \leq y \leq f(x)].$$

In a similar manner, it can be shewn that $\underline{\int}_{(E)} f(x)\,dx$ is the interior measure of the set $[x \text{ in } E;\ 0 \leq y \leq f(x)]$.

255. In accordance with the definition of a generalized integral given by W. H. Young (I, § 389), the integral $\int_{(E)} f(x)\,dx$ exists when the upper and lower integrals have the same value. Let us consider the case in which $f(x) \geqq 0$; the case of a function which is not necessarily positive, but has a finite lower boundary is at once reduced to the case in which $f(x) \geqq 0$, by adding to $f(x)$ a properly chosen constant.

The integral $\int_{(E)} f(x)\,dx$ exists only when the upper and lower generalized integrals have one and the same finite value. It will be shewn that, when $\int_{(E)} f(x)\,dx$ exists, so also does $\int_{(E)} f^{(N)}(x)\,dx$, where N has any positive value; when $f(x)$ is bounded, N may, of course, be restricted to have no value greater than the upper boundary of $f(x)$ in E.

Let N' be a number greater than N; then E may be divided in four different ways into measurable sets $e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}$, so that

$$0 \leq \Sigma u^{(1)} m(e^{(1)}) - \overline{\int}_{(E)} f^{(N)}(x)\,dx < \epsilon,$$

$$0 \leq \underline{\int}_{(E)} f^{(N)}(x)\,dx - \Sigma l^{(2)} m(e^{(2)}) < \epsilon,$$

$$0 \leq \Sigma u^{(3)} m(e^{(3)}) - \overline{\int}_{(E)} f^{(N')}(x)\,dx < \epsilon,$$

$$0 \leq \underline{\int}_{(E)} f^{(N')}(x)\,dx - \Sigma l^{(4)} m(e^{(4)}) < \epsilon.$$

If we define the set e_{pqrs} as the set common to all the four sets $e_p^{(1)}$, $e_q^{(2)}$, $e_r^{(3)}$, $e_s^{(4)}$, the division of E into the sets e_{pqrs} may be conceived of as a continuation of each one of the four sets of sub-divisions employed above; and the four inequalities will all be strengthened by the adoption of this new method of sub-division of E. It is thus seen that there exists a set of sub-divisions of E into measurable parts e, such that, for this system (e), the four conditions are satisfied that

$$\overline{\int}_{(E)} f^{(N)}(x)\,dx = \Sigma um(e) - \theta\epsilon,$$

$$\underline{\int}_{(E)} f^{N}(x)\,dx = \Sigma lm(e) + \theta'\epsilon,$$

$$\overline{\int}_{(E)} f^{(N')}(x)\,dx = \Sigma u'm(e) - \theta''\epsilon,$$

$$\underline{\int}_{(E)} f^{(N')}(x)\,dx = \Sigma l'm(e) + \theta'''\epsilon,$$

where θ, θ', θ'', θ''' are all in the interval $(0, 1)$, and u, l denote the upper and lower boundaries of $f^{(N)}(x)$, in e, and also u', l' denote the upper boundaries of $f^{N'}(x)$, in e.

We have now

$$\left\{\overline{\int}_{(E)} f^{(N')}(x)\,dx - \underline{\int}_{(E)} f^{(N')}(x)\right\} - \left\{\overline{\int}_{(E)} f^{(N)}(x)\,dx - \underline{\int}_{(E)} f^{(N)}(x)\,dx\right\}$$
$$= \Sigma(u' - l')\,m(e) - \Sigma(u - l)\,m(e) - (\theta'' + \theta''' - \theta - \theta')\,\epsilon.$$

In case e consists entirely of points at which $f^{(N)}(x) = N$, we have $u = l = N$, $u' - l' \geqq 0$; if e contains no points for which $f^{(N)}(x) = N$, we have $u = u'$, $l = l'$; and, if e contains both species of points, we have $l' = l$, $u' \geqq u$. Thus, in all cases $u' - l' \geqq u - l$; and the number

$$\Sigma(u' - l')\,m(e) - \Sigma(u - l)\,m(e)$$

is accordingly positive or zero, for each value of ϵ. If ϵ now converge to zero, it is seen that the expression on the left-hand side of the above equation is certainly $\geqq 0$. Therefore $\overline{\int}_{(E)} f^{(N)}(x)\,dx - \underline{\int}_{(E)} f^{(N)}(x)\,dx$ is a monotone non-diminishing function of N, as N is increased.

It now follows that the excess of the exterior over the interior measure of the set $[x \text{ in } E;\ 0 \leqq y \leqq f^{(N)}(x)]$ does not diminish as N increases. If, when N is infinite, this difference is zero, it follows that it must be zero for every value of N.

Thus, if $\int_{(E)} f(x)\,dx$ exists, so also does $\int_{(E)} f^{(N)}(x)\,dx$, for every positive value of N, and all the sets $[x \text{ in } E;\ 0 \leqq y \leqq f^{(N)}(x)]$ are measurable. In

accordance with the theorem established in I, § 427, the section of the set $[x \text{ in } E, 0 \leq y \leq f^{(N)}(x)]$ by $y = \bar{N}$ is measurable, for almost all values of $\bar{N}$ in the interval $(0, N)$. This section is congruent with the set of those points of E such that $f(x) \geqq \bar{N}$, and $< N$. Therefore, for almost all values of $\bar{N}$, the set of points for which $\bar{N} \leq f^{(N)}(x) \leq N$ is measurable; and from the theorem established in I, § 383, it follows that $f^{(N)}(x)$ is a measurable function. Since this holds for all values of N, it follows that $f(x)$ is measurable (see I, § 400).

It has now been proved that:

For any bounded function $f(x)$, and for any function with a finite lower boundary, the definition of an integral given in I, § 389, *is completely equivalent to the definition of Lebesgue.*

There remains for consideration only the case in which $f(x)$ is unbounded, both in the positive and the negative directions, in the measurable set E. Let it be assumed that $\overline{\int}_{(E)} f(x)\,dx$ has a finite value; and consider $\sum\limits_{\iota=1} u_\iota m(e_\iota)$, for any set of sub-divisions of E into measurable parts, such that, in each part, u_ι is finite. If $\sum\limits_{\iota=1} u_\iota m(e_\iota)$ is not absolutely convergent, the terms of the series can be so rearranged that the series diverges to $-\infty$ (see § 26); and in that case the lower boundary of $\sum\limits_{\iota=1} u_\iota m(e_\iota)$, for all possible sets of sub-divisions of E, would not exist as a finite number. Therefore the series $\sum\limits_{\iota=1} u_\iota m(e_\iota)$ must be absolutely convergent; and it can be arranged as two series, consisting respectively of positive, and of negative, terms. Let $f_+(x)$ denote the function defined by $f_+(x) = f(x)$, when $f(x) \geqq 0, f_+(x) = 0$, when $f(x) < 0$; and let $f_-(x)$ denote the function defined by $f_-(x) = f(x)$, when $f(x) \leq 0, f_-(x) = 0$, when $f(x) > 0$. The value of $\overline{\int} f_+(x)\,dx$ is defined as the lower boundary of the sum of the positive terms of $\sum\limits_{\iota=1} u_\iota m(e_\iota)$, when all sets of sub-division of E are considered; and $\overline{\int} f_-(x)\,dx$ is the lower boundary of the sum of the negative terms of $\sum\limits_{\iota=1} u_\iota m(e_\iota)$; and both these lower boundaries are finite, since $\overline{\int}_{(E)} f(x)\,dx$ is assumed to exist. Sets e, $e^{(1)}$, $e^{(2)}$ of sub-divisions of E can be so determined that

$$\sum_{\iota=1} u_\iota m(e_\iota) - \overline{\int}_{(E)} f(x)\,dx, \quad \sum_{\iota=1} u_\iota^{(1)} m(e_\iota^{(1)}) - \overline{\int}_{(E)} f_+(x)\,dx,$$

and

$$\sum_{\iota=1} u_\iota^{(2)} m(e_\iota^{(2)}) - \overline{\int}_{(E)} f_-(x)\,dx,$$

are all three in the interval $(0, \epsilon)$. If we take a new set of sub-divisions of E, of which the type is the set of points common to e_p, $e_q^{(1)}$, $e_r^{(2)}$; we obtain

a sub-division of E, which is a continuation of each of those above employed, and for which the three conditions

$$0 \leqq \Sigma u_\iota m(e_\iota) - \overline{\int}_{(E)} f(x)\,dx < \epsilon,$$

$$0 \leqq \Sigma \bar{u}_\iota m(e_\iota) - \overline{\int}_{(E)} f_+(x)\,dx < \epsilon,$$

$$0 \leqq \Sigma \bar{\bar{u}}_\iota m(e_\iota) - \overline{\int}_{(E)} f_-(x)\,dx < \epsilon,$$

hold good, where in the second inequality, only the positive values of u_ι, viz. $\bar{u}_\iota$ are taken, and in the third inequality, only the negative values of u_ι, viz. $\bar{\bar{u}}_\iota$ are taken. We thus find, since

$$\Sigma u_\iota m(e_\iota) = \Sigma \bar{u}_\iota m(e_\iota) + \Sigma \bar{\bar{u}}_\iota m(e_\iota),$$

that $\overline{\int}_{(E)} f_+(x)\,dx + \overline{\int}_{(E)} f_-(x)\,dx$ differs from $\overline{\int}_{(E)} f(x)\,dx$ by less than ϵ. Since ϵ is arbitrary, it follows that

$$\overline{\int}_{(E)} f_+(x)\,dx + \overline{\int}_{(E)} f_-(x)\,dx = \overline{\int}_{(E)} f(x)\,dx.$$

Similarly, assuming that $\underline{\int}_{(E)} f(x)\,dx$ has a finite value, we find that

$$\underline{\int}_{(E)} f_+(x)\,dx + \underline{\int}_{(E)} f_-(x)\,dx = \underline{\int}_{(E)} f(x)\,dx.$$

We now see that

$$\overline{\int}_{(E)} f(x)\,dx - \underline{\int}_{(E)} f(x)\,dx$$

is the sum of

$$\overline{\int}_{(E)} f_+(x)\,dx - \underline{\int}_{(E)} f_+(x)\,dx \text{ and } \overline{\int}_{(E)} f_-(x)\,dx - \underline{\int}_{(E)} f_-(x)\,dx.$$

All three of these are $\geqq 0$; it thus follows that, if $\int_{(E)} f(x)\,dx$ exists, in accordance with W. H. Young's definition, so also do $\int_{(E)} f_+(x)\,dx$ and $\int_{(E)} f_-(x)\,dx$.

It now follows, since $f_+(x)$ has a finite lower boundary, that $\int_{(E)} f_+(x)\,dx$ and $\int_{(E)} f_-(x)\,dx$ exist as L-integrals, having the same values as before. Hence $\int_{(E)} f(x)\,dx$ is the sum of the two L-integrals $\int_{(E)} f_+(x)\,dx$, $\int_{(E)} f_-(x)\,dx$, and this sum defines the L-integral $\int_{(E)} f(x)\,dx$. Therefore, when $f(x)$ has a generalized integral over E, in accordance with the definition in I, § 389, it is summable over E, and the L-integral has the same value as the generalized integral.

Conversely, let it be assumed that $f(x)$ is summable over E. In accordance with the definition of the L-integral, given in I, § 388, if e_r is the set of points at which $a_{r-1} \leqq f(x) < a_r$, the two sums $\sum\limits_{-\infty}^{\infty} a_{r-1} m(e_r)$, $\sum\limits_{-\infty}^{\infty} a_r m(e_r)$ differ from one another by less than $\eta m(E)$, and the L-integral $\int_{(E)} f(x)\,dx$ is between the values of the two sums. If u_e, l_e denote the boundaries of $f(x)$ in e_r, we have $u_e \leqq a_r$, $u_l \geqq a_{r-1}$; and it follows that

$$\sum_{-\infty}^{\infty} u_r m(e_r) - \sum_{-\infty}^{\infty} l_r m(e_r) < \eta m(E).$$

Moreover the lower boundary of $\sum\limits_{-\infty}^{\infty} u_r m(e_r)$, and the upper boundary of $\sum\limits_{-\infty}^{\infty} l_r m(e_r)$, for the system of nets corresponding to a sequence of values of η, converging to zero, are both the L-integral $\int_{(E)} f(x)\,dx$. It then follows that

$$\overline{\int}_{(E)} f(x)\,dx \leqq \int_{(E)} f(x)\,dx, \text{ and } \underline{\int}_{(E)} f(x)\,dx \geqq \int_{(E)} f(x)\,dx.$$

Since $\overline{\int}_{(E)} f(x)\,dx \geqq \underline{\int}_{(E)} f(x)\,dx$, it follows from these relations that $\int_{(E)} f(x)\,dx = \underline{\int}_{(E)} f(x)\,dx = \overline{\int}_{(E)} f(x)\,dx$. Therefore the generalized integral of $f(x)$, over E, exists, and has the value of the L-integral.

The theorem has now been completely established, that:

The definition of an integral, of any function, bounded or unbounded, over a measurable set E, of finite measure, given in I, § 389, *is completely equivalent to the definition of Lebesgue.*

Moreover, utilizing results obtained in § 254, we have the following theorem:

The lower generalized integral of a non-negative function $f(x)$, defined in a measurable set E of finite measure, is the upper boundary of the lower generalized integrals, over E, of all upper semi-continuous functions $\phi(x)$, defined in E, and such that $0 \leqq \phi(x) \leqq f(x)$.

If the bounded function $f(x)$ be no longer restricted to be non-negative, a number c can be so chosen that $f(x) + c$ is non-negative, and if $\phi(x)$ be any upper semi-continuous function $\leqq f(x)$, the function $\phi(x) + c$ will also be upper semi-continuous, and $\leqq f(x) + c$. Applying the above theorem to the function $f(x) + c$, it is seen that the upper boundary of all the integrals $\int_{(E)} \{\phi(x) + c\}\,dx$ is $\underline{\int}_{(E)} \{f(x) + c\}\,dx$, and hence that $\underline{\int}_{(E)} \phi(x)\,dx$ has for its upper boundary $\int_{(E)} f(x)\,dx$. Thus we have the theorem that:

If $f(x)$ be any bounded function, defined in the measurable set E, of finite measure, the lower generalized integral of $f(x)$ over the set E is the upper boundary of the lower generalized integrals of all upper semi-continuous functions defined in E, such that $\phi(x) \leqq f(x)$.

If we make use of the fact that the exterior measure of a $(p+1)$-dimensional set is obtained by subtracting the interior measure of the complement of the set with respect to a $(p+1)$-dimensional cell which contains the set from the measure of that cell, we obtain the following theorem:

If $f(x)$ be any bounded function, defined in the measurable set E, of finite measure, the generalized upper integral of the function over E is the lower boundary of the upper generalized integrals of all lower semi-continuous functions, defined in E, such that $\phi(x) \geqq f(x)$.

THE METHOD OF MONOTONE SEQUENCES

256. A method of investigating properties of integrals, depending upon the use of integrals of monotone sequences of functions of special types, has been developed* by W. H. Young. This method consists of the extensions of properties of the integrals of the functions which constitute the monotone sequence to the integral of the function to which the monotone sequence converges. A general account of this method will be given here; it is possible to use this method conversely as the basis of a general theory of integration of all functions capable of analytical definition.

If $f(x)$ be a bounded non-negative function, defined in a measurable set E, of finite measure, it has been shewn in § 255, that $\underline{\int}_{(E)} f(x)\,dx$ is the upper boundary of the integrals $\int_{(E)} \phi(x)\,dx$, for all upper semi-continuous functions $\phi(x)$ such that $\phi(x) \leqq f(x)$. It will be shewn that $\phi(x)$ is summable in E, so that $\underline{\int}_{(E)} \phi(x)\,dx = \int_{(E)} \phi(x)\,dx$.

It should be observed that there is no loss of generality in restricting the functions $\phi(x)$ to be non-negative. For, if $\phi(x)$ be any upper semi-continuous function, it remains so if all its negative values be changed to zero.

If $\{H_n\}$ denotes a sequence of closed sets, contained in E, and such that each set of the sequence is contained in the next, the sequence can be so determined that, if H_ω be the outer limiting set of the sequence,

$$m(H_\omega) = m(E).$$

* See "A new method in the theory of integration," *Proc. Lond. Math. Soc.* (2), vol. IX (1910); also "The general theory of integration," *Phil. Trans.* (A), vol. CCIV (1905). See further *Mess. of Math.* vol. XXVII (1907), p. 148; *Proc. Camb. Phil. Soc.* vol. XIV (1908), p. 520; *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 298; and *Comptes Rendus*, vol. CLXII (1916), p. 909.

It is known (see I, § 230) that the set K_n, of points of H_n at which $\phi(x) \geqq \alpha$, is closed, whatever value α may have. The sequence $\{K_n\}$ has accordingly an outer limiting set K_ω, which is measurable. Each point of H_ω at which $\phi(x) \geqq \alpha$ is contained in H_n, for some value of n, and for all greater values; and the set of points of H_ω at which $\phi(x) \geqq \alpha$ is measurable. The set of points of $E - H_\omega$ having measure zero, the set of points of $E - H_\omega$ at which $\phi(x) \geqq \alpha$ has measure zero. Therefore the set of points of E at which $\phi(x) \geqq \alpha$ is measurable; and therefore $\phi(x)$ is summable in E.

If $f(x)$ be any bounded function, it can be reduced to a non-negative function by the addition of a suitable constant. It has therefore been proved that:

If $f(x)$ be a bounded function, defined in a measurable set E, of finite measure, of any number of dimensions, the lower generalized integral of $f(x)$ over E is equal to the upper boundary of the integrals of all upper semi-continuous functions, defined in E, and such that $f(x) \geqq \phi(x)$.

If the function $U - f(x)$ be considered, instead of $f(x)$, it can be shewn at once that:

The upper generalized integral of the bounded function $f(x)$ over E is equal to the lower boundary of the integrals of all lower semi-continuous functions $\psi(x)$, such that $\psi(x) \geqq f(x)$.

257. A sequence $\{\phi_n(x)\}$, of upper semi-continuous functions, can be so determined that $\lim_{n\to\infty}\int_{(E)} \phi_n(x)\,dx = \underline{\int}_{(E)} f(x)\,dx$. It will be shewn that the sequence $\{\phi_n(x)\}$ can be so determined as to be monotone. Taking the two functions $\phi_n(x)$, $\phi_{n+1}(x)$, let $\chi_{n+1}(x)$ be the function which has, at each point x, the value of the greater of the two functions $\phi_n(x)$, $\phi_{n+1}(x)$. This function $\chi_{n+1}(x)$ is upper semi-continuous (see § 111), and it is $\leqq f(x)$; moreover $\int_{(E)} \chi_{n+1}(x)\,dx \geqq \int_{(E)} \phi_{n+1}(x)\,dx$. Starting with $\phi_1(x)$, $\phi_2(x)$ a monotone sequence $\{\chi_n(x)\}$ will be formed, which has the same property, in relation to $\underline{\int}_{(E)} f(x)\,dx$, as the original sequence. It has thus been shewn that:

A monotone non-diminishing sequence of functions, all upper semi-continuous in the measurable set E can be so determined that, if $\{\phi_n(x)\}$ denote the sequence, $\lim_{n\sim\infty}\int_{(E)} \phi_n(x)\,dx = \underline{\int}_{(E)} f(x)\,dx$; where $f(x)$ is any bounded function. Moreover $\phi_n(x) \leqq f(x)$.

In a similar manner, it can be shewn that:

A monotone non-increasing sequence of functions, all lower semi-continuous in E, can be so determined that, if $\{\psi_n(x)\}$ denote the sequence,

$$\lim_{n\sim\infty}\int_{(E)} \psi_n(x)\,dx = \int_{(E} f(x)\,dx.$$

The monotone sequence of functions $\{\phi_n(x)\}$ will converge to a function $\Phi(x)$, which is an *lu*-function (see § 111), such that $\Phi(x) \leqq f(x)$. Since $|\phi_n(x)|$ is bounded for all values of n and x, we have

$$\int_{(E)} \Phi(x)\,dx = \lim_{n\sim\infty} \int_{(E)} \phi_n(x)\,dx;$$

and therefore $$\int_{(E)} \Phi(x)\,dx = \underline{\int}_{(E)} f(x)\,dx.$$

The function $\Phi(x)$ may be termed a bounding *lu*-function of $f(x)$.

Similarly if $\Psi(x)$ be the lower limit of the sequence $\{\psi_n(x)\}$, of lower semi-continuous functions, we have $\int_{(E)} \Psi(x)\,dx = \overline{\int}_{(E)} f(x)\,dx$; and $\Psi(x)$ is a bounding *ul*-function of $f(x)$.

In case $f(x)$ is summable in E, we have

$$\int_{(E)} \Psi(x)\,dx = \int_{(E)} f(x)\,dx = \int_{(E)} \Phi(x)\,dx;$$

and since $\Psi(x) \geqq f(x) \geqq \Phi(x)$, we see that $\Psi(x) = f(x)$, almost everywhere, and $\Phi(x) = f(x)$, almost everywhere.

Thus it has been shewn that:

When $f(x)$ is summable in E, functions $\Phi(x)$, $\Psi(x)$ which are bounding lu-functions and bounding ul-functions of $f(x)$ are such that almost everywhere in E, $\Phi(x) = \Psi(x) = f(x)$.

The functions $\Phi(x)$, $\Psi(x)$ are not unique, but it follows from this theorem that two functions which are both bounding *ul*-functions of $f(x)$ differ from one another only at points of a set of measure zero. A similar statement applies to two bounding *lu*-functions.

258. It will now be shewn that, if $f(x)$ be a bounded summable function, defined in the measurable set E, a bounding *lu*-function and a bounding *ul*-function of $f(x)$ can be constructed.

If U and L are the upper and lower boundaries of $f(x)$ in E, let (L, U) be divided into n equal parts of lengths $(U - L)/n$; and let a_r denote $L + \frac{r}{n}(U - L)$, where $r = 0, 1, 2, \ldots n$. Let g_r denote the measurable set of points of E at which $f(x) \geqq a_r$. The sets $g_{n-1}, g_{n-2}, \ldots g_0$ are such that each set is contained in the next; closed parts $h_{n-1}, h_{n-2}, \ldots h_0$ of the sets $g_{n-1}, g_{n-2}, \ldots g_0$ can be determined, each one of which is contained in the next, and such that $m(g_0) - m(h_0)$ is less than $\frac{1}{n}$.

Let the function $\phi_n(x)$ be defined by means of the specifications

$$\phi_n(x) = a_{n-1} \text{ in } h_{n-1};\ \phi_n(x) = a_{n-2} \text{ in } h_{n-2} - h_{n-1}, \ldots,$$
$$\phi_n(x) = a_r \text{ in } h_r - h_{r+1}, \ldots;\ \phi_n(x) = a_1 \text{ in } h_1 - h_2;$$

and $\phi_n(x) = a_0 \equiv L$, in all the remaining points of E.

Let the function $\psi_n(x)$ be defined by means of the specifications

$$\psi_n(x) = a_1 \text{ in } h_0 - h_1;\ \psi_n(x) = a_2 \text{ in } h_1 - h_2, \ldots,$$
$$\psi_n(x) = a_{r+1} \text{ in } h_r - h_{r+1}, \ldots;\ \psi_n(x) = a_{n-1} \text{ in } h_{n-2} - h_{n-1},$$

and $\psi_n(x) = a_n \equiv U$ in all the remaining points of E.

The function $\phi_n(x)$ is a u-function, and $\psi_n(x)$ is an l-function; the sequence $\{\phi_n(x)\}$ is monotone non-diminishing, and the sequence $\{\psi_n(x)\}$ is monotone non-increasing. Moreover $\psi_n(x) - \phi_n(x) = \dfrac{U-L}{n}$, except at points of a set of measure less than $\dfrac{1}{n}$, at which $\psi_n(x) - \phi_n(x) = U - L$. It follows that, except at the points of a set of measure zero, we have

$$\Phi(x) = \lim_{n\sim\infty} \phi_n(x) = \lim_{n\sim\infty} \psi_n(x) = \Psi(x).$$

Moreover $\Phi(x)$, $\Psi(x)$ are both equal to $f(x)$, almost everywhere in E, and are thus bounding lu- and ul-functions, corresponding to $f(x)$.

In case the set E is a linear interval (a, b), and the function $f(x)$ is monotone in E, the semi-continuous functions $\phi_n(x)$, $\psi_n(x)$, constructed as above, are also monotone in (a, b). Let $f(x)$ be monotone non-diminishing, then the sets $g_{n-1}, g_{n-2}, \ldots g_0$ consist of intervals (α_{n-1}, b), (α_{n-2}, b), $\ldots$ (α_0, b) which are closed at the end-point b, but not necessarily at the end-points $\alpha_{n-1}, \alpha_{n-2}, \ldots \alpha_0$. The closed sets $h_{n-1}, h_{n-0}, \ldots h_0$ may be taken to consist of the closed intervals (β_{n-1}, b) $(\beta_{n-2}, b), \ldots (\beta_0, b)$, each of which is contained in the next; where $\beta_0 - a < \dfrac{1}{n}$. The function $\phi_n(x) = a_{n-1}$, when $\beta_{n-1} \leqq x \leqq b$; $\phi_n(x) = a_{n-2}$, when $\beta_{n-2} \leqq x < \beta_{n-1}$; $\phi_n(x) = a_r$, when $\beta_r \leqq x < \beta_{r+1}$; $\phi_n(x) = a$, when $a \leqq x < \beta_1$. Thus $\phi_n(x)$ is monotone non-diminishing, and is constant in each interval of a finite set. A similar remark applies to the function $\psi_n(x)$.

259. There remains for consideration the case of an unbounded summable function $f(x)$. First let $f(x) \geqq 0$, then $\int_{(E)} f(x)dx$ is $\lim_{r\sim\infty} \int_{(E)} f^{(N_r)}(x)dx$; where $\{N_r\}$ is a monotone divergent sequence of positive numbers, and $f^{(N_r)}(x)$ is the function which has the value of $f(x)$, when $f(x) \leqq N_r$, and which has the value N_r when $f(x) > N_r$. Let $\Phi_r(x)$ be a bounding lu-function for the function $f^{(N_r)}(x)$, then $\int_{(E)} \Phi_r(x)\,dx = \int_{(E)} f^{(N_r)}(x)\,dx$; and thus we have $\int_{(E)} f(x)\,dx = \lim_{r\sim\infty} \int_{(E)} \Phi_r(x)\,dx$.

That the bounding functions $\Phi_r(x)$ for the functions $f^{(N_r)}(x)$ may be so determined that $\{\Phi_r(x)\}$ is a monotone non-diminishing sequence is a consequence of the following lemma:

If $f^{(1)}(x), f^{(2)}(x)$ are bounded summable functions such that $f^{(1)}(x) \leqq f^{(2)}(x)$, in E, then the corresponding bounding lu-functions $\Phi^{(1)}(x)$, $\Phi^{(2)}(x)$ can be so

determined that $\Phi^{(1)}(x) \leqq \Phi^{(2)}(x)$. ***Similarly the corresponding bounding ul-functions*** $\Psi^{(1)}(x)$, $\Psi^{(2)}(x)$ ***can be so determined that*** $\Psi^{(1)}(x) \geqq \Psi^{(2)}(x)$.

Let $\Phi^{(1)}(x) = \lim_{n\sim\infty} \phi_n^{(1)}(x)$, $\Phi^{(2)}(x) = \lim_{n\sim\infty} \phi_n^{(2)}(x)$; where $\{\phi_n^{(1)}(x)\}$, $\{\phi_n^{(2)}(x)\}$ are monotone non-diminishing sequences of u-functions. A new monotone sequence $\{g_n(x)\}$ of u-functions can be so determined that $g_n(x)$ has as its value, at each point, the greater of the two functions $\phi_n^{(1)}(x)$, $\phi_n^{(2)}(x)$. At any point at which $g_n(x)$ differs from $\phi_n^{(2)}(x)$, it is equal to $\phi_n^{(1)}(x)$ and is therefore $\leq f^{(1)}(x)$, and therefore also $\leqq f^{(2)}(x)$. Also since $\phi_n^{(2)}(x) \leqq g_n(x)$, $\int_{(E)} \phi_n^{(2)}(x)\,dx \leqq \int_{(E)} g_n(x)\,dx$; and thus

$$\lim_{n\sim\infty} \int_{(E)} \phi_n^{(2)}(x)\,dx = \lim_{n\sim\infty} \int_{(E)} g_n(x)\,dx,$$

since the limit on the left is the highest possible. Thus the sequence $\{g_n(x)\}$ may be taken instead of the sequence $\{\phi_n^{(2)}(x)\}$, in which case we have $\phi_n^{(1)}(x) \leqq \phi_n^{(2)}(x)$, or $\Phi^{(1)}(x) \eqslantless \Phi^{(2)}(x)$. The second part of the lemma may be deduced by means of a change of sign of the functions.

It has now been shewn that there exists a monotone non-diminishing sequence $\{\Phi_r(x)\}$ of lu-functions such that $\int_{(E)} f(x)\,dx = \lim_{r\sim\infty} \int_{(E)} \Phi_r(x)\,dx$.

If $\Phi(x)$ is the limit of $\{\Phi_r(x)\}$, it is an llu-function, i.e. an lu-function.

Thus, if $f(x)$ be a positive unbounded summable function, there exists an lu-function $\Phi(x)$ such that the integrals of $f(x)$ and of $\Phi(x)$ over the set E are identical.

If $f(x)$ be an unbounded summable function having both signs, it can be expressed as the difference of two positive summable functions, and $\int_{(E)} f(x)\,dx$ can be expressed as the difference of the integrals of two lu-functions, or as the sum of an lu-function and a ul-function. Therefore $f(x)$ can be replaced, for the purpose of determining its integral, either by an ulu-function, or by a lul-function.

260. The process of deducing properties of integrals of summable functions from those of the integrals of continuous functions, or even from those of finite polynomials, consists of the following stages, in each of which a monotone sequence is employed.

(1) Let E be a closed set of points in any number of dimensions, and $f(x)$ a continuous function defined in E. If Δ be a fundamental cell which contains E, a continuous function may be defined in Δ which has at every point of E the value of $f(x)$ (see § 108). Applying to Δ the theorem of § 161, a monotone sequence of polynomials can be constructed which converges in Δ, and therefore in E, uniformly to the value $f(x)$. It then follows that $\int_{(E)} f(x)\,dx$ is representable as $\lim_{n\sim\infty} \int_{(E)} P_n(x)\,dx$, where $\{P_n(x)\}$ is the monotone sequence of polynomials.

(2) If $f(x)$ be an upper semi-continuous bounded function defined in the measurable set E, there exists (see § 105) a monotone non-increasing sequence $\{f_n(x)\}$, of continuous functions which converges to $f(x)$. Then $\int_{(E)} f(x)\,dx$ has the value $\lim_{n\sim\infty}\int_{(E)} f_n(x)\,dx$.

Similarly, if $f(x)$ be a bounded l-function, there exists a monotone non-diminishing sequence of continuous functions $f_n(x)$, such that

$$\lim_{n\sim\infty}\int_{(E)} f_n(x)\,dx = \int_{(E)} f(x)\,dx.$$

(3) If $f(x)$ be any bounded function, summable in the measurable set E, there exists a monotone non-diminishing sequence $\{\phi_n(x)\}$ of u-functions, such that $\int_{(E)} f(x)\,dx = \lim_{n\sim\infty}\int_{(E)} \phi_n(x)\,dx$.

Also there exists a monotone non-increasing sequence $\{\psi_n(x)\}$, of l-functions, such that $\int_{(E)} f(x)\,dx = \lim_{n\sim\infty}\int_{(E)} \phi_n(x)\,dx$.

Thus $f(x)$ can be replaced, either by an lu-function, or by a ul-function, without alteration of the value of its integral.

(4) If $f(x)$ be an unbounded function, summable in the measurable set E, the function $f(x)$ may be replaced, without altering the value of its integral, either by a lul-function, i.e. by the limit of a monotone non-diminishing sequence of ul-functions, or by an ulu-function, i.e. by the limit of a monotone non-increasing sequence of lu-functions.

It has been pointed out by W. H. Young that, starting with the definition of the integral of a continuous function (or even of a polynomial), the integral of a u-function, or of an l-function may be defined as the limit of the integrals of the continuous functions of the monotone sequences of continuous functions which converge to the u-function or the l-function. Similarly the integral of a ul-function, or of an lu-function, is defined as the limit of the integrals of the l-functions, or of the u-functions of the monotone sequence which converges to the given ul-function, or the given lu-function. The integral of a bounded summable function is then defined as that of either the ul-function, or the lu-function which is equivalent to the summable function. Finally the integral of an unbounded summable function may be defined as equal to that of the equivalent ulu-function, or of the equivalent lul-function; the integral of either of these latter being defined as the limit of the integrals of the ul-functions, or of the lu-functions of the monotone sequences which converge to them respectively.

In this manner the method of monotone sequences is employed to build up successively the definitions of the integrals of the successive types of functions. Examples of the application of this method have been given by W. H. Young*.

* *Proc. Lond. Math. Soc.* (2), vol. IX (1910), pp. 36–50.

TONELLI'S THEORY OF INTEGRATION

261. Lebesgue's theory of integration depends upon the theory of measurable sets of points, and this theory makes use of the multiplicative axiom, as has been seen in I, § 130. A mode of defining the integral of a function in a finite interval has been developed* by Tonelli, which is independent of the general theory of the measure of sets of points. This independence also appertains to the theory of integration suggested by W. H. Young (§ 260), in which sequences of semi-continuous functions are employed, and in which the definition of an integral ultimately depends on that of the integral of a continuous function. We proceed to give an account of Tonelli's theory of integration, which also consists of an extension of the notion of the integral of a continuous function.

A set Δ, of non-overlapping intervals contained in the interval (a, b), is spoken of by Tonelli as *un plurintervallo*; the set Δ may contain a finite, or infinite, number of distinct intervals. The intervals of the set Δ may be either closed or open; an interval $a \leqq x < \alpha$, or an interval $\beta < x \leqq b$, of which a and b are end-points, may be regarded as open. The sum, or limiting sum, of the lengths of the intervals of Δ is taken to be the length of Δ, whether the intervals are closed or open.

A function $f(x)$, defined in (a, b), is said to be quasi-continuous in (a, b) if a sequence $\{\Delta_n\}$, of sets of open intervals, exists such that the length of Δ_n is less than $\frac{1}{n}$, and is such that $f(x)$ is, for each value of n, continuous in the part of (a, b) which remains when all the points of Δ_n are removed from the interval. The sets Δ_n are said to be associated with $f(x)$.

It is easily seen that a function which is discontinuous only at points belonging to a finite, or to an enumerable, set of points in (a, b) is quasi-continuous, but this does not exhaust the class of quasi-continuous functions.

Let $f(x)$ be quasi-continuous and bounded in (a, b), and consider Δ_n one of the open sets of non-overlapping intervals associated with $f(x)$. Let $f_{\Delta_n}(x)$ be the function which has the value $f(x)$ at each point x that does not belong to Δ_n; and let $f_{\Delta_n}(x)$ be linear in each interval δ, of Δ_n; the linear function having at the end-points of δ the values of $f(x)$ at those end-points. The function $f_{\Delta_n}(x)$ is then continuous in (a, b). We have thus a sequence $\{f_{\Delta_n}(x)\}$ of functions, all of which are continuous in (a, b). The functions $f(x), f_{\Delta_n}(x)$ differ from one another only at the points of Δ_n, where the length of Δ_n, being $\frac{1}{n}$, converges to zero, as $n \sim \infty$.

The functions $\{f_{\Delta_n}(x)\}$ are said to be *associated* with $f(x)$.

* *Annali di Mat.* (4), vol. I (1923), p. 105; see also the treatise *Fondamenti di Calcolo della Variazioni*, Bologna, 1912.

The integral $\int_a^b f(x)\,dx$, *of the quasi-continuous bounded function* $f(x)$, *in* (a, b), *is defined to be the limit of the sequence of integrals* $\int_a^b f_{\Delta_n}(x)\,dx$, *as* n *is increased indefinitely.*

It can easily be shewn that the value of $\int_a^b f(x)\,dx$, so defined, is independent of the particular set of associated functions which is employed in forming the sequence.

The definition of the integral of a continuous function which can be employed is that of Cauchy, of which the most general form is that of Riemann.

The ordinary properties of $\int_a^b f(x)\,dx$ such as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx,$$

$$\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

can be easily established in accordance with the above definition.

Tonelli has extended his definition of an integral to the case of unbounded quasi-continuous functions by the method of de la Vallée Poussin (I, § 387).

If $f_{N,N'}(x) = f(x)$, when $-N' \leq f(x) \leq N$; $f_{N,N'}(x) = N$, when $f(x) > N$; and $f_{N,N'}(x) = -N'$, when $f(x) < -N'$, the integral of the quasi-continuous function $f(x)$ is defined to have the value

$$\lim_{N\sim\infty,\, N'\sim\infty} \int_a^b f_{N,N'}(x)\,dx,$$

whenever the double limit exists; and it is then denoted by $\int_a^b f(x)\,dx$.

The necessary and sufficient condition for the existence of $\int_a^b f(x)\,dx$ can easily be shewn to be that $\left|\int_a^b f_{N,N'}(x)\,dx\right|$ is bounded, with respect to (N, N').

The theorem $\quad \lim\limits_{m\sim\infty} \int_a^b f_m(x)\,dx = \int_a^b f(x)\,dx,$

where $|f_m(x)|$ is bounded with respect to (m, x), and the sequence of quasi-continuous functions $f_m(x)$ converges to $f(x)$ has been established by Tonelli on the basis of his definition. He has also proved the theorems of integration by parts and other properties of integrals.

Tonelli's definition is applicable to the class of quasi-continuous functions, and this class certainly includes all those functions which are defined by ordinary processes. If the theory of Lebesgue integration be assumed, it follows that, as has been shewn in § 179, a measurable function is continuous relatively to a set of points G which is perfect, and of measure less than $b - a$ by less than an arbitrarily fixed positive number ϵ. The complementary set $C(G)$ can be enclosed in intervals of a set of which the measure is $< \epsilon$. It thus follows that every measurable function is quasi-continuous, in accordance with Tonelli's definition of quasi-continuity. Hence every Lebesgue integral is also an integral in accordance with Tonelli's definition, the Lebesgue theory of measure being assumed.

PERRON'S DEFINITION OF AN INTEGRAL

262. A new definition of the integral of a function in a finite linear interval, which is independent of the general theory of measurable sets of points, was given* by Perron. It has as its starting point the conception of the inverse relation between the integration and derivation of a function.

If $f(x)$ be a function defined in the linear interval (a, b), the greater of the upper derivatives $D^+ f(x)$, $D^- f(x)$, on the right and left of the point x may be denoted by $\bar{D}f(x)$, so that $\bar{D}f(x) = \overline{\lim\limits_{h\sim 0}} \dfrac{f(x+h) - f(x)}{h}$, as h converges to zero, when h is unrestricted as regards sign. Similarly $\underline{D}f(x)$ may be taken to denote the smaller of the numbers $D_+(x)$, $D_- f(x)$.

In the first instance, let $f(x)$ be any bounded function, defined in (a, b), and let U and L be its upper and lower boundaries in the interval. A continuous function $\phi(x)$, defined in (a, b), and such that $\bar{D}\phi(x) \leqq f(x)$, at all points of (a, b), and also such that $\phi(a) = 0$, is said to be *a minor function associated with* $f(x)$. Similarly, a function $\psi(x)$ such that $\underline{D}\psi(x) \geqq f(x)$, in (a, b), and such that $\psi(a) = 0$, is said to be *a major function associated with* $f(x)$.

It is clear that major and minor functions always exist; for example, $U(x - a)$ is a major function, and $L(x - a)$ is a minor function.

If $\phi(x)$ be any minor function associated with $f(x)$, we see that (I, § 280) $\dfrac{\phi(b) - \phi(a)}{b - a}$ does not exceed the upper boundary of $\bar{D}\phi(x)$ in (a, b); that is $\phi(b) \leqq U(b - a)$.

It follows that the upper boundary of $\phi(b)$, for all minor functions, is a finite number g; and thus there exists a minor function $\phi(x)$ such that $\phi(b) > g - \epsilon$, where ϵ is an arbitrarily prescribed positive number.

* *Sitzungsber. d. Heidelberger Akad.* vol. v *a* (1914). No. 14.

Similarly there exists a number G which is the lower boundary of $\psi(b)$, for all major functions; and thus there exists a major function for which $\psi(b) < G + \epsilon$.

Since $$\underline{D}\psi(x) \geqq f(x) \geqq \bar{D}\phi(x),$$
we have $$\underline{D}\{\psi(x) - \phi(x)\} \geqq \underline{D}\psi(x) - \bar{D}\phi(x) \geqq 0,$$
for all values of x in (a, b). It then follows (I, § 280) that
$$\frac{(\psi(x_2) - \phi(x_2)) - (\psi(x_1) - \phi(x_1))}{x_2 - x_1} \geqq 0,$$
whatever pair of values, in (a, b), x_1, x_2 may have. From this it follows that $\psi(x) - \phi(x)$ is monotone non-diminishing in (a, b). Since $\phi(a) = \psi(a) = 0$, it follows that $\psi(x) \geqq \phi(x)$, $\psi(b) \geqq \phi(b)$, and $G \geqq g$.

In case $G = g$, the function $f(x)$ is said to be integrable in (a, b), and G or g is taken to define the value of $\int_a^b f(x)\,dx$.

It has thus been shewn that:

For every minor function $\phi(x)$, and for every major function $\psi(x)$, associated with $f(x)$, the relation $\phi(b) \leqq \int_a^b f(x)\,dx \leqq \psi(b)$ holds good.

The condition for the existence of $\int_a^b f(x)\,dx$ is that minor and major functions $\phi(x)$, $\psi(x)$ can be so determined that $\psi(b) - \phi(b) < \eta$, in which case $\psi(x) - \phi(x) < \eta$, in (a, b); where η is an arbitrarily prescribed positive number.

Perron himself gave an investigation of the principal properties of $\int_a^x f(x)\,dx$, in accordance with this definition.

263. The definition can be extended to the case in which $f(x)$ is unbounded in (a, b), provided, in such a case, major and minor functions exist. The general definition may be stated as follows:

In the interval (a, b), a function $f(x)$ is defined which has everywhere (or almost everywhere) a finite value. Let it be assumed that continuous functions $\phi(x)$, $\psi(x)$ exist such that $\underline{D}\psi(x) \geqq f(x) \geqq \bar{D}\phi(x)$, when $\bar{D}\phi(x)$ has nowhere the value $+\infty$, and $\underline{D}\psi(x)$ has nowhere the value $-\infty$; and
$$\phi(a) = \psi(a) = 0;$$
then if $$\underline{\lim}\,\psi(x) = \overline{\lim}\,\phi(x) = F(x),$$
the function $F(x)$ defines the indefinite integral $\int_a^x f(x)\,dx$ in (a, b), and $F(b)$ defines the definite integral $\int_a^b f(x)\,dx$.

The proof in § 262 is applicable to shew that $\psi(x) - \phi(x)$, for every pair of functions, is monotone non-diminishing in (a, b); it follows that

$\psi(x) - F(x)$, $F(x) - \phi(x)$ are monotone non-diminishing in (a, b). Since $\phi(x)$, $\psi(x)$ can be so determined that $\psi(b) - F(b) < \eta$, it follows that $\psi(x) - F(x) < \eta$, and thus a sequence of values of $\psi(x)$ converges to $F(x)$ uniformly in (a, b), and therefore $F(x)$ is continuous in (a, b). Thus the functions $\psi(x) - F(x)$, $F(x) - \phi(x)$ are monotone non-diminishing continuous functions. It may thus be stated that:

In order that the continuous function $F(x)$, where $F(a) = 0$, may be the indefinite integral of $f(x)$, it is necessary and sufficient that, if the positive number ϵ be arbitrarily prescribed, a pair of continuous functions $\psi(x)$, $\phi(x)$ which satisfy the conditions in the above definition should exist, which satisfy the conditions $0 < \psi(x) - F(x) < \epsilon$, $0 < F(x) - \phi(x) < \epsilon$.

The relation of the integral so defined with the integrals defined by Lebesgue and Denjoy has been investigated by Bauer*, Hake†, Alexandroff‡, and Looman§. It was proved by de la Vallée Poussin that every L-integral is also an integral in accordance with Perron's definition; this proof is given in § 437. It was proved by Bauer that a bounded function is integrable in accordance with Perron's definition when, and only when, it is measurable, and accordingly integrable in accordance with Lebesgue's definition. It was proved by Hake, and again later by Alexandroff, that a function, integrable in accordance with Denjoy's definition, is always integrable in accordance with Perron's definition. The converse of this was established by Alexandroff. Consequently it is known that:

There is complete equivalence between the definition of Perron and Denjoy.

In view of the simplicity of the definition of Perron as compared with that of Denjoy, and of the fact that the former makes no use of the theory of the measure of sets of points, this theorem may prove to be of great importance in future developments of the conception of an integral.

Perron's definition was extended by Bauer to the case of functions of any number of variables, and it was shewn by him that every Lebesgue integral of a function of any number of variables is a Perron integral, in accordance with the extended definition.

THE SUMMABILITY OF INTEGRALS

264. If the integral $\int_a^x f(t)\,dt$ exists for all finite values of $x > a$, the integral $\int_a^\infty f(t)\,dt$ exists in the ordinary sense when $\int_a^x f(t)\,dt$ has a definite limit, as $x \sim \infty$. In analogy with the case of infinite series, various conventional definitions of $\int_a^\infty f(t)\,dt$ can be employed which assign to it a

* *Monatshefte f. Math. u. Physik*, vol. XXVI (1915), p. 153.

† *Math. Annalen*, vol. LXXXIII (1921), p. 119. ‡ *Math. Zeitschr.* vol. XX (1924), p. 213.

§ *Math. Annalen*, vol. XCIII (1924), p. 153.

definite meaning in cases when it does not exist in the ordinary sense. Such a definition should satisfy the condition of consistency, in accordance with which the value of the integral, with the conventional definition, should coincide with its value in the ordinary sense when the latter exists. The integral $\int_a^{t_1} f(t)\,dt$ may be regarded as analogous to a partial sum of an infinite series, and $\frac{1}{x}\int_a^x dt_1 \int_a^{t_1} f(t)\,dt$ may be regarded as analogous to the arithmetic mean of a partial sum of an infinite series. Thus the integral $\int_a^\infty f(t)\,dt$ is said to exist $(C, 1)$ when $\lim_{x\sim\infty} \frac{1}{x}\int_a^x dt_1 \int_a^{t_1} f(t)\,dt$ has a definite value. The integral $\int_a^\infty f(t)\,dt$ is then said to be *summable* $(C, 1)$, and its sum $(C, 1)$ is defined to be the value of the limit.

The extension of Cesàro's method of summation to summation (C, r), where r is a positive integer, is made by defining the sum (C, r) of the integral $\int_a^\infty f(t)\,dt$ to be

$$\lim_{x\sim\infty} \frac{r!}{x^r} \int_a^x dt_r \int_a^{t_r} dt_{r-1} \dots \int_a^{t_2} dt_1 \int_a^{t_1} f(t)\,dt \qquad \text{......(1)},$$

when this limit exists.

Since

$$\int_a^{t_2} dt_1 \int_a^{t_1} f(t)\,dt = \int_a^{t_2} (t_2 - t) f(t)\,dt,$$

$$\int_a^{t_3} dt_2 \int_a^{t_2} dt_1 \int_a^{t_1} f(t)\,dt = \frac{1}{2}\int_a^{t_3} (t_3 - t)^2 f(t)\,dt,$$

we have, proceeding in the same manner, for the sum (C, r) of the integral $\int_a^\infty f(t)\,dt$, the expression

$$\lim_{x\sim\infty} \int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\,dt \qquad \text{......(2)}.$$

This expression (2) is analogous to the sum of a series by Riesz's method, which has been shewn in § 60 to be equivalent to the sum (C, r). As in the case of series, the expression (2) may be taken to define the sum (C, r) of the integral $\int_a^\infty f(t)\,dt$, when r is not restricted to be a positive integer. When the integral exists in the ordinary sense, it is summable $(C, 0)$.

The method of Hölder for defining the sum (H, r) of an infinite series, for positive integral values of r, may be extended by analogy to the case of integrals. Thus the sum (H, r) of the integral $\int_a^\infty f(t)\,dt$ may be defined to be

$$\lim_{x\sim\infty} \frac{1}{x} \int_a^x \frac{dt_r}{t_r} \int_a^{t_r} \frac{dt_{r-1}}{t_{r-1}} \dots \int_a^{t_1} dt_1 \int_a^{t_1} f(t)\,dt \qquad \text{......(3)}$$

when this limit exists.

That the definitions (2) and (3) are equivalent when r is a positive integer, has been proved* by Landau. Both of the expressions (1) and (3) were† considered by Du Bois-Reymond.

265. In order to prove that, for $r > 0$, the definition (2), of summability (C, r), satisfies the condition of consistency, let it be assumed that $\int_a^\infty f(t)\,dt$ exists as $\lim\limits_{x\sim\infty}\int_a^x f(t)\,dt$.

If ϵ be an arbitrarily chosen positive number, we have

$$\left|\int_A^{A'} f(t)\,dt\right| < \epsilon,$$

for all values of $A' > A$, provided A be sufficiently large. Also

$$\int_A^x \left(1 - \frac{t}{x}\right)^r f(t)\,dt = \left(1 - \frac{A}{x}\right)^r \int_A^\alpha f(t)\,dt,$$

for $x > A$, where α is a number in the interval (A, x). It follows that the integral on the left-hand side is numerically less than ϵ, for all values of x.

We thus have

$$\left|\int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\,dt - \int_a^A \left(1 - \frac{t}{x}\right)^r f(t)\,dt\right| < \epsilon,$$

for all values of x. Also, since $\left(1 - \frac{t}{x}\right)^r$ is a monotone function of x, we have

$$\lim_{x\sim\infty}\int_a^A \left(1 - \frac{t}{x}\right)^r f(t)\,dt = \int_a^A f(t)\,dt$$

(see § 205). Hence

$$\left|\int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\,dt - \int_a^A f(t)\,dt\right| < 2\epsilon,$$

for all sufficiently large values of x, provided A is sufficiently large. It now follows that $\lim\limits_{x\sim\infty}\int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\,dt$ exists, and is equal to $\int_a^\infty f(t)\,dt$. Thus the condition of consistency is satisfied.

It may be shewn that:

The necessary and sufficient condition that the integral $\int_a^\infty f(t)\,dt$, *when summable* $(C, 1)$, *should be convergent is that* $\lim\limits_{x\sim\infty}\frac{1}{x}\int_a^x tf(t)\,dt = 0$.

Denoting $\int_a^x f(t)\,dt$ by $f_1(x)$, and $\int_a^x f_1(t)\,dt$ by $f_2(x)$, we have

$$\frac{f_2(x)}{x} = \frac{1}{x}\int_a^x f_1(t)\,dt = f_1(x) - \frac{1}{x}\int_a^x tf(t)\,dt,$$

from which the theorem at once follows.

* *Leipz. Sitzungsber.* vol. LXV (1913), p. 131.

† *Crelle's Journal,* vol. C (1887), p. 356.

The following is analogous to the theorem given in § 54:

If $|xf(x)| < K$, *and* $\int_a^x f(t)\,dt$ *is summable* $(C, 1)$, *it is also convergent.*

Let
$$g(x) = xf(x), \quad G(x) = \int_a^x g(t)\,dt;$$
then
$$f_1(x) = \int_a^x f(t)\,dt = \int_a^x \frac{1}{t} G'(t)\,dt = \frac{G(x)}{x} - \frac{G(a)}{a} + \int_a^x \frac{G(t)}{t^2}\,dt;$$
and therefore
$$\frac{1}{x}\int_a^x f_1(t)\,dt = f_1(x) - \frac{G(x)}{x} = -\frac{G(a)}{a} + \int_a^x \frac{G(t)}{t^2}\,dt.$$

It follows that, with the hypothesis of summability $(C, 1)$, the integral $\int_a^x \frac{G(t)}{t^2}\,dt$ is convergent; and it will be shewn that this cannot be the case unless $\frac{G(x)}{x}$ converges to zero, as $x \sim \infty$, from which it follows, by the last theorem, that $\int_a^\infty f(t)\,dt$ exists. If $\frac{G(x)}{x}$ does not converge to zero, a positive number K_1 exists such that $G(x) > K_1 x$, or $G(x) < -K_1 x$, for all sufficiently large values of x. It will be assumed that $G(x) > K_1 x$, for such values of x; we may take $K_1 < K$. Let X be a value of x such that $G(X) > K_1 X$, and let $X_1 = \left(1 - \frac{K_1}{2K}\right) X$; then, for $X_1 \leqq x \leqq X$, we have
$$|G(x) - G(X)| = \left|\int_x^X tf(t)\,dt\right| < K(X - x) \leqq K(X - X_1);$$
and therefore $\quad G(x) \geqq G(X) - |G(x) - G(X)| > \frac{1}{2}K_1 X.$

We now have
$$\int_{X_1}^X \frac{G(t)}{t^2}\,dt > \tfrac{1}{2}K_1 X \int_{X_1}^X \frac{dt}{t^2} = \tfrac{1}{2}K_1\left(\frac{X}{X_1} - 1\right) = \frac{\frac{1}{2}K_1^2}{2K - K_1};$$
and therefore $\int_{X_1}^X \frac{G(t)}{t^2}$ is greater than some fixed number K_2; clearly this is inconsistent with the convergence of the integral $\int_a^\infty \frac{G(t)}{t^2}\,dt$.

266. A general method of summation, of importance in connection with the theory of Fourier's integrals, has been treated in detail* by Hardy, C. N. Moore†, Bromwich‡, and others.

Let a function $\phi(x)$, defined in the interval $(0, \infty)$, satisfy the conditions (1), that $\phi''(x)$ is continuous, and is positive when x is greater than some fixed number, (2), that $\phi(x)$ has only a finite set of maxima and minima, (3), that $\int_0^\infty \phi(x)\,dx$ exists, and (4), that $\phi(0) = 1$.

* *Camb. Phil. Trans.* vol. XXI (1912), p. 431; see also the same volume, p. 39.

† *Trans. Amer. Math. Soc.* vol. VIII (1907), p. 312.

‡ *Math. Annalen*, vol. LXV (1908), p. 367.

It follows from these assumptions that $\phi'(x)$ is negative, for all sufficiently large values of x, and that it then increases steadily, converging to 0, as $x \sim \infty$. It follows also that $\phi(x)$ is positive, for all sufficiently large values of x, and decreases steadily to zero. The integral $\int_a^\infty \phi'(x)\,dx$ exists, where $a \geqq 0$ and since $\phi'(x)$ is monotone for all sufficiently large values of x,

$$\lim_{x\sim\infty} x\phi'(x) = 0.$$

Further, since

$$\int_a^x t\phi''(t)\,dt = x\phi'(x) - a\phi'(a) + \phi(a) - \phi(x),$$

it follows that $\int_a^\infty x\phi''(x)\,dx\ (a \geqq 0)$ exists.

If the function $\phi(x)$ be defined so as to satisfy the above conditions, the integral $\int_a^\infty f(x)\,dx\ (a \geqq 0)$ will be said to be *summable* (ϕ), and to have the sum s, if $\lim_{k\sim 0}\int_a^\infty \phi(kx) f(x)\,dx = s$, the convergence of k to zero being through positive values.

Important special cases of summation (ϕ) are when $\phi(x) = e^{-x}$, or $\phi(x) = e^{-x^2}$.

This definition satisfies the condition of consistency, for if $\int_a^\infty f(x)\,dx$ exists, and has the value s, whether the convergence be absolute or not, we see that $\int_a^\infty \phi(kx) f(x)\,dx$ converges to $\int_a^\infty f(x)\,dx$, or s, since $1 - \phi(kx)$ satisfies the conditions of the last theorem in § 281, when k has any sequence $\{k_n\}$ of values converging to zero.

The following theorems are given by the writers referred to above:

(1) *If* $\int_a^\infty f(x)\,dx$ *is summable* $(C, 1)$, *and has sum* s, *and if*

$$\lim_{x\sim\infty} \phi(kx)\int_a^x f(t)\,dt = 0,$$

for every positive value of k, *then* $\int_a^\infty f(x)\,dx$ *is summable* (ϕ), *and its sum* (ϕ) *is* s.

(2) *If* $\int_a^\infty f(x)\,dx$ *is summable* (ϕ), *and* $|xf(x)|$ *is less than a fixed positive number* K, *then* $\int_a^\infty f(x)\,dx$ *is convergent, and has as its value the sum* (ϕ).

This is the analogue of the theorem in § 54 for series.

CHAPTER VI

THE CONSTRUCTION OF FUNCTIONS WITH ASSIGNED SINGULARITIES

THE CONDENSATION OF SINGULARITIES

267. A method of constructing functions which possess, at an infinitely numerous set of points in a linear interval, singularities in relation to continuity, derivatives, or oscillations, has been given by Hankel. The method depends upon the employment of functions which at a single point possess one of the singularities in question, and consists in building up, by the use of such a function of a simple type, the more complicated analytical representations of a function which possess the singularity at an everywhere-dense set of points. To this method, Hankel* has given the name "Principle of condensation of singularities" (das Prinzip der Verdichtung der Singularitäten); the name may however be conveniently applied to other methods of constructing functions capable of analytical representation, which have been given more recently by other writers.

Let $\phi(y)$ be a function defined for the interval $(-1, +1)$, bounded in that interval, and continuous at every point of the interval, including $-1, +1$, except at the point $y = 0$, where however $\phi(0) = 0$. The function $\phi(\sin n\pi x)$ is finite and continuous for every value of x which is not a rational fraction m/n, with n as denominator, and it vanishes for all points at which x has this form.

The series $\sum_{n=1}^{\infty} \frac{\phi(\sin n\pi x)}{n^s}$, where $s > 1$, is, in accordance with the fact that $\phi(y)$ is bounded, uniformly convergent in every interval; and its sum is a bounded function of x which is continuous for all irrational values of x.

If $\phi(y)$ were also continuous for $y = 0$, the function represented by the series would be continuous also for rational values of x, but when $\phi(y)$ is discontinuous at $y = 0$, the properties of the function

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\phi(\sin n\pi x)}{n^s}$$

in relation to continuity or discontinuity at the points where x has rational values require investigation.

The series being uniformly convergent, it follows from the theorem of § 86, that $f(x)$ is continuous at every point at which all the functions

* See his memoir "Untersuchungen über die unendlich oft unstetigen im oscillierenden Functionen," Inaugural dissertation (1870), reproduced in *Math. Annalen*, vol. XX (1882), p. 20. The method has been treated in a rigorous manner by Dini, *Grundlagen*, p. 157.

$\phi(\sin n\pi x)$ are continuous, i.e. for all irrational values of x. Let us consider the values of the function $f(x)$ in the neighbourhood of a point $x = p/q$, where p and q are relative primes. We may write the value of $f(x)$ in the form

$$\sum_{n_q=1}^{\infty} \frac{\phi(\sin n_q \pi x)}{n_q{}^s} + \frac{1}{q^s}\sum_{m=1}^{\infty} \frac{\phi(\sin qm\pi x)}{m^s},$$

where n_q has those integral values only which are not multiples of q.

The first of these series is uniformly convergent, and its sum is continuous at the point p/q; we therefore find that

$$f(p/q+h) - f(p/q) = \eta_h + \frac{1}{q^s}\sum_{m=1}^{\infty} \frac{\phi(\overline{-1}|^{mp} \sin qm\pi h)}{m^s},$$

where η_h converges to zero when h does so.

Case I. Let $\phi(y)$ have an ordinary discontinuity at $y = 0$; we then have

$$f(p/q+0) - f(p/q) = \frac{\phi(\pm 0)}{q^s}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^s} + \frac{\phi(+0)}{q^s}\sum_{r=1}^{\infty}\frac{1}{(2r)^s},$$

$$f(p/q-0) - f(p/q) = \frac{\phi(\mp 0)}{q^s}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^s} + \frac{\phi(-0)}{q^s}\sum_{r=1}^{\infty}\frac{1}{(2r)^s},$$

where the upper or lower of the ambiguous signs are to be taken, according as p is even or odd.

If $\phi(+0)$, $\phi(-0)$ are different from one another, and from zero, these relations shew that, at a point p/q, for which p is even, the function $f(x)$ has ordinary discontinuities both on the right and on the left, the measures of the two being not identical. Moreover the same statement may be made for a point p/q at which p is odd, unless $\phi(+0)$, $\phi(-0)$ have such values that one or other of the above expressions vanishes, in which case there is an ordinary discontinuity on one side, and the function is continuous on the other side. It is easily seen to be impossible that the two expressions can simultaneously vanish, and therefore there is an ordinary discontinuity on one side at least.

If $\phi(+0) \neq 0$, $\phi(-0) = 0$, there is discontinuity on the right at the points $x = 2p'/q$, and continuity on the left; and at the points $x = (2p'+1)/q$, there are discontinuities on both sides, with different measures.

If $\phi(+0) = \phi(-0)$, so that $\phi(y)$ has only a removable discontinuity at the point $y = 0$, then the function $f(x)$ has removable discontinuities at all the points $x = p/q$.

In every case the function $f(x)$ is a point-wise discontinuous function, because its discontinuities are all ordinary ones (see I, § 239).

Case II. Let $\phi(y)$ have a discontinuity of the second kind, at $y = 0$, on

one side at least. In this case it will be assumed that $s > 2$. Denoting by A the upper boundary of $|\phi(y)|$ in the interval $(-1, +1)$, we have

$$\left|\sum_{m=1}^{\infty} \frac{\phi(\overline{-1}|^{mp}\sin qm\pi h)}{m^s} - \phi(\overline{-1}|^p \sin q\pi h)\right|$$
$$< \frac{A}{2^{s-2}} \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} < \frac{A}{2^{s-2}}\left(\frac{\pi^2}{6} - 1\right),$$

and hence

$$f(p/q + h) - f(p/q) = \frac{1}{q^s}\phi(\overline{-1}|^p \sin q\pi h) + \eta_h + \frac{\zeta A}{2^{s-2}} \cdot \frac{1}{q^s},$$

where ζ is such that $-1 < \zeta < 1$, and is dependent on h.

If $\phi(y)$ have a discontinuity of the second kind on both sides of the point $y = 0$, there are finite oscillations in arbitrarily small neighbourhoods of the point on the two sides; if then s be chosen so great that $A/2^{s-2}$ is less than half the saltus at $y = 0$, we see that $f(x)$ has discontinuities of the second kind on both sides at all the points $x = p/q$.

If $\phi(y)$ have a discontinuity of the second kind at $y = 0$, on the right, and have a discontinuity of the first kind, or be continuous, on the left, there is, at each of the points $x = p/q$, where p is even, a discontinuity of $f(x)$ of exactly the same kind as that of $\phi(y)$ at $y = 0$. On the other hand, if s be sufficiently large, there is at each of the points $x = p/q$, where p is odd, a discontinuity of the second kind on both sides. For we may express $f(p/q + h) - f(p/q)$ in the form

$$\eta_h + \frac{1}{q^s}\sum_{r=0}^{\infty} \frac{\phi(-\sin \overline{2r+1}\, q\pi h)}{(2r+1)^s} + \frac{1}{q^s}\sum_{r=0}^{\infty} \frac{\phi(\sin 2rq\pi h)}{(2r)^s},$$

which can, as in the previous case, be reduced to the form

$$\eta_h + \frac{1}{q^s}\phi(-\sin q\pi h) + \frac{1}{q^s}\phi(\sin 2q\pi h) + \frac{A\zeta_1}{q^s 3^{s-2}} + \frac{A\zeta_2}{q^s 2^{s-2}},$$

where ζ_1, ζ_2 are both in the interval $(-1, 1)$. We thus see that, if s be sufficiently great, there are finite oscillations in arbitrarily small neighbourhoods of p/q on both sides.

The existence of the factor $1/q^s$ in the expression for $f(p/q + h) - f(p/q)$ shews that there are only a finite number of points p/q at which the saltus of $f(x)$ is $\geqq k$, where k is an arbitrarily chosen positive number; and thus $f(x)$ belongs to the special class of point-wise discontinuous functions for which the set K is a finite set, for each value of k.

EXAMPLES

(1) Let $\phi(y) = \sin\frac{1}{y}$, and $\phi(0) = 0$; the function $f(x)$ is then defined by

$$f(x) = \sum_1^{\infty} \frac{1}{n^s} \sin(\operatorname{cosec} n\pi x),$$

where, when $x = p/q$, the terms for which n is a multiple of q are to be omitted. This function

is, at least when $s>2$, a point-wise discontinuous function which is continuous at all the irrational points, and has discontinuities of the second kind, on both sides, at the rational points.

(2) Let
$$\phi(y)=\frac{4}{\pi}\sum_{r=0}^{\infty}\frac{1}{2r+1}\sin(2r+1)\frac{\pi y}{a},$$
where $a>1$. For $0<y\leqq 1$, we have $\phi(y)=1$; for $-1\leqq y<0$, we have $\phi(y)=-1$; also $\phi(0)=0$; and thus $\phi(y)$ has an ordinary discontinuity at $y=0$.

The function
$$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^s}\left[\sum_{r=0}^{\infty}\frac{1}{2r+1}\sin\left\{(2r+1)\frac{\pi}{a}\sin n\pi x\right\}\right]$$
is a point-wise discontinuous function, which is continuous for all irrational values of x, and has ordinary discontinuities on both sides at all the rational points.

(3) With the same value of $\phi(y)$ as in Ex. (2), let
$$\chi(x)=\sum_1^{\infty}\frac{1}{n^s[\phi(\sin n\pi x)]^s},$$
where $s>1$. For any irrational value of x, $\chi(x)$ has the value $\sum\limits_1^{\infty}\frac{1}{n^s}$, and for any rational value of x, the function is indefinitely great. Now let
$$f(x)=\frac{\sum\limits_1^{\infty}\frac{1}{n^s}}{\chi(x)};$$
then $f(x)=1$, for all irrational values of x, and $f(x)=0$, for all rational values of x. The function $f(x)$ is accordingly totally discontinuous. The values of $f(x)$ are improperly defined at the rational points.

268. Let us next assume that $\phi(y)$ is continuous throughout the interval $(-1, 1)$, and has no differential coefficient at the point $y=0$, where $\phi(0)=0$, but that, at every other point in the interval $(-1, +1)$, it has a differential coefficient which is numerically less than some fixed finite number A. In this case $\frac{\phi(h)}{h}$ has no definite limit for $h\sim 0$, either when h is positive, or when h is negative, or in both cases; or else the two limits both exist, but have different values.

The numbers $\frac{\phi(h)}{h}$ which are equal to $|\phi'(\theta h)|$, where $\theta>0$ (see I, § 262), have a definite upper boundary U ($\leqq A$) for all values of h.

Assuming that $s>2$, we see that the series
$$\pi\sum_1^{\infty}\frac{\phi'(\sin n\pi x)}{n^{s-1}}\cos n\pi x$$
converges for all irrational values of x, since the general term is numerically less than B/n^{s-1}, where B is some fixed number.

Consider the series
$$\sum_1^{\infty}\frac{\phi[\sin n\pi(x+h)]-\phi(\sin n\pi x)}{hn^s},$$

where x has an irrational value. It will be shewn that this series converges uniformly for all values of h. Unless n and h are such that $\sin n\pi (x+h)$ and $\sin n\pi x$ are equal, in which case $\phi [\sin n\pi (x+h)] - \phi (\sin n\pi x) = 0$, we can write the general term of the series in the form

$$\frac{\pi}{n^{s-1}} \cdot \frac{\phi [\sin n\pi (x+h)] - \phi (\sin n\pi x)}{\sin n\pi (x+h) - \sin n\pi x} \cdot \frac{\sin \frac{1}{2} n\pi h}{\frac{1}{2} n\pi h} \cdot \cos n\pi (x + \tfrac{1}{2}h).$$

It then follows that the general term of the series is numerically less than $\frac{\pi V}{n^{s-1}}$, where V is the upper boundary of the absolute values of the incrementary ratios of the function. Since the series $\Sigma 1/n^{s-1}$ is convergent, it follows that the above series converges uniformly for all values of h which are $\neq 0$; and consequently, in accordance with the last theorem of § 234, the function $f(x)$ has a finite differential coefficient for any irrational value of x.

Next let x have the rational value p/q. We may then express $\dfrac{f\left(\frac{p}{q}+h\right) - f\left(\frac{p}{q}\right)}{h}$ in the form

$$\sum_{n_q=1}^{\infty} \frac{\phi [\sin n_q\pi (x+h)] - \phi (\sin n_q\pi x)}{hn_q^s} + \frac{1}{q^s}\sum_{m=1}^{\infty} \frac{\phi (\overline{-1}|^{mp} \sin mq\pi h)}{hm^s},$$

where n_q has all positive integral values which are not multiples of q. In accordance with the above proof we see that $\sum\limits_{n_q=1}^{\infty} \frac{\phi (\sin n_q\pi x)}{n_q^s}$ has, for the value $x = p/q$, a finite differential coefficient which is the sum

$$\pi \sum_{n_q=1}^{\infty} \frac{\phi' (\sin n_q \pi p/q)}{n_q^{s-1}} \cos n_q \pi p/q.$$

We have now shewn that

$$\frac{f\left(\frac{p}{q}+h\right) - f\left(\frac{p}{q}\right)}{h}$$

$$= \pi \sum_{n_q=1}^{\infty} \frac{\phi' (\sin n_q \pi p/q)}{n_q^{s-1}} \cos n_q \pi p/q + \eta_h + \frac{1}{q^s}\sum_{m=1}^{\infty} \frac{\phi (\overline{-1}|^{mp} \sin mq\pi h)}{hm^s},$$

where η_h is a number which converges to zero when h is indefinitely diminished.

Case I. Let $\phi(y)$ have definite derivatives on the right and on the left when $y = 0$; and thus $\frac{\phi(h)}{h}$ has one limit $\phi'(+0)$ for positive values of h

converging to zero, and another limit $\phi'(-0)$ for negative values of h so converging. We thus have, when p is even,

$$\lim_{h\sim 0}\frac{f\left(\frac{p}{q}+h\right)-f\left(\frac{p}{q}\right)}{h}=\pi\sum_{n_q=1}^{\infty}\frac{\phi'(\sin n_q\pi p/q)}{n_q^{\,s-1}}\cos n_q\pi p/q+\frac{\pi\phi'(+0)}{q^{s-1}}\sum_{m=1}^{\infty}\frac{1}{m^{s-1}},$$

$$\lim_{h\sim 0}\frac{f\left(\frac{p}{q}-h\right)-f\left(\frac{p}{q}\right)}{h}=\pi\sum_{n_q=1}^{\infty}\frac{\phi'(\sin n_q\pi p/q)}{n_q^{\,s-1}}\cos n_q\pi p/q+\frac{\pi\phi'(-0)}{q^{s-1}}\sum_{m=1}^{\infty}\frac{1}{m^{s-1}}.$$

For an uneven value of p, we find

$$\lim_{h\sim 0}\frac{f\left(\frac{p}{q}+h\right)-f\left(\frac{p}{q}\right)}{h}=\pi\sum_{n_q=1}^{\infty}\frac{\phi'(\sin n_q\pi p/q)}{n_q^{\,s-1}}\cos n_q\pi p/q+\frac{\pi\phi'(+0)}{q^{s-1}}\sum_{r=1}^{\infty}\frac{1}{(2r)^{s-1}}-\frac{\pi\phi'(-0)}{q^{s-1}}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^{s-1}},$$

$$\lim_{h\sim 0}\frac{f\left(\frac{p}{q}-h\right)-f\left(\frac{p}{q}\right)}{h}=\pi\sum_{n_q=1}^{\infty}\frac{\phi'(\sin n_q\pi p/q)}{n_q^{\,s-1}}\cos n_q\pi p/q-\frac{\pi\phi'(+0)}{q^{s-1}}\sum_{r=1}^{\infty}\frac{1}{(2r+1)^{s-1}}+\frac{\pi\phi'(-0)}{q^{s-1}}\sum_{r=1}^{\infty}\frac{1}{(2r)^{s-1}}.$$

From these results it is seen that $f(x)$ has, at the rational points, definite derivatives on the right and on the left, differing in value from one another, and therefore, at all these points, the function has a singularity of the same kind as $\phi(y)$ possesses at the point $y=0$.

Case II. Let $\phi(y)$ have, on one side of $y=0$ at least, no definite derivative. Unless mqh is an integer, in which case $\phi(\overline{-1|}^{mp}\sin mq\pi h)=0$, we have

$$\frac{\phi(\overline{-1|}^{mp}\sin mq\pi h)}{hm^s}=\frac{\phi(\overline{-1|}^{mp}\sin mq\pi h)}{\overline{-1|}^{mp}\sin mq\pi h}\cdot\frac{\sin mq\pi h}{mq\pi h}(-1)^{mp}\frac{q\pi}{m^{s-1}};$$

and this is numerically less than $\dfrac{q\pi U}{m^{s-1}}$. It follows that

$$\frac{1}{q^s}\sum_{m=1}^{\infty}\frac{\phi(\overline{-1|}^{mp}\sin mq\pi h)}{hm^s}=\frac{1}{q^s}\frac{\phi(\overline{-1|}^{p}\sin q\pi h)}{h}+P,$$

where P is numerically less than $\dfrac{\pi U}{q^{s-1}}\displaystyle\sum_{m=2}^{\infty}\frac{1}{m^{s-1}}$. By taking a sufficiently large value of s, the number P may be made as small as we please, and therefore

$$\frac{1}{q^s}\sum_{m=1}^{\infty}\frac{\phi(\overline{-1|}^{mp}\sin mq\pi h)}{hm^s}$$

will, for a sufficiently large value of s, oscillate in the same manner as

$$\frac{1}{q^s}\frac{\phi\left(\overline{-1}|^p \sin q\pi h\right)}{h},$$

as h is diminished indefinitely. It is thus seen that $\dfrac{f\left(\dfrac{p}{q}+h\right)-f\left(\dfrac{p}{q}\right)}{h}$ has, on one side, or on both sides, of $h=0$, no definite limit; and thus $f(x)$ has no differential coefficient at any of the rational points, providėd a sufficiently large value of s be chosen.

EXAMPLES

(1) Let $\phi(y)=y$ or $-y$, according as y is positive or negative. The corresponding function $f(x)$ is $\sum\limits_1^\infty \frac{1}{n^s}\sqrt{\sin^2 n\pi x}$, where the positive value of the square root is to be taken. This function is continuous, and has a differential coefficient for all irrational values of x. At the rational points it has no differential coefficient, but has definite derivatives on both sides.

(2) Let $\quad \phi(y)=y\sin(\log y^2)$, then $f(x)=\sum\limits_1^\infty \dfrac{\sin n\pi x\,[\log \sin^2 n\pi x]}{n^s}$.

The function $f(x)$ is continuous, and has a finite differential coefficient for all irrational values of x. If s be sufficiently large, it has no definite derivatives either on the right or on the left, for rational values of x; the four derivatives at such a point are all finite.

269. Let it next be assumed that $\phi(y)$ is continuous in the interval $(-1, 1)$, and has a finite differential coefficient at every point except at $y=0$, but that this differential coefficient has no upper boundary to its absolute magnitude in any neighbourhood of the point $y=0$. In this case $\phi(y)$ may either have a differential coefficient at $y=0$, which is finite or indefinitely great; or it may have indefinitely great derivatives, on the right and on the left, of opposite signs; or it may have no definite derivatives. When $\phi(y)$ is a function of this type, it is not certain that $f(x)$ has differential coefficients for irrational values of x; for the differential coefficients $\phi'(\sin n\pi x)$ are not all numerically less than a fixed finite number, for such a value of x, and for all values of n; and thus the argument of § 268 does not apply.

For a rational point $x=p/q$, we have as before,

$$\frac{f\left(\dfrac{p}{q}+h\right)-f\left(\dfrac{p}{q}\right)}{h}=\sum_{n_q=1}^{\infty}\frac{\phi\left[\sin n_q\pi(x+h)\right]-\phi(\sin n_q\pi x)}{hn_q{}^s},$$
$$+\frac{1}{q^s}\sum_{m=1}^{\infty}\frac{\phi\left(\overline{-1}|^{mp}\sin mq\pi h\right)}{hm^s}.$$

The theorem of § 235 will be applied to shew that the function $\sum\limits_{n_q=1}^{\infty}\dfrac{\phi(\sin n_q\pi x)}{n_q{}^s}$ has, for the value p/q of x, a differential coefficient obtained by means of term by term differentiation of the series.

The first condition required by the theorem in question, viz. that the terms of the series

$$\sum_{n_q=1}^{\infty} \frac{\pi\phi'(\sin n_q \pi p/q)}{n_q^{s-1}} \cos n_q \frac{p}{q}\pi$$

shall be definite, and that this series shall converge, is certainly satisfied.

To shew that the conditions relating to

$$\left|\frac{R_m\left(\frac{p}{q}+h\right)}{h}\right|, \quad \left|\frac{R_m\left(\frac{p}{q}\right)}{h}\right|$$

are satisfied, we observe that $R_m(x) < \frac{U}{m^{s-1}}$, where U denotes the upper limit of $|\phi(y)|$ in the interval $(-1, 1)$. Let t be so chosen that $1 > t > \frac{1}{s-1}$, $(s > 2)$, and let m be that integer next greater than $|h|^{-t}$ which is not a multiple of q; we then have $|hm^{s-1}| > |h|^{1-(s-1)t}$. It follows that, for each fixed value of h, m has been so chosen that

$$\left|\frac{R_m\left(\frac{p}{q}+h\right)}{h}\right|, \quad \left|\frac{R_m\left(\frac{p}{q}\right)}{h}\right|$$

are both less than $U|h|^{(s-1)t-1}$, and are therefore both less than ϵ, provided $|h| < \delta$; where δ is fixed so that $U\delta^{(s-1)t-1} < \epsilon$. It is clear that δ may be chosen so small that m exceeds an arbitrarily prescribed integer m', for all the values of h such that $|h| < \delta$.

We have lastly to prove that the sum of the first m terms of the series of which the general term is

$$\frac{\phi\left[\sin n_q\pi\left(\frac{p}{q}+h\right)\right] - \phi\left(\sin n_q\pi\frac{p}{q}\right)}{hn_q^{s}} - \frac{\pi\phi'\left(\sin n_q\pi\frac{p}{q}\right)}{n_q^{s-1}}\cos n_q\frac{p}{q}\pi,$$

is numerically less than ϵ.

This series may be divided into two portions $\sum_{1}^{m_1-1}$ and $\sum_{m_1}^{m}$, where m_1 is a fixed number independent of h, so chosen that the sum $\sum_{m_1}^{\infty}\frac{1}{n_q^{s-1}}$ is less than an arbitrarily chosen number η. The sum of the first m_1 terms of the series under consideration can be made arbitrarily small, by taking δ sufficiently small; for the number of terms is independent of h. We have then only to consider the sum $\sum_{m_1}^{m}$.

Since $n_q\frac{p}{q}$ differs from an integer by at least $\frac{1}{q}$, it follows that

$\phi'\left(\sin n_q\pi\frac{p}{q}\right)$ is numerically less than some fixed positive number U', for all values of n_q. We therefore see that

$$\left|\sum_{m_1}^{m}\frac{\pi\phi'\left(\sin n_q\pi\frac{p}{q}\right)\cos n_q\frac{p}{q}}{n_q^{s-1}}\right| < \pi\eta U'.$$

Further, m has been so chosen that $m-|h|^{-t}<1$; from which we have $m\,|h| = |h|^{1-t}+\theta\,|h|$, where $0<\theta<1$. If δ be now so chosen that $\delta^{1-t}+\delta<1/2q$, the two numbers $n_q\frac{p}{q}$, $n_q\left(\frac{p}{q}+h\right)$ differ from one another by less than $\frac{1}{2q}$; moreover they are never integers, and contain no integer between them, and they differ from the nearest integer by more than $\frac{1}{2q}$. It follows that, for all values of y between $\sin n_q\pi\frac{p}{q}$ and $\sin n_q\pi\left(\frac{p}{q}+\delta\right)$, where n_q has the values belonging to it in the series, $\sum\limits_{m_1}^{m}$, $\phi(y)$ has a differential coefficient numerically less than some fixed number U''.

Writing $\dfrac{\phi\left[\sin n_q\pi\left(\frac{p}{q}+h\right)\right]-\phi\left(\sin n_q\pi\frac{p}{q}\right)}{hn_q^{s}}$ in the form

$$\frac{\pi}{n_q^{s-1}}\,\frac{\phi\left[\sin n_q\pi\left(\frac{p}{q}+h\right)\right]-\phi\left(\sin n_q\pi\frac{p}{q}\right)}{\sin n_q\pi\left(\frac{p}{q}+h\right)-\sin\left(n_q\pi\frac{p}{q}\right)}\cos n_q\pi\left(\frac{p}{q}+\frac{1}{2}h\right)\frac{\sin\frac{1}{2}n_q\pi h}{\frac{1}{2}n_q\pi h};$$

we see that this term is numerically less than $\frac{\pi}{n_q^{s-1}}U''$. It now follows that

$$\left|\sum_{m}^{m_1}\right| < \pi\eta\,(U''+U');$$

and this is numerically as small as we please, if we choose η and δ sufficiently small. It is therefore possible to choose δ so small that the last of the requisite conditions is satisfied, for all values of $|h|<\delta$.

It has now been proved that

$$\frac{f\left(\frac{p}{q}+h\right)-f\left(\frac{p}{q}\right)}{h}=\pi\sum_{1}^{\infty}\frac{\phi'\left(\sin n_q\pi\frac{p}{q}\right)}{n_q^{s-1}}\cos n_q\pi\frac{p}{q}+\sigma$$
$$+\frac{1}{q^s}\sum_{m=1}^{\infty}\frac{\phi\,(\overline{-1}|^{mp}\sin mq\pi h)}{hm^s},$$

where σ and h converge together to zero.

The second series on the right-hand side of this equation can be written in the form

$$\frac{1}{q^s}\sum_{n=1}^{n=m}\frac{\phi\,(\overline{-1}|^{np}\sin nq\pi h)}{hn^s}+\frac{1}{q^s}\sum_{n=m+1}^{\infty}\frac{\phi\,(\overline{-1}|^{np}\sin nq\pi h)}{hn^s},$$

where m is fixed as before, for each value of h. The second sum is arbitrarily small, for a sufficiently small value of δ. We have then to consider the first sum, which may be written in the form

$$\frac{\pi}{q^{s-1}} \sum_{n=1}^{n=m} \frac{\overline{-1}|^{np}}{n^{s-1}} \frac{\phi\left(\overline{-1}|^{np} \sin nq\pi h\right)}{\overline{-1}|^{np} \sin nq\pi h} \frac{\sin nq\pi h}{nq\pi h};$$

and we now consider this sum in the different cases which arise when various assumptions are made as to the nature of the derivatives of $\phi(y)$ at the point $y = 0$.

Case I. Let $\phi(y)$ have the derivative $+\infty$, at $y = 0$ on the right, and the derivative $-\infty$, at $y = 0$ on the left. It is clear that, for positive values of h, all the terms of the series have one and the same sign, δ having been chosen so small that mh is also sufficiently small; also it is clear that the first term of the series becomes numerically arbitrarily great for sufficiently small values of h. It therefore appears that the sum of the series becomes indefinitely great, as h approaches the limit zero from the right-hand side. If h be negative, the terms of the series all have the same sign, the opposite one from that which they have when h is positive, and as before, the sum of the series is indefinitely great as h converges to zero.

It has therefore been shewn that

$$\frac{f\left(\frac{p}{q}+h\right)-f\left(\frac{p}{q}\right)}{h}$$

has the limit $+\infty$ on one side of the point $\frac{p}{q}$, and $-\infty$ on the other side. The singularities of the derivative of $f(x)$ at the rational points have the same peculiarity as that of $\phi(y)$ at the point $y = 0$; i.e. derivatives on the right and on the left exist, which are infinite, but of opposite signs.

Case II. Let $\phi(y)$ have a differential coefficient at $y = 0$, which is either $+\infty$, or $-\infty$.

It is then clear that, in case p be even,

$$\frac{f\left(\frac{p}{q}+h\right)-f\left(\frac{p}{q}\right)}{h}$$

has the same limit $+\infty$, or $-\infty$, as $\frac{\phi(h)}{h}$. If p be odd, the terms of the series under examination have alternate signs, and no conclusion can in general be drawn as to the nature of the derivatives of $f(x)$ at the point $x = \frac{p}{q}$.

Case III. Let $\phi(y)$ have a finite differential coefficient at $y = 0$.

In this case, as is easily seen, $f(x)$ has, at the point $\frac{p}{q}$, a definite differential coefficient of which the value is

$$\pi \sum_{m=1}^{\infty} \frac{\phi'\left(\sin n\pi \frac{p}{q}\right)}{n^{s-1}} \cos n\pi \frac{p}{q}.$$

Case IV. Let $\phi(y)$ have finite derivatives at $y = 0$ on the right, and on the left, which differ from one another. In this case $f(x)$ has, at each rational point, finite derivatives on the right, and on the left, which differ from one another.

Case V. Let $D^+\phi(0)$, $D_+\phi(0)$, $D^-\phi(0)$, $D_-\phi(0)$ be all finite and different from one another. The function $f(x)$ has then at $\frac{p}{q}$, at least when p is even, the same peculiarity as $\phi(y)$ at $y = 0$.

EXAMPLES

(1) Let $\phi(y) = y \sin \frac{1}{y}$, $\phi(0) = 0$. The corresponding function $f(x)$ is given by

$$f(x) = \sum_1^{\infty} \frac{\sin n\pi x \sin \frac{1}{n\pi x}}{n^s}, \text{ where } s > 2.$$

This function is continuous, but has no definite derivatives at the rational points. No assertion can be made as to the derivatives at the irrational points, because the differential coefficient $\phi'(y)$ has indefinitely great values in every neighbourhood of $y = 0$.

(2) Let $\phi(y) = (y^2)^{\frac{\alpha}{\beta}}$, where α, β are positive integers such that $2\alpha < \beta$, and the real positive values of the root are taken. We then have

$$f(x) = \sum_1^{\infty} \frac{(\sin^2 n\pi x)^{\frac{\alpha}{\beta}}}{n^s}, \text{ where } s > 2.$$

This function is continuous, and has, at all rational points, indefinitely great derivatives on the right, and on the left, of opposite signs. No assertion can be made as to the derivatives at the irrational points.

CANTOR'S METHOD OF CONDENSATION OF SINGULARITIES

270. A method of constructing a function which exhibits, at an everywhere-dense set of points, some singularity, either in relation to continuity, or to its derivatives, has been given by Cantor*. Let $\phi(y)$ denote a function which is continuous for all values of y in the interval $(-1, 1)$, except $y = 0$; and let $\phi(0) = 0$. Let G denote an enumerable set of points $\omega_1, \omega_2, \omega_3, \ldots$, which may be everywhere-dense. The method of condensation consists of the construction of the function

$$f(x) = \sum_{n=1}^{\infty} c_n \phi(x - \omega_n),$$

* *Math. Annalen*, vol. XIX (1882), p. 588. See also Dini's *Grundlagen*, p. 188.

where $c_1, c_2, \ldots, c_n, \ldots$ are positive numbers, so chosen that the series $\sum\limits_1^\infty c_n$ is convergent, and that $\sum\limits_{n=1}^\infty c_n\phi(x-\omega_n)$ converges absolutely for each value of x, and uniformly in every interval.

This method has two advantages over that of Hankel. In the first place, the points $\omega_1, \omega_2, \ldots$ do not necessarily consist of the rational points of the interval $(-1, +1)$, but may form any enumerable aggregate. In the second place, for a value ω_n of x, the singularity in question is exhibited by the one term $c_n\phi(x-\omega_n)$ only, of the series which represents $f(x)$; whereas in Hankel's method, the singularity of $\phi(y)$ at $y=0$ is exhibited, for $x = p/q$, by an indefinitely great number of terms of the series which represents the function formed by condensation.

Let now $\phi(y)$ be discontinuous at $y=0$; then, for any value of x_0 of x, which is not one of the values of G, the terms of the series $\Sigma c_n\phi(x-\omega_n)$ are all continuous; hence, since the series converges uniformly in any interval containing x_0, it follows that $f(x)$ is continuous at x_0. Again, in order to consider the continuity of $f(x)$ at the point ω_n, we may separate the term $c_n\phi(x-\omega_n)$ from the rest of the series. As before, the series which consists of all the terms except the one $c_n\phi(x-\omega_n)$ represents a function which is continuous at $x=\omega_n$, but $c_n\phi(x-\omega_n)$ has at ω_n a discontinuity of the same character as that of $\phi(y)$ at $y=0$. It has therefore been shewn that $f(x)$ is continuous at every point which does not belong to G, but has at every point of G a discontinuity of the same character as that of $\phi(y)$ at the point $y=0$. If $\phi(y)$ have a finite saltus k at $y=0$, the saltus of $c_n\phi(x-\omega_n)$ at ω_n is kc_n. Hence, on account of the convergence of Σc_n, there are only a finite number of points ω_n at which the saltus of $f(x)$ exceeds any fixed positive number. The function $f(x)$ is therefore a function that is integrable (R).

Let it next be assumed that $\phi(y)$ is continuous throughout $(-1, 1)$, and possesses a differential coefficient for every value of y except $y=0$; and that the differential coefficients are all numerically less than some fixed positive number B. It then follows that the four derivatives of $\phi(y)$ at $y=0$ are all finite; it also follows that $\left|\frac{\phi(h)}{h}\right|$ is less than some fixed number A, for all values of h which are numerically less than some fixed number δ.

We now see that for any pair of points y_1, y_2 such that $|y_1-y_2|<\delta$, we have $\left|\frac{\phi(y_1)-\phi(y_2)}{y_1-y_2}\right| <$ the greater of the numbers A and B, which may be denoted by C.

If x be not a point of G, the sum

$$\sum_{n=m+1}^{\infty} c_n \left| \frac{\phi(x+h-\omega_n)-\phi(x-\omega_n)}{h} \right| \text{ is } < C \sum_{n=m+1}^{\infty} c_n,$$

provided $|h| < \delta$; hence the series is uniformly convergent for all values of h such that $0 < |h| < \delta$, and therefore it represents the value of $f'(x)$.

In case x be a point ω_n of G, we separate from the series which represents $f(x)$, the term $c_n\phi(x-\omega_n)$. It appears then that the remaining part of the series represents a function which has a definite differential coefficient $\lambda(\omega_n)$ at ω_n.

We have therefore

$$\frac{f(\omega_n+h)-f(\omega_n)}{h} = c_n\frac{\phi(h)}{h} + \lambda(\omega_n) + \zeta,$$

where ζ converges to zero when h does so. It thus appears that $f(x)$ has no definite derivatives at $x = \omega_n$, but that it has at the point the same kind of singularity as $\phi(y)$ has at the point $y = 0$.

EXAMPLES

(1) Let $\phi(y) = y - \frac{1}{2}y \sin(\frac{1}{2}\log y^2)$. This function has a differential coefficient $\phi'(y)$ for every point except $y=0$; and $\phi'(y)$ oscillates between the values $1-1/\sqrt{2}$, $1+1/\sqrt{2}$.

The corresponding function $f(x) = \Sigma c_n\phi(x-\omega_n)$ has a differential coefficient at every point not belonging to G_n. At the point $x=\omega_n$, its derivative oscillates between values $\frac{1}{2}c_n + \lambda(\omega_n)$ and $\frac{1}{2}c_n + \lambda(\omega_n)$.

(2*) Let $\phi(y) = y^{\frac{1}{3}}$; then $\phi'(0) = +\infty$. The corresponding function $\Sigma c_n(x-\omega_n)^{\frac{1}{3}}$ has differential coefficients which are finite at a set of points not belonging to G. At a point ω_n of G, we have $f'(\omega_n) = +\infty$. This example does not fall under the case considered above, because $|\phi'(y)|$, for $|y|>0$, has no upper limit.

THE CONSTRUCTION OF NON-DIFFERENTIABLE FUNCTIONS

271. It has been pointed out in I, § 259, that a function $f(x)$ may be continuous at a particular point x, and yet may not possess, at that point, a differential coefficient, either finite, or infinite with a fixed sign. A simple example of such a function is $x\sin\frac{1}{x}$, which at the point $x=0$ is continuous, but whose derivatives, both on the right and on the left, oscillate in the interval $(-1, 1)$; similarly the function $x^{\frac{1}{3}}\sin\frac{1}{x}$ is continuous at the point $x=0$, but the derivatives, both on the right and on the left, oscillate through the interval $(-\infty, \infty)$. The question of the existence

* This function $\Sigma c_n(x-\omega_n)^{\frac{1}{3}}$ has been studied by Brodén, see his paper "Ueber das Weierstrass-Cantor'sche Condensationverfahren," *Stockholm Öfv.*, 1896, p. 583; also *Math. Annalen*, vol LI (1899), p. 318. See further Pompeiu, *Math. Annalen*, vol. LXIII (1907), p. 326, where it is shewn that, if the series be denoted by t, the inverse function $x=G(t)$ is a continuous function with a limited differential coefficient, which is zero at an everywhere-dense set of points, provided the series $\Sigma c_n^{\frac{1}{3}}$ be convergent. This function is accordingly everywhere-oscillating.

of a continuous function which at no point has a differential coefficient, either finite, or infinite with fixed sign, was settled affirmatively by the construction by Weierstrass* of such a non-differentiable function. This example of a non-differentiable function, namely the function

$$y = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where b is an odd integer, and a is such that $0 < a < 1$, and $ab > 1 + \frac{3}{2}\pi$, was first published† by du Bois-Reymond, with Weierstrass' own proof.

Attention has however been directed‡ by M. Jašek of Pilsen to the existence of a manuscript by Bolzano, said to date from the year 1834, in which Bolzano defined a function, continuous in a finite interval, which he proved to possess no finite differential coefficient at any point belonging to a certain set everywhere-dense in the interval. It has been shewn by§ K. Rychlik that, in point of fact, Bolzano's function possesses no differential coefficient, either finite or infinite (with fixed sign), at any interior point of the interval for which the function is defined; at the left-hand end-point of the interval there is a derivative on the right of value $+\infty$, and at the right-hand end-point the derivatives on the left are oscillatory.

An example of a non-differential function was published|| in 1890, due to Cellérier, namely, $y = \sum_{n=1}^{\infty} a^{-n} \sin a^n x$, where a is a sufficiently large even integer. There is evidence that Cellérier discovered that function as early as 1830. This function is however not non-differentiable in the same strict sense as in the case of Weierstrass' function, for, although it has no finite differential coefficient at any point, it has a differential coefficient $+\infty$, at the points of an everywhere-dense set of points x, and a differential coefficient $-\infty$, at the points of another everywhere-dense set.

A general theory of the construction of non-differentiable functions was given¶ by Dini, which includes that of the Weierstrassian function as a special case. Methods of construction of such functions, dependent upon the employment of assigned functional values at the points of enumerable everywhere-dense sets, have been developed by Faber** and by Steinitz††.

* *Werke*, vol. II, pp. 92, 97, 223.

† *Crelle's Journal*, vol. LXXIX (1875), pp. 21–37.

‡ *Sitz. berichte der k. Böhm Ges. der Wiss.* (1920–21).

§ *Ibid.* (1921–22).

|| *Bull. des Sc. Math.* (2), vol. XIV (1890), p. 152. A discussion of Cellérier's function has been published by G. C. Young, *Quarterly Journal of Math.* vol. XLVII (1916), pp. 137, 171; see also Falanga, *Giorn. di Mat.* vol. LIX (1921), p. 137. In both of these writings however the existence of the infinite differential coefficients was overlooked.

¶ *Annali di Mat.* (2), vol. VIII (1877), p. 121; see also Dini-Lüroth, *Grundlagen füreine Theorie der Funktionen einen veränderlichen reelen Grösse*, Leipzig, 1892, pp. 205–29.

** *Math. Annalen*, vol. LXVI (1909), p. 81, and vol. LXIX (1910), p. 372.

†† *Ibid.* vol. LII (1899), p. 58.

It has been shewn* by E. H. Moore that the space-fitting curves given by Peano, Hilbert, and Moore (see I, §§ 326–328) are at each point devoid either of a unique derivative on the right or of a unique derivative on the left, and thus may be regarded as non-differentiable functions; although these curves do not represent single-valued functions, and thus do not belong to the class considered here.

A simple method of constructing functions which are non-differentiable in the strict sense has been given† by Knopp. This method, of which a short account will be given below, is applicable to obtain Weierstrass' function and various other such functions which have been obtained by other mathematicians.

272. Let the incrementary ratio $\frac{f(x_2)-f(x_1)}{x_2-x_1}$, for the continuous function $f(x)$, corresponding to the two points x_1, x_2, be denoted, as in I, § 277, by $I(x_1, x_2)$. If x be a fixed point, and $x_2 < x < x_1$, we have $I(x_1, x_2) = I(x_1, x)\frac{x_1-x}{x_1-x_2} + I(x_2, x)\frac{x-x_2}{x_1-x_2}$; and it follows that $I(x_1, x_2)$ lies between the two numbers $I(x_1, x)$ and $I(x_2, x)$; and when these have equal values, $I(x_1, x_2)$ has the same value.

If $\lim\limits_{x_1\sim x} I(x_1, x)$, $\lim\limits_{x_2\sim x} I(x_2, x)$ have one and the same unique value, either finite, or $+\infty$, or $-\infty$, $\lim\limits_{x_1\sim x,\, x_2\sim x} I(x_1, x_2)$ is unique, and has the same value. Conversely, if $\lim\limits_{x_1\sim x,\, x_2\sim x} I(x_1, x_2)$ has a unique value, independent of the modes in which x_1 and x_2 converge to x, then $I(x_1, x)$, $I(x_2, x)$ each converges or diverges to that value, and there is a differential coefficient, finite or infinite, at the point x.

It follows that, in order that the function may be non-differentiable at the point x, it must be possible to obtain two pairs of sequences of x_1, x_2, where each of the four sequences converges to x, such that $I(x_1, x_2)$ does not, for the two pairs of sequences, converge or diverge to one and the same value. This is applicable as a criterion to establish the non-differentiability of a function at a particular point. In particular, it will be sufficient to shew that $I(x_1, x)$, $I(x_2, x)$ have not one and the same unique limit as $x_1 \sim x$, $x_2 \sim x$, or that this is the case for $I(x_1, x)$ and $I(x_1, x_2)$.

As an important example of the use of this criterion, we shall first consider Weierstrass' function $f(x) = \sum\limits_{r=0}^{\infty} a^r \cos(b^r \pi x)$. Let x have a fixed value, and let c_n be the integer, corresponding to each value of n, such that $c_n - \frac{1}{2} \leqq b^n x < c_n + \frac{1}{2}$.

* *Trans. Amer. Math. Soc.* vol. I (1900), p. 72.

† *Math. Zeitschr.* vol. II (1918), p. 1. This memoir contains a very full reference to the literature of the subject.

First, let $$x_2 = \frac{c_n - 1}{b^n}, \quad x_1 = \frac{c_n + \frac{1}{2}}{b^n},$$

then $$I(x_1, x_2) = \tfrac{2}{3} b^n \left\{ \sum_{r=0}^{r=n-1} a^r (\cos b^r \pi x_1 - \cos b^r \pi x_2) + (-1)^{c_n} \frac{a^n}{1-a} \right\},$$

it being assumed that b is an odd integer.

Since $$|\cos b^r \pi x_1 - \cos b^r \pi x_2| \leqq b^r \pi (x_1 - x_2) \leqq \frac{3}{2} \frac{\pi b^r}{b^n},$$

we have $$I(x_1, x_2) = \tfrac{2}{3} a^n b^n \frac{(-1)^{c_n}}{1-a} + \lambda\pi \frac{a^n b^n - 1}{ab - 1},$$

where $-1 \leqq \lambda \leqq 1$. In case

$$\frac{2}{3} \frac{1}{1-a} > \frac{\pi}{ab-1}, \text{ or } ab > 1 + \tfrac{3}{2}\pi(1-a),$$

we have $I(x_1, x_2) = (-1)^{c_n} a^n b^n N_n - \dfrac{\lambda\pi}{ab-1}$, where $N_n > 0$. In a similar manner, if we take $x_2' = \dfrac{c_n - \frac{1}{2}}{b^n}$, $x_1' = \dfrac{c_n + 1}{b^n}$, we find that

$$I(x_1'\ x_2') = -(-1)^{c_n} a^n b^n N_n' - \frac{\lambda'\pi}{ab-1};$$

where $N_n' > 0$, and $-1 \leqq \lambda' \leqq 1$, where, as before, $ab > 1 + \frac{3}{2}\pi(1-a)$.

In case there is in $\{c_n\}$ a sequence of even integers, it is seen that, as c_n has successively the values in this sequence, $I(x_1, x_2)$ is positive and increases indefinitely, and $I(x_1', x_2')$ is negative and increases indefinitely in numerical value. It follows that there is no differential coefficient at the point x. The same conclusion can be made in case $\{c_n\}$ contains a sequence of odd integers. The theorem of Weierstrass has thus been established that, b being an odd integer, if $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$ the continuous function $\sum\limits_{n=0}^{\infty} a^n \cos(b^n \pi x)$ is non-differentiable. That the inequality

$$ab > 1 + \tfrac{3}{2}\pi$$

may be, as is shewn above, replaced by the less stringent condition

$$ab > 1 + \tfrac{3}{2}\pi(1-a)$$

was proved by Bromwich*.

G. H. Hardy has shewn†, by a method which is much more abstruse than the one which has been employed above, that, if $0 < a < 1$, $ab \geqq 1$, whether b be integral or not, the function $\sum\limits_{n=0}^{\infty} a^n \cos b^n x$ has no finite differential coefficient, and he has obtained other properties of this and similar functions.

It will now be shewn‡ that Weierstrass' function, when the integer b is subject to the condition $ab > 1$, has, at an everywhere-dense set of points, a unique derivative on the right equal to $-\infty$, and a unique derivative on

* *Theory of Infinite Series*, p. 490. † *Trans. Amer. Math. Soc.* vol. XVII (1916), p. 301.

‡ See G. C. Young's memoir, *Quarterly Journal*, vol. XLVII (1916), p. 167, where this is established.

the left, equal to $+\infty$; and that at the points of another everywhere-dense set, there are unique infinite derivatives, $+\infty$ on the right and $-\infty$ on the left. In geometrical language, the function has cusps at the points of two everywhere-dense sets; in these sets the cusps point in opposite directions.

Consider the point $x = 0$, we have then

$$\frac{f(0) - f(h)}{h} = \sum_{n=0}^{\infty} a^n \{1 - \cos b^n \pi h\} = 2 \sum_{n=0}^{\infty} a^n \sin^2 \tfrac{1}{2} b^n \pi h;$$

let m be the positive integer such that $|h| b^m \leq 1 < |h| b^{m+1}$, we have then

$$\frac{f(0) - f(h)}{h} > 2b^m \sum_{n=0}^{n=m} a^n \sin^2 \tfrac{1}{2} b^n \pi |h|;$$

and since $\sin \frac{1}{2} b^n \pi |h| > \frac{1}{\pi}(b^n \pi |h|) > b^{n-m-1}$, we have

$$\frac{f(0) - f(h)}{h} > \frac{2}{b^{m+2}} \sum_{n=0}^{n=m} a^n b^{2n} > \frac{2}{b^{m+2}} \frac{a^m b^{2m}}{ab^2 - 1} > \frac{2}{b^2} \frac{a^m b^m}{ab^2 - 1}.$$

As m increases indefinitely, h converges to zero, and since for $ab > 1$, $a^m b^m$ increases indefinitely, we have

$$D^+ f(0) = D_+ f(0) = -\infty, \text{ and } D^- f(0) = D_- f(0) = +\infty.$$

Let $x = x' + \frac{2r}{b^m}$, where r is any positive or negative integer, and m is a positive integer; we have then

$$f(x) = \sum_{n=0}^{m-1} a^n \cos b^n \pi x + \sum_{n=m}^{\infty} a^n \cos b^n \pi x'.$$

The first term on the right-hand side has a finite differential coefficient at the point $x = \frac{2r}{b^m}$, and the second term has a unique derivative $-\infty$, on the right, and a unique derivative $+\infty$, on the left. Thus at the everywhere-dense set of points $x = \frac{2r}{b^m}$, we have $D^+ f(x) = D_+ f(x) = -\infty$, and $D^- f(x) = D_- f(x) = \infty$. If we take $x = x' + \frac{2r+1}{b^m}$, we have

$$f(x) = \sum_{n=0}^{m-1} a^n \cos b^n \pi x - \sum_{n=m}^{\infty} a^n \cos b^n \pi x';$$

therefore, at the everywhere-dense set of points $x = \frac{2r+1}{b^m}$, we have

$$D^+ f(x) = D_+ f(x) = +\infty; \; D^- f(x) = D_- f(x) = -\infty.$$

It does not appear to be definitely known whether a non-differentiable function can exist which has no cusps.

273. The function given by Cellérier will be now discussed.

Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \sin a^n x$, where a is an even integer. We have

$$f(x+h) - f(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \{\sin a^n (x+h) - \sin a^n x\};$$

if now $h = \frac{2\pi}{a^m}$, where m is a positive integer, all the terms on the right-hand side vanish, except the first $m-1$ terms; thus

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=1}^{m-1} \cos a^n x \frac{\sin(2\pi a^{n-m})}{2\pi a^{n-m}} - \sum_{n=1}^{m-1} \sin a^n x \frac{1-\cos(2\pi a^{n-m})}{2\pi a^{n-m}}.$$

The first sum on the right-hand side differs from $\sum_{n=1}^{m-1} \cos a^n x$ by less than $\sum_{n=1}^{m-1} \frac{1}{6}\left(\frac{2\pi}{a^{m-n}}\right)^2$, if we assume that $a > 2$; and this is less than $\frac{2\pi^2}{3a} \cdot \frac{1}{a-1}$; the general term in the second sum is numerically less than πa^{n-m}, hence the sum is numerically less than $\frac{\pi}{a-1}$. We thus have

$$I_m \equiv \frac{f(x+h) - f(x)}{h} = \sum_{n=1}^{m-1} \cos a^n x + \lambda\theta,$$

where $|\theta| < 1$, and λ is a positive number dependent only on a, and which may be made as small as we please by taking a sufficiently large.

If we take $h = \frac{\pi}{a^m}$, the mth term in the incrementary ratio I_m' does not vanish, but has the value $-\frac{2}{\pi} \sin a^m x$; the succeeding terms all vanish, and we find that $I_m' = \sum_{n=1}^{m-1} \cos a^n x - \frac{2}{\pi} \sin a^m x + \lambda'\theta'$, where $|\theta'| < 1$, and λ' is a positive number dependent only on a, and which becomes as small as we please by taking a sufficiently large. From the above results we have

$$I_{m+1} - I_m = \cos a^m x + 2\lambda\theta'',$$

$$I_m - I_m' = \frac{2}{\pi} \sin a^m x + 2\lambda'\theta''';$$

when $|\theta''|$, $|\theta'''|$ are less than 1. It follows that

$$\frac{\pi^2}{4}(I_m - I_m')^2 + (I_{m+1} - I_m)^2$$

is, for all values of m, greater than $1 - 2\{4\lambda^2\theta''^2 + \pi^2\lambda'^2\theta'''^2\}^{\frac{1}{2}}$, or than $1 - 4\pi(\lambda^2 + \lambda'^2)^{\frac{1}{2}}$, which is certainly positive, if a be large enough.

It is consequently impossible that I_m and I_m' should both have unique finite limits which have the same value, for it is impossible that both the conditions $\lim_{m\sim\infty}(I_m - I_m') = 0$, $\lim_{m\sim\infty}(I_{m+1} - I_m) = 0$ should be satisfied. Therefore, if a be a sufficiently large even integer, $f(x)$ has at no point a finite differential coefficient. In order that I_m and I_m' may both have the

same infinite limit, $+\infty$, or $-\infty$, it is not necessary that these conditions should be satisfied. It will be shewn here that there is an everywhere-dense set of points in which Cellérier's function has a differential coefficient $+\infty$. In geometrical language, the curve represented by Cellérier's function has a set of points of inflexion. At the point $x = 0$, we have

$$\frac{f(h) - f(0)}{h} = \sum_{n=1}^{\infty} \frac{1}{a^n h} \sin(a^n h);$$

let m be such that $a^m |h| \leqq \frac{1}{2}\pi < a^{m+1} |h|$; then the sum of the first m terms on the right-hand side is positive, and greater numerically than $\frac{2m}{\pi}$. The sum of the remaining terms is numerically less than $\sum_{n=m+1}^{\infty} \frac{1}{a^n |h|}$, or than $\frac{2a}{\pi(a-1)}$. If m be indefinitely increased $\frac{f(h) - f(0)}{h}$ diverges to $+\infty$, whether h be positive or negative. It follows that, at the point $x = 0$, the function has a differential coefficient $+\infty$. Let $x = x' + \frac{r\pi}{a^m}$, where r is any integer and m is a positive integer. We have then

$$f(x) = \sum_{n=1}^{m} \frac{1}{a^n} \sin(a^n x) + \sum_{n=m+1}^{\infty} \frac{1}{a^n} \sin a^n x';$$

and this function has a differential coefficient $+\infty$ at the point $x = \frac{r\pi}{a^m}$, since the first sum has a finite differential coefficient. It has been shewn (I, § 298) that the set of all the points at which the differential coefficient is infinite has a measure zero.

The method which has been applied above to shew that Cellérier's function has, for a sufficiently large even integer a, no finite differential coefficient, may also be applied* to prove the same property of Weierstrass' function $\sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$, where $0 < a < 1$, and b is an odd integer such that $ab \geqq 1$.

274. An account will now be given of the mode of construction of non-differentiable functions developed† by Knopp, and which has been already referred to in § 271.

Let $u_n(x)$ be a continuous function, defined for each value of n (0, 1, 2, 3, ...) for the indefinite interval $(-\infty, \infty)$, as a periodic function, of period $2l$, so that $u_n(x) = u_n(x + 2l)$.

* This has been carried out in detail by Falanga (*loc. cit.*), where however the condition that b must be odd is omitted, although it is necessary in the process. The possibility is also overlooked that the function may have an infinite differential coefficient.

† *Math. Zeitschr.* vol. II (1918), p. 1. In this memoir geometrical illustrations of the method of construction are given.

If c_n be the greatest value of $|u_n(x)|$, it will be assumed that the series $\sum_{n=0}^{\infty} c_n$ is convergent, so that, in accordance with Weierstrass' test, the series $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly to a continuous function $f(x)$. Let the partial sum $\sum_{n=0}^{n} u_n(x)$ be denoted by $f_n(x)$, so that $f(x) = \lim_{n\sim\infty} f_n(x)$; we thus have $f_n(x) - f_{n-1}(x) = u_n(x)$, $f_0(x) = u_0(x)$.

It will further be assumed that each function $u_n(x)$ has, in a complete period $(0, 2l)$ of x, a finite set of maxima and minima, the number of which increases indefinitely with n, and so that the greatest interval between a minimal point x, of $u_n(x)$, and either of the adjacent maximal points x, diminishes indefinitely as n is increased indefinitely. Let ξ be any value of x, then ξ is in an interval $(x_r^{(n)}, x_{r+1}^{(n)})$, where $x_r^{(n)}$ and $x_{r+1}^{(n)}$ are two consecutive minimal points of $u_n(x)$; the point ξ may coincide with $x_r^{(n)}$ or with $x_{r+1}^{(n)}$. Let $x'^{(n)}_{r-1}$ be the maximal point of $u_n(x)$ next to, and on the left of, the point $x_r^{(n)}$; and let $x'^{(n)}_{r+1}$ be the maximal point of $u_n(x)$ next to, and on the right of, the point $x_{r+1}^{(n)}$, and let us consider the two incrementary ratios

$$\frac{f(x'^{(n)}_{r+1}) - f(x_r^{(n)})}{x'^{(n)}_{r+1} - x_r^{(n)}}, \quad \frac{f(x_{r+1}^{(n)}) - f(x'^{(n)}_{r-1})}{x_{r+1}^{(n)} - x'^{(n)}_{r-1}},$$

of the function $f(x)$. Since the interval $(x_r^{(n)}, x_{r+1}^{(n)})$ is determinate for each value of n, for a fixed point ξ, we have, as n is increased, two sets of incrementary ratios of $f(x)$ such as are considered in § 272, in expressing the condition that the function $f(x)$ shall be non-differentiable at the point x. Let it be assumed that, from and after some value m, of n, the conditions

$$f(x'^{(n)}_{r+1}) > f(x_r^{(n)}), \quad f(x'^{(n)}_{r-1}) > f(x_{r+1}^{(n)})$$

are both satisfied, for $n \geqq m$; the two incrementary ratios then have opposite signs. In case both the incrementary ratios increase indefinitely in numerical value, as $n \sim \infty$, they diverge to ∞ and $-\infty$ respectively, and there is consequently no differential coefficient, finite or infinite, at the point ξ. In order to ensure that this is the case for all points ξ, let A_n be the upper boundary of the set of absolute values of the incrementary ratios of $u_n(x)$ for every pair of points; this is the same as the upper boundary of the absolute values of any one of the four derivatives of $u_n(x)$ (see I, § 280). It then follows that the values of all incrementary ratios for the function $f_{n-1}(x) \equiv u_0(x) + u_1(x) + \ldots + u_{n-1}(x)$, lie in the interval bounded by the two numbers $\pm (A_0 + A_1 + \ldots + A_{n-1})$.

Let it now be assumed that

$$u_n(x'^{(n)}_{r+1}) > u_n(x_r^{(n)}), \text{ and } u_n(x'^{(n)}_{r-1}) > u_n(x_{r+1}^{(n)}),$$

from and after some value m, of n, wherever the point ξ may be; since the

functions are periodic these conditions are finite in number, being all obtained by assigning a finite set of values to r.

It follows that

$$\begin{cases} f(x'^{(n)}_{r+1}) - f(x^{(n)}_r) > f_n(x'^{(n)}_{r+1}) - f_n(x^{(n)}_r) \\ f(x'^{(n)}_{r-1}) - f(x^{(n)}_{r+1}) > f_n(x'^{(n)}_{r-1}) - f_n(x^{(n)}_{r+1}) \end{cases}, \text{ where } n \geqq m.$$

Let B_n denote the smallest of the finite set of numbers

$$\frac{u_n(x'^{(n)}_{r+1}) - u_n(x^{(n)}_r)}{x'^{(n)}_{r+1} - x^{(n)}_r}, \quad \frac{u_n(x'^{(n)}_{r-1}) - u_n(x^{(n)}_{r+1})}{x'^{(n)}_{r-1} - x^{(n)}_{r+1}},$$

where r has the finite set of values required for points ξ in the interval $(0, 2l)$.

We see then, since $f_n(x) = f_{n-1}(x) + u_n(x)$, that

$$\frac{f(x'^{(n)}_{r+1}) - f(x^{(n)}_r)}{x'^{(n)}_{r+1} - x^{(n)}_r} > B_n - (A_0 + A_1 + \ldots + A_{n-1}),$$

$$\frac{f(x'^{(n)}_{r-1}) - f(x^{(n)}_r)}{x'^{(n)}_{r-1} - x^{(n)}_{r+1}} < -B_n + (A_0 + A_1 + \ldots + A_{n-1}).$$

If now $\lim\limits_{n\sim\infty} \{B_n - (A_0 + A_1 + \ldots + A_{n-1})\} = +\infty$ the required conditions are satisfied by the two incrementary ratios, and the function $f(x)$ has consequently at no point a differential coefficient. It has thus been proved that:

It is sufficient, in order that $f(x)$ may be non-differentiable, that (1), $u_n(x'^{(n)}_{r+1}) > u_n(x^{(n)}_r)$, *and* $u_n(x'^{(n)}_{r-1}) > u_n(x^{(n)}_{r+1})$ *for* $n \geqq m$, *where* $x^{(n)}_r$, $x^{(n)}_{r+1}$ *are any two consecutive minimal points of* $u_n(x)$, *in order from left to right,* $x'^{(n)}_{r+1}$ *is the maximal point of* $u_n(x)$ *next on the right of* $x^{(n)}_{r+1}$, *and* $x'^{(n)}_{r-1}$ *the maximal point next on the left of* $x^{(n)}_r$; *and* (2), *that*

$$\lim_{n\sim\infty} \{B_n - (A_0 + A_1 + \ldots + A_{n-1})\} = \infty;$$

where A_n is the upper boundary of the absolute values of all incrementary ratios of $u_n(x)$, and B_n is the smallest of the finite sets of numbers

$$\frac{u_n(x'^{(n)}_{r+1}) - u_n(x^{(n)}_r)}{x'^{(n)}_{r+1} - x^{(n)}_r}, \quad \frac{u_n(x'^{(n)}_{r-1}) - u_n(x^{(n)}_{r+1})}{x'^{(n)}_{r-1} - x^{(n)}_{r+1}}.$$

275. There are four specially simple types of non-differentiable functions which may be defined by the method developed above.

(1) Let the minima of $u_n(x)$, for $n \geqq 1$, be all zero, from which it follows that $u_n(x) \geqq 0$, for all values of x. Further, let all the maximal and minimal points of $u_{n-1}(x)$ be at zeros of $u_n(x)$. In this case the condition (1), of the above theorem, is certainly satisfied, since

$$u_n(x^{(n)}_r) = 0, \quad u_n(x^{(n)}_{r+1}) = 0, \quad u_n(x'^{(n)}_{r+1}) > 0, \quad u_n(x'^{(n)}_{r-1}) > 0.$$

(2) Let the maximal and minimal points of $u_{n-1}(x)$ be all at zeros of $u_n(x)$, for $n = 1, 2, 3, \ldots$. In this case $f(x) = f_{n-1}(x)$, at any maximal or minimal point of $u_{n-1}(x)$.

(3) Let the minima of $u_n(x)$, for $n \geqq 1$, be all zero; and thus $u_n(x) \geqq 0$, and let all the minima of $u_{n-1}(x)$ be at zeros of $u_n(x)$, but not as in (1) the maxima.

In this case we have $f(x) = f_{n-1}(x)$ at any minimal point of $u_{n-1}(x)$.

(4) At every minimal point of $u_{n-1}(x)$, let

$$u_n(x), \quad u_n(x) + u_{n+1}(x), \ldots, u_n(x) + u_{n+1}(x) + \ldots + u_{n+m}(x), \ \ldots$$

all have negative values; and at any maximal point of $u_{n-1}(x)$ let the same expressions all have positive values.

In this case $f_{n+m-1}(x) < f_{n-1}(x)$ at a minimal point of $u_{n-1}(x)$, and $f_{n+m-1}(x) > f_{n-1}(x)$ at a maximal point of $u_{n-1}(x)$.

As a simple example of type (1), let $\psi(x)$ denote the polygonal function which is defined in the interval $(0, 1)$ by

$$\psi(x) = x, \text{ for } 0 \leqq x \leqq \tfrac{1}{2}; \quad \psi(x) = 1 - x, \text{ for } \tfrac{1}{2} \leqq x \leqq 1;$$

and which is defined for all other values of x by the law that it is periodic, with period 1.

Let $u_n(x) = a^n \psi(b^n x)$, where $a < 1$, and b is an even integer; since $0 \leqq \psi(b^n x) \leqq \frac{1}{2}$, it is clear that $\sum\limits_{n=1} a^n \psi(b^n x)$ converges uniformly to a continuous function $f(x)$. The maximal and minimal points of $\psi(b^{n-1}x)$ are given by $x = r \cdot \frac{1}{2b^{n-1}}$, where r is a positive or negative integer, or zero; and all these points are zeros of $b^n x$; hence the function is of type (1), and therefore the condition (1) of the theorem of § 274 is satisfied. The value of A_n is the maximum of $|a^n b^n \psi'(b^n x)|$ which is $a^n b^n$; also $B_n = \frac{1}{3} a^n b^n$, since $0, \frac{1}{b^n}$ are consecutive minima of $u_n(x)$, and $\frac{3}{2b^n}$ is the distance of the minimal point $x = 0$ from the maximal point $x = \frac{3}{2b^n}$. The requisite condition (2), of § 274, is that

$$\lim_{n \sim \infty} \left(\tfrac{1}{3} a^n b^n - ab - a^2 b^2 - \ldots - a^{n-1} b^{n-1}\right) = \infty,$$

or

$$\lim_{n \sim \infty} a^n b^n \left(\frac{1}{3} - \frac{1}{ab - 1}\right) = \infty,$$

which is satisfied if $ab > 4$.

It has thus been shewn that:

The function $\sum\limits_{n=1}^{\infty} a^n \psi(b^n x)$, *where* $a < 1$, *and* b *is an even integer, is non-differentiable, if* $ab > 4$.

The function $\psi(x)$ was first employed* by Faber, for the purpose of constructing a non-differentiable function. The function actually constructed by Faber was the function $\sum_{n=1}^{\infty} \frac{1}{10^n} \psi(2^{n!} x)$; he shewed that this function satisfies the wider condition that $\left|\frac{f(x+h)-f(x)}{|h|^\epsilon}\right|$, for an arbitrarily small positive value of ϵ, has arbitrarily large values.

In general, the function $\sum_{n=1}^{\infty} \alpha_n \psi(\beta_n x)$, where $\sum_{n=1}^{\infty} \alpha_n$ is convergent, and β_n an integer such that β_n/β_{n-1} is an even integer, is non-differentiable if $\frac{1}{2}\alpha_n\beta_n - (\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_{n-1}\beta_{n-1})$ diverges to $+\infty$, as $n \sim \infty$.

If we take instead of $\psi(x)$ the function $|\sin \pi x|$, we obtain the function $\Sigma a^n |\sin b^n \pi x|$; when $a < 1$, and b is an even integer, then $\sum_{n=1}^{\infty} a^n |\sin b^n \pi x|$ is a non-differentiable function, of type (1), provided the second condition of the theorem of § 274 is satisfied. In this case it is found that $A_n = a^n b^n \pi$, $B_n = \frac{2}{3} a^n b^n$, and thus the condition is fulfilled if $\frac{2}{3} a^n b^n - \frac{a^n b^n \pi}{ab - 1}$ diverges to $+\infty$, which will be the case if $ab > 1 + \frac{3}{2}\pi$.

As an example of a non-differentiable function of type (2), let $\chi(x)$ denote the polygonal function obtained by joining

$$\chi(x) = x, \text{ for } 0 \leq x \leq \tfrac{1}{2}, \quad \chi(x) = 1 - x, \text{ for } \tfrac{1}{2} \leq x \leq \tfrac{3}{2},$$
$$\chi(x) = x - 2, \text{ for } \tfrac{3}{2} \leq x \leq 2,$$

and extending the function so that it is periodic, and of period 2.

If $f^{(1)}(x) = \sum_{n=1}^{\infty} a^n \chi(b^n x)$, $f^{(2)}(x) = \sum_{n=1}^{\infty} (-1)^n a^n \chi(b^n x)$, where $0 < a < 1$, and b is an even integer, the maximal and minimal points of

$$u_{n-1}(x) \equiv a^{n-1}\chi(b^{n-1}x), \text{ or } (-1)^{n-1}a^{n-1}\chi(b^{n-1}x)$$

are zeros of $u_n(x)$.

If $ab > 1$, as in the former case the condition (2) is fulfilled, and it is clear that the condition (1) is satisfied. Therefore, when $ab > 1$, the functions $f^{(1)}(x)$, $f^{(2)}(x)$ are non-differentiable.

As before it is seen that the two functions

$$\sum_{n=1}^{\infty} a^n \sin b^n \pi x, \quad \sum_{n=1}^{\infty} (-1)^n a^n \sin b^n \pi x,$$

where $0 < a < 1$, and b is an even integer, are non-differentiable if

$$ab > 1 + \tfrac{3}{2}\pi.$$

Examples of non-differentiable functions of type (3) are

$$\sum_{n=1}^{\infty} a^n \psi(b^n x), \quad \sum_{n=1}^{\infty} a^n |\sin b^n \pi x|,$$

* *Math. Annalen*, vol. LXVI (1909), p. 81, vol. LXIX (1910), p. 372; see also *Jahresber. der deutschen Math. Vereinig.* vol. XVI (1907).

where $0 < a < 1$, b is an odd integer, and in the first case $ab > 1$, in the second $ab > 1 + \frac{3}{2}\pi$.

Examples of functions of type (4) are:

$$\sum_{n=1}^{\infty} a^n \chi(b^n x), \text{ where } 0 < a < 1,\ b = 4m + 1,\ ab > 4,$$

$$\sum_{n=1}^{\infty} a^n \sin b^n \pi x, \text{ where } 0 < a < 1,\ b = 4m + 1,\ ab > 1 + \tfrac{3}{2}\pi,$$

$$\sum_{n=1}^{\infty} (-1)^n a^n \chi(b^n x), \text{ where } 0 < a < 1,\ b = 4m + 3,\ ab > 4,$$

$$\sum_{n=1}^{\infty} (-1)^n a^n \sin(b^n \pi x), \text{ where } 0 < a < 1,\ b = 4m + 3,\ ab > 1 + \tfrac{3}{2}\pi.$$

It is easily verified that, subject to the stated conditions, the conditions of the theorem of § 274 are satisfied. If, in the second and fourth of these functions we change x into $x + \frac{1}{2}$, the functions become the Weierstrassian function $\sum_{n=1}^{\infty} a^n \cos b^n \pi x$, where b is any odd integer, $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$.

THE CONSTRUCTION OF A DIFFERENTIABLE EVERYWHERE-OSCILLATING FUNCTION

276. The first attempt to construct a function with maxima and minima in every interval, which should have at every point a finite differential coefficient, was made by Hankel*. The function which he constructed is however not an everywhere-oscillating function. By Du Bois-Reymond† the view was expressed that no such function can exist, but Dini ‡ regarded the existence of such functions as highly probable. The first actual construction of such a function is due to Köpcke, who having first § constructed an everywhere-oscillating function with derivatives on the right and on the left at every point, in a subsequent memoir || obtained a function having the required properties. Köpcke's construction has been simplified by Pereno¶, and the account here given is based upon the work of the latter.

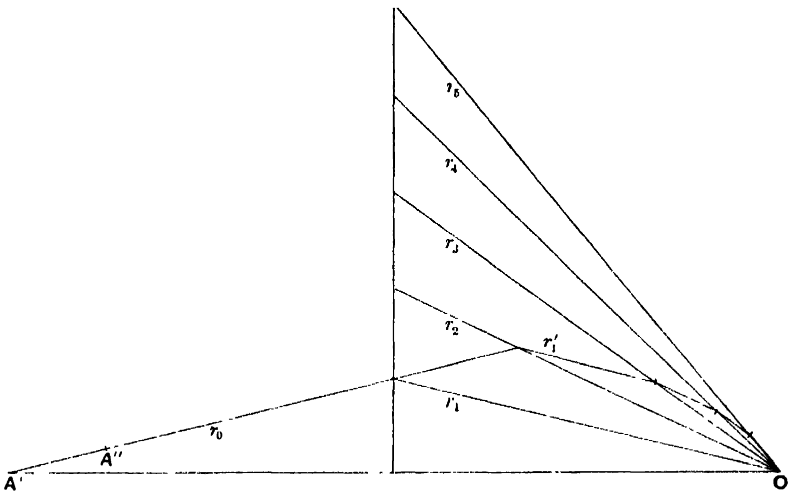
On a straight line AB measure off segments AA', $B'B$, each equal to $\frac{1}{20}AB$. Let O be the middle point of AB, and draw through O straight lines $r_1, r_2, r_3, \ldots r_{2^n+1}$, making angles with OA of which the tangents are

$$1/2^n, \quad 2/2^n, \quad 3/2^n, \ldots (2^n + 1)/2^n$$

respectively. Through A' draw a straight line r_0 making with $A'O$ an angle of tangent $1/2^n$. Through the intersection (r_0, r_2) of r_0 and r_2, draw a straight line r_1' parallel to r_1: through (r_1', r_3) draw a straight line r_2' parallel to r_2,

* *Math. Annalen*, vol. XX (1882), p. 81.
† *Crelle's Journal*, vol. LXXIX (1875), p. 32.
‡ *Grundlagen*, p. 383.
§ *Math. Annalen*, vol. XXIX (1887), p. 123.
|| *Math. Annalen*, vols. XXXIV (1889), p. 161, and XXXV (1890), p. 104.
¶ *Giorn. di Mat.* vol. XXXV (1897), p. 132.

and so on. The straight lines $r_0, r_1', r_2', \ldots r_{2^n-1}\, r_{2^n+1}$ form an unclosed polygon above $A'O$. On OB' describe a precisely similar polygon on the other side of AB. The figure is drawn for the case $n = 2$, and shews the half of the figure belonging to $A'O$. The two polygons form a single polygon joining $A'B'$, and crossing it at O. On r_0 take $A'A'' = AA'$, and describe an arc of a circle touching AB at A, and r_0 at A''. At each vertex of the polygon which has been constructed, mark off on the sides adjacent to that vertex lengths equal to $\frac{1}{20}$ of the shorter side, and construct an arc of a circle touching the two sides at the extremities of these segments so marked off. We have now a figure joining A and B, and composed of arcs of circles and of straight lines. This figure, by means of its ordinates perpendicular to AB, defines a



continuous differentiable function, with a continuous differential coefficient which is zero at A and B, and is $-(2^n+1)/2^n$ at O. This function may be denoted by $(A \mid B)_n$.

Let x, y be a system of coordinate axes in a plane, and draw a quadrant of a circle passing through the points (0, 0) and (1, 0), in the positive quadrant. Let $F_0(x)$ be the function represented by this quadrant, for the interval (0, 1) of x. The function $F_0(x)$ has a maximum at $x = \frac{1}{2}$; also $F_0'(0) = 1$, $F_0'(1) = -1$. If a_0 denote the value of $F_0'(x)$ at $x_0 = \frac{1}{4}$, describe the curve of which the ordinates are $a_0\,(0 \mid \frac{1}{2})_1$, from $x = 0$ to $x = \frac{1}{2}$, and $-a_0\,(\frac{1}{2} \mid 1)_1$, from $x = \frac{1}{2}$ to $x = 1$. This curve represents a continuous function $f_1(x)$; and we have

$$f_1'(0) = f_1'(\tfrac{1}{2}) = f_1'(1) = 0,$$

and
$$f_1'(\tfrac{1}{4}) = -\tfrac{3}{2}a_0, \quad f_1'(\tfrac{3}{4}) = \tfrac{3}{2}a_0.$$

The function $$F_1(x) = F_0(x) + f_1(x)$$
is such that

$F_1'(\frac{1}{4}) = -\frac{1}{2}a_0$, $F_1'(\frac{3}{4}) = \frac{1}{2}a_0$, $F_1'(0) = 1$, $F_1'(1) = -1$, $F_1'(\frac{1}{2}) = 0$.
Thus $F_1(x)$ has a maximum in the interval $(0, \frac{1}{4})$, a minimum in $(\frac{1}{4}, \frac{1}{2})$, a maximum at $x = \frac{1}{2}$, a minimum in $(\frac{1}{2}, \frac{3}{4})$, and a maximum in $(\frac{3}{4}, 1)$.

Let the interval $(0, 1)$ be divided into sub-intervals, by means of the points at which $F_1'(x) = 0$; then, in each of these sub-intervals, $F_1(x)$ is monotone. Then divide each of these sub-intervals into 2, 4, 8, ... equal parts, until the fluctuation of $F_1'(x)$ in each of these parts is $\leq \frac{1}{2}$: this is always possible, since $F_1'(x)$ is a continuous function. Let $c_1^{(1)}, c_1^{(2)}, c_1^{(3)}, \ldots$ denote all the points in which $(0, 1)$ has been divided in this manner. In any one part $(c_1^{(s-1)}, c_1^{(s)})$, $F_1(x)$ is monotone, and its differential coefficient has a fluctuation $\leq \frac{1}{2}$. Let $a_1^{(1)}, a_1^{(2)}, \ldots$ denote the values of $F_1'(x)$ at the middle points of the intervals $(0, c_1^{(1)})$, $(c_1^{(1)}, c_1^{(2)})$, ... Describe the curves

$$a_1^{(1)}(0 \mid c_1^{(1)})_2, \quad a_1^{(2)}(c_1^{(1)} \mid c_1^{(2)})_2, \quad a_1^{(3)}(c_1^{(2)} \mid c_1^{(3)})_2 \ldots;$$

these form together a continuous curve which represents a function $f_2(x)$. Let

$$F_2(x) = F_0(x) + f_1(x) + f_2(x);$$

then $F_2(x)$ has, in every interval $(c_1^{(s-1)}, c_1^{(s)})$, a new maximum and a new minimum. The length of each interval is $< 1/2^2$.

Proceeding in this manner, let us suppose that the function $F_n(x)$ has been formed. Take the points at which $F_n'(x)$ vanishes, and, in case $F_n(x)$ has lines of invariability, the limiting points of those lines; these points divide $(0, 1)$ into sub-intervals in each of which $F(x)$ is monotone. Then divide each of these sub-intervals into 2, 4, 8, ... parts, until the fluctuation of $F_n'(x)$ in each part is $\leq 1/2^n$; let $c_n^{(1)}, c_n^{(2)}, c_n^{(3)} \ldots$ be all the points of division of $(0, 1)$ thus formed. In any interval $(c_n^{(s-1)}, c_n^{(s)})$, the function $F_n(x)$ is monotone, and the fluctuation of $F_n'(x)$ is $\leq 1/2^n$. Let

$$a_n^{(1)}, a_n^{(2)}, \ldots a_n^{(s)}, \ldots$$

be the values of $F_n'(x)$ at the middle points of the intervals; and, in the case of a line of invariability, take as the corresponding value of the a_n, $1/2^n$ or $-1/2^n$, according as the line of invariability is in the interval $(0, \frac{1}{2})$, or in $(\frac{1}{2}, 1)$. Let the curves $a_n^{(s)}(c_n^{(s-1)} \mid c_n^{(s)})_{n+1}$ be described, and let the function represented by the totality of these curves be denoted by $f_{n+1}(x)$. Then the function

$$F_{n+1}(x) = F_n(x) + f_{n+1}(x)$$

has a new maximum, and a new minimum, in every interval $(c_n^{(s-1)}, c_n^{(s)})$, and the length of each of these intervals is less than $1/2^{n+1}$.

If this law of generation of the functions $f_n(x)$ be employed indefinitely, we have a series

$$F_0(x) + f_1(x) + f_2(x) + \dots + f_n(x) + \dots;$$

and it will be shewn that this series represents a continuous function which is everywhere differentiable, and which has an everywhere-dense set of maxima and minima.

277. Let $F_0'(x) + f_1'(x) + f_2'(x) + \dots + f_n'(x) = S_n(x)$; it will then be shewn that, for every value of n and x, $S_n(x)$ is numerically less than $\prod_{n=1}^{\infty}\left(1+\frac{1}{2^n}\right)$, which may be denoted by P. Let us assume that $|S_n(x)|$ is, for every value of x, less than $\prod_1^n\left(1+\frac{1}{2^n}\right)$, which may be denoted by P_n: it will then be shewn that $|S_{n+1}(x)| < P_{n+1}$.

Let the point x be in the interval $(c_n^{(s-1)}, c_n^{(s)})$, where $x < c_n^{(s)}$, the number s depending upon the value of x; we have then, in accordance with the construction of the functions,

$$S_{n+1}(x) = S_n(x) + \alpha_n \frac{a_n^{(s)}}{2^{n+1}},$$

where

$$1 \geqq \alpha_n \geqq -(2^{n+1}+1).$$

In the interval $(c_n^{(s-1)}, c_n^{(s)})$, $S_n(x)$ has a fixed sign, the same as that of $a_n^{(s)}$, but this is not the case for $S_{n+1}(x)$. If α_n is positive, we have

$$|S_{n+1}(x)| < P_n\left(1+\frac{1}{2^n}\right) < P_{n+1}.$$

If α_n is negative, we have

$$|S_{n+1}(x)| < |S_n(x)| < P_n < P_{n+1};$$

it has thus been shewn that if $|S_n(x)| < P_n$, then also $|S_{n+1}(x)| < P_{n+1}$. Now $|F_1'(x)|$ is, everywhere in $(0, 1)$, less than $(1+\frac{1}{2})$, and therefore the theorem $|S_n(x)| < P_n$ follows by induction. *A fortiori* $|S_n(x)|$ is, for every value of n and x, $< P$.

The numerically greatest value of $f_{n+1}(x)$ in the interval $(c_n^{(s-1)}, c_n^{(s)})$ is at some point on the left of the middle point of the interval, and that value is consequently $< \frac{1}{2}\cdot\frac{1}{2^{n+1}}\cdot\frac{a_n^{(s)}}{2^{n+1}}$, since the length of the interval is less than $1/2^{n+1}$. Also, as has been shewn above, $a_n^{(s)} < P$, and therefore

$$|f_{n+1}(x)| < \frac{P}{2^{2n+3}};$$

and hence, since the terms of the series $f_1(x) + f_2(x) + \dots$ are numerically less than the corresponding terms of the absolutely convergent series

$$\frac{P}{2^5} + \frac{P}{2^7} + \dots + \frac{P}{2^{2n+3}} + \dots,$$

it follows that the series $f_1(x) + f_2(x) + \dots$ is uniformly convergent in the interval $(0, 1)$. It follows that the function $F(x)$, defined as the sum-function of the series $F_0(x) + f_1(x) + f_2(x) + \dots$, is a continuous function.

In order to prove that the function $F(x)$ is everywhere differentiable, we shall shew that it satisfies the conditions stated in the theorem of § 235.

We have first of all to shew that the series $f_1'(x) + f_2'(x) + \dots$ is convergent for all values of x in $(0, 1)$. In case, for any value of x, all the numbers $S_n(x), S_{n+1}(x), \dots$, from and after some value of n, have all the same sign, say the positive sign, we have

$$S_{m+1}(x) \leqq S_m(x) + \frac{a_m^{(s_1)}}{2^{m+1}},$$

where m is the value of n in question. Also

$$S_{m+2}(x) \leqq S_{m+1}(x) + \frac{a_{m+1}^{(s_2)}}{2^{m+2}},$$

with similar inequalities involving higher indices. From these inequalities, we find

$$S_{m+p}(x) - S_m(x) \leqq \frac{a_m^{(s_1)}}{2^{m+1}} + \frac{a_{m+1}^{(s_2)}}{2^{m+2}} + \dots + \frac{a_{m+p-1}^{(s_p)}}{2^{m+p}} \leqq \frac{P}{2^m};$$

and since m may be taken so great that $P/2^m$ is arbitrarily small, we see that m may be so chosen that $S_{m+p} - S_m(x)$ is arbitrarily small, whatever positive integral value p may have. It has thus been shewn that, in the case considered, the series is convergent.

It may happen that $S_n(x)$ is zero, owing to x being at a point of division $a_n^{(s)}$; in this case all the functions $f_n'(x)$ with higher indices vanish, and therefore all the functions $S_n(x)$ vanish, from and after the particular value of n. It may happen that $S_n(x)$ vanishes, owing to x being a point of invariability of $F_n(x)$; in this case $S_{n+1}(x)$ may vanish if x is an extreme of $f_{n+1}(x)$, and then x is a point of division $a_{n+1}^{(s)}$, and all the functions $S_m(x)$ for indices $m > n$ vanish. Thus if, for any value of x, $S_n(x)$, $S_{n+1}(x)$ both vanish, then $S_m(x)$ vanishes for all values of $m \geqq n$. If $S_n(x)$ vanishes, but not $S_{n-1}(x)$ or $S_{n+1}(x)$, x is a point of invariability of $F_n(x)$, and

$$S_{n+1}(x) \leqq \frac{1}{2^n}\left(1 + \frac{1}{2^{n+1}}\right) < \frac{2P}{2^{n+1}},$$

and the same reasoning is applicable as before. Let us next suppose that the functions $S_n(x)$ are never all of the same sign, from and after any value n, and that for some values of n they vanish; let $n_1, n_2, \dots$ be the values of n for which $S_n(x)$ has a change of sign, for example, let $S_{n_1}(x)$ be negative or zero, and $S_{n_1+1}(x)$ be positive, and $S_{n_2}(x)$ positive or zero, and $S_{n_2+1}(x)$ negative, and so on. If $S_{n_1}(x)$ is negative, we have

$$S_{n_1+1}(x) = S_{n_1}(x) + \frac{a_{n_1}^{(s_1)}}{2^{n_1+1}}\alpha_{n_1},$$

where $$1 \geqq \alpha_{n_1} \geqq -(2^{n_1+1} + 1),$$

and since α_{n_1} is negative, we have

$$S_{n_1+1}(x) < \frac{1}{2^{n_1}} + \frac{P}{2^{n_1+1}} < \frac{P}{2^{n_1}},$$

account being taken of the fact that the fluctuation of $F_{n_1}'(x)$ in the interval in which x lies is $\leqq \frac{1}{2^{n_1}}$. If $S_{n_1}(x)$ is zero, so that x is a point of invariability of $F_{n_1}(x)$, we have

$$S_{n_1+1}(x) \leqq \frac{1}{2^{n_1+1}}\left(1 + \frac{1}{2^{n_1+1}}\right) < \frac{P}{2^{n_1}}.$$

In any case we find that

$$S_{n_1+p}(x) < S_{n_1+1}(x) + \frac{P}{2^{n_1+1}} < \frac{P}{2^{n_1}} + \frac{P}{2^{n_1+1}},$$

where
$$p = 1, 2, \ldots n_2 - n_1.$$

Similarly, we find that

$$|S_{n_2+1}(x)| < \frac{P}{2^{n_2}}, \text{ if } S_{n_2}(x) = 0;$$

and if $S_{n_2}(x) > 0$, we have

$$|S_{n_2+p}(x)| < |S_{n_2+1}(x)| + \frac{P}{2^{n_2+1}} < \frac{P}{2^{n_2}} + \frac{P}{2^{n_2+1}},$$

for
$$p = 1, 2, 3, \ldots n_3 - n_2.$$

It is seen from these results that $|S_n(x)|$ becomes arbitrarily small for all sufficiently great values of n, and thus $\lim_{n=\infty} S_n(x) = 0$. It has now been shewn that in every case the series

$$F_0'(x) + f_1'(x) + f_2'(x) + \ldots$$

converges for each value of x in the interval $(0, 1)$.

278. It must next be proved that, if ϵ be an arbitrarily chosen positive number, then, for a given x, a number $\delta > 0$ can be found, such that, for each value of h numerically less than δ, and for which $x + h$ is in the interval $(0, 1)$, there exists an integer m, variable with h, and not less than a prescribed integer m', such that the three numbers

$$\frac{F_m(x+h) - F_m(x)}{h} - S_n(x), \quad \frac{R_m(x+h)}{h}, \quad \frac{R_m(x)}{h}$$

are all numerically less than ϵ; $R_m(x)$ denoting the remainder of the series which represents $F(x)$, that is, $F(x) - F_{m-1}(x)$.

The case may be left out of account in which x coincides with one of the points of division of $(0, 1)$; for the function $F(x)$ is then represented by a finite series, and is differentiable, since $f'_{n+p}(c_n^{(s)}) = 0$, for $p > 1$.

Let ϵ, m' be fixed, and let us consider a point x in $(0, 1)$; then a number $n \geqq m'$ can be so determined that

$$\frac{P}{2^{n-2}} < \frac{1}{3}\epsilon, \text{ and } |S_{n+p}(x) - S_{n+q}(x)| < \frac{1}{3}\epsilon,$$

where p, q are any positive integers. For any value of h, such that $x+h$ falls within the interval $(c_n^{(s-1)}, c_n^{(s)})$, the number m can be determined. Let h be positive, and determine n_1 so that $x < c_{n_1+1}^{(s')} \leqq x+h \leqq c_{n_1}^{(s'')} \leqq c_n^{(s)}$; then it can be shewn that n_1+2 is a suitable value for m. We have

$$f_{n_1+1+p}(c_{n_1+1}^{(s)}) = 0;$$

and
$$|f_{n_1+1+p}(c_{n_1+1}^{(s')} \pm k)| < \frac{P}{2^{n_1+1+p}} k, \text{ for } p \geqq 1.$$

The point $c_{n_1+1}^{(s')}$ is in general between x and $x+h$, and therefore it determines two segments, k_1, k_2, where

$$x = c_{n_1+1}^{(s')} - k_1, \quad x+h = c_{n_1+1}^{(s')} + k_2.$$

We have therefore

$$|f_{n_1+1+1}(x)| < \frac{P}{2^{n_1+1+1}} k_1, \quad |f_{n_1+1+2}(x)| < \frac{P}{2^{n_1+1+2}} k_1,$$

and so on; and from these inequalities we find that

$$|R_{n_1+2}(x)| < Pk_1\left\{\frac{1}{2^{n_1+2}} + \frac{1}{2^{n_1+3}} + \dots\right\} < \frac{Pk_1}{2^{n_1+1}},$$

and similarly that

$$|R_{n_1+2}(x+h)| < \frac{Pk_2}{2^{n_1+1}}.$$

Since k_1, k_2 are less than h, we have

$$\left|\frac{R_{n_1+2}(x)}{h}\right| < \frac{P}{2^{n_1+1}} < \epsilon, \text{ and } \left|\frac{R_{n_1+2}(x+h)}{h}\right| < \frac{P}{2^{n_1+1}} < \epsilon.$$

It has thus been shewn that $m = n_1+2$ is a value of m which satisfies the required condition. The case in which h is negative can be treated in the same manner.

We have now to prove that

$$\left|\frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} - S_{n+2}(x)\right| < \epsilon.$$

We see that

$$\frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} - S_{n+2}(x)$$
$$= \left\{\frac{F_{n_1}(x+h) - F_{n_1}(x)}{h} - S_{n_1+1}(x)\right\} + \left\{\frac{f_{n_1+1}(x+h) - f_{n_1}(x)}{h} - f'_{n_1+1}(x)\right\};$$

and if x, $x+h$ are points in $(c_{n_1}^{(s-1)}, c_{n_1}^{(s)})$, the absolute value of the first term on the right-hand side is not greater than $1/2^{n_1}$.

We consider therefore

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} - f'_{n_1+1}(x).$$

From the construction for $f_{n_1+1}(x)$, we have

$$\frac{f_{n_1+1}(x+h) - f_{n_1}(x)}{h} \leq \frac{a_{n_1}^{(s)}}{2^{n_1+1}},$$

since x, $x+h$ are in the interval $(c_{n_1}^{(s-1)}, c_{n_1}^{(s)})$. Let us take the case in which $F_{n_1}(x)$ increases from $c_{n_1}^{(s-1)}$ to $c_{n_1}^{(s)}$; then, for any point x in the interval between these two points, we have

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \leq \frac{a_{n_1}^{(s)}}{2^{n_1+1}}.$$

We shall find also a lower limit for this incrementary ratio. The point x is such that the ordinate of $a_{n_1}^{(s)}(c_{n_1}^{(s-1)} \mid c_{n_1}^{(s)})_{n_1+1}$ is below the x-axis, and if, for that point, the differential coefficient is negative, we have

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \geqq f'_{n_1+1}(x).$$

Let the sides of the rectilinear polygon which was employed in the construction of $a_{n_1}^{(s)}(c_{n_1}^{(s-1)} \mid c_{n_1}^{(s)})_{n_1+1}$ be denoted by

$$r_0', r_1', \ldots r'_{2^{n_1+1}-1}, r'_{2^{n_1+1}+1} \cdot s'_{2^{n_1+1}-1} \ldots s_2', s_1', s_0',$$

where r_m' is equal and parallel to s_m'. On r_2', produced beyond (r_1', r_2'), take a segment equal to r_2'; then this segment is equal and parallel to s_2', and the line joining the end of this segment with (s_2', s_3') is parallel to r_3, and will cut r_1' in a point p_1. But s_3' is parallel to r_2, and passes through (s_2', s_3'); therefore this segment is the prolongation of s_3', and is consequently inclined to the x-axis at an angle whose tangent is $-3\frac{a_{n_1}^{(s)}}{2^{n_1+1}}$. Hence, for a point between $c_{n_1}^{(s)}$ and p_1, for which the ordinate is positive, we have

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > -3\frac{a_{n_1}^{(s)}}{2^{n_1+1}}.$$

But the greatest value of $f'_{n_1+1}(x)$, in this case, is $\frac{a_{n_1}^{(s)}}{2^{n_1+1}}$; and therefore

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > f'_{n_1+1}(x) - 4\frac{a_{n_1}^{(s)}}{2^{n_1+1}}.$$

If a point p_2 on r_2' be determined, by making a similar construction with r_3' instead of r_2', then, for every point on the arc p_1, p_2, except p_2,

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > -4\frac{a_{n_1}^{(s)}}{2^{n_1+1}}.$$

But the maximum value of the differential coefficient is, in this case, $-\frac{a_{n_1}^{(s)}}{2^{n_1+1}}$; therefore also in this case,

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > f'_{n_1+1}(x) - 4\frac{a_{n_1}^{(s)}}{2^{n_1+1}}.$$

This condition holds for every point on the curve which has a positive ordinate. It holds also for points with a negative ordinate; because for such points with a negative differential coefficient the relation

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \geqq f'_{n_1+1}(x)$$

holds; and for points where the differential coefficient is positive, the expression on the left-hand is positive, and that on the right-hand is negative.

It has now been established that

$$f'_{n_1+1}(x) - 4\frac{a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \leqq \frac{a_{n_1}^{(s)}}{2^{n_1+1}},$$

and it has already been proved that

$$\frac{F_{n_1}(x+h) - F_{n_1}(x)}{h} = F'_{n_1}(x) + \frac{\theta}{2^{n_1}}, \text{ where } 1 \geqq \theta \geqq -1.$$

We now see that

$$F'_{n_1+1}(x) + \frac{\theta}{2^{n_1}} - 4\frac{a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{F_{n_1+1}(x+h) - F_{n_1}(x)}{h} \leqq F'_{n_1}(x) + \frac{\theta}{2^{n_1}} + \frac{a_{n_1}^{(s)}}{2^{n_1+1}},$$

and hence $\left|\frac{F_{n_1+1}(x+h) - F_{n_1}(x)}{h} - S_{n_1+2}(x)\right| < \epsilon,$

since $a_{n_1}^{(s)} < P$, and $P/2^{n_1-2} < \frac{1}{3}\epsilon$, and $|\theta/2^{n_1}| < P/2^{n_1} < \frac{2}{3}\epsilon$.

It has now been established that the function $F(x)$ has at every point a finite differential coefficient which is the sum of the convergent series

$$F_0'(x) + f_1'(x) + f_2'(x) + \dots.$$

Lastly, it must be proved that $F(x)$ has an everywhere-dense set of maxima and minima.

It has been shewn that, in every interval $(c_{n-1}^{(s-1)}, c_{n-1}^{(s)})$, the function $F_n(x)$ has a new maximum and a new minimum, and that the length of the interval is less than $1/2^n$. If x_0 is a maximum of $F_n(x)$, we have

$$F_n(x_0) = F_{n+1}(x_0), \text{ and } F_n'(x_0) = F'_{n+1}(x_0) = 0.$$

Moreover $f_{n+1}(x)$ is negative in the neighbourhood of the point x_0, and therefore $F_{n+1}(x_0 + h) - F_{n+1}(x_0)$ is negative or zero, provided $|h|$ is less than some number k. It thus appears that $F_{n+1}(x)$ has also a maximum at x_0. If x_0 is a point of invariability of $F_n(x)$, it is no longer one for $F_{n+1}(x)$, and cannot be a point of invariability of all the functions with higher indices. If x_0 is a limiting point of a line of invariability, $F_{n+1}(x)$ will have a maximum or a minimum, or else a point of inflexion at x_0. In every case $F_{n+1}(x)$ will have a maximum and a minimum in every line of invariability

of $F_n(x)$. For any given interval, as small as we please, n can be determined so great that the interval contains one of the intervals $(c_{n-1}^{(s-1)}, c_{n-1}^{(s)})$ in its interior, and all the functions $F_n(x)$, $F_{n+1}(x)$, ... have maxima in this interval; and it follows that $F(x)$ also has maxima therein.

It may be remarked that $F'(x)$, although definite at every point, has discontinuities of the second kind at an everywhere-dense set of points. At every point of continuity, this differential coefficient must vanish (see I, § 285). The function $F'(x)$ is not integrable in accordance with Riemann's definition.

CHAPTER VII

THE REPRESENTATION OF FUNCTIONS AS LIMITS OF INTEGRALS

THE GENERAL CONVERGENCE THEOREM

279. In the theory of the representation of a function $f(x)$ as the sum of a series of some special type the method of procedure usually consists of the partial summation of the series; the partial sum being expressed as an integral which involves the number n of terms of the series as a parameter, followed by the determination of the nature of the limit of the integral as n is indefinitely increased.

The general theory of the evaluation of a limit of the form

$$\lim_{n \sim \infty} \int_a^b f(x') \, \Phi(x', x, n) \, dx',$$

or more particularly of the form

$$\lim_{n \sim \infty} \int_a^b f(x') \, \Phi(x' - x, n) \, dx',$$

is, in its modern form, due to the investigations of Hobson* and of Lebesgue†, but an earlier theory, of a less general character, was given by Du Bois-Reymond‡ and Dini§. Further developments have been given|| by Hahn.

In this chapter the two investigations are welded together into a unified form, with a view to the attainment of the greatest possible degree of generality. The theory is in part extended to cover the case of functions of any number of variables, and to the case in which the function of a single variable is non-summable, but has either a D-integral or an HL-integral.

The following theorem, which may be referred to as the general convergence theorem, together with specializations and generalizations of it, is of fundamental importance in this connection:

THEOREM I. *Let $f(x')$ be a bounded or unbounded function, summable in the interval (a, b) of the variable x'. Let $\Phi(x', x, n)$ be a function defined for all values of x' in the interval (a, b), for all values of n in a sequence of increasing numbers without an upper limit (in particular the sequence of integers), and for all values of x in some set of points G. Further let $\Phi(x', x, n)$ satisfy the following conditions:*

* *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 349, and (2), vol. XII (1912), p. 166.

† *Annales de Toulouse* (3), vol. I (1909), p. 25.

‡ *Crelle's Journal*, vol. LXIX (1868), p. 93, and vol. LXXIX (1875), p. 38.

§ *Serie di Fourier*, Pisa (1880).

|| *Denkschr. d. Wiener Akad.* vol. XCIII (1916), pp. 585, 657; also *Wiener Ber.* vol. CXXVII (1918), p. 1763.

(1) *For each pair of numbers x, n for which $\Phi(x', x, n)$ is defined, that function of x' is equivalent* (see I, § 394) *to a function which does not exceed in absolute value a fixed number K, independent of the particular values of x and n. The trivial case in which, for a finite set of values of n, this condition is not satisfied may clearly be disregarded, since such values of n may be removed from the sequence.*

(2) *For each pair of values of α and β, such that $a \leqq \alpha \leqq \beta \leqq b$,*

$$\int_\alpha^\beta \Phi(x', x, n)\, dx'$$

exists as an L-integral, for each pair of values of n and x (in G), and it converges to zero, uniformly for all values of x in G, as $n \sim \infty$.

Then $\int_a^b f(x')\,\Phi(x', x, n)\, dx'$ converges to zero as $n \sim \infty$, uniformly for all values of x in G.

It is clear that there will be no loss of generality if the condition $|\Phi(x', x, n)| \leqq K$ is taken to hold for all the values of x', x, n without exception.

It should be observed that, in case $\Phi(x', x, n) \geqq 0$, for all values of x', x, and n, the condition (2) may be replaced by the condition that $\int_a^b \Phi(x', x, n)\, dx'$ should converge to zero, as $n \sim \infty$, uniformly for all values of x in G.

To prove the theorem, we observe that, in accordance with the theorem in I, § 430, a continuous function $\phi(x')$, defined in (a, b), can be so determined that $\int_a^b |f(x') - \phi(x')|\, dx' < \frac{\epsilon}{K}$; where ϵ is a prescribed positive number. The interval (a, b) may be divided into a number of parts (a, α_1), (α_1, α_2), … (α_{r-1}, b), so chosen that the fluctuation of $\phi(x')$ in each of these parts is less than $\frac{\epsilon}{K(b-a)}$. Let $\psi(x')$ be a function which, in the interior of each part (α_{s-1}, α_s), where $s = 1, 2, 3, \dots r$, has the constant value $c_s \equiv \phi\left(\frac{\alpha_s + \alpha_{s-1}}{2}\right)$. At the extremities of the parts we may take $\psi(x')$ to have the value zero. Thus $\psi(x')$ has the finite set of values $c_1, c_2, \dots c_r, 0$.

Since $|\phi(x') - \psi(x')| < \frac{\epsilon}{K(b-a)}$, everywhere except at the end-points of the r parts of (a, b), we have

$$\int_a^b |\phi(x') - \psi(x')|\, dx' < \frac{\epsilon}{K};$$

and therefore
$$\int_a^b |f(x') - \psi(x')|\, dx' < \frac{2\epsilon}{K}.$$

The integral $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ may be expressed by

$$\int_a^b \{f(x') - \psi(x')\}\,\Phi(x', x, n)\,dx' + \sum_{s=1}^{s=r} c_s \int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'.$$

Hence we have

$$\left|\int_a^b f(x')\,\Phi(x', x, n)\,dx'\right| < 2\epsilon + \sum_{s=1}^{s=r} |c_s| \left|\int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'\right|.$$

From the condition (2), of the theorem, a number n_ϵ, belonging to the sequence of values of n, can be so determined that

$$\left|\int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'\right| < \frac{\epsilon}{\sum\limits_{s=1}^{s=r} |c_s|}, \text{ for } s = 1, 2, 3, \dots r;$$

and for all values of x in G, provided $n \geqq n_\epsilon$. It now follows that

$$\left|\int_a^b f(x')\,\Phi(x', x, n)\,dx'\right| < 3\epsilon, \text{ provided } n \geqq n_\epsilon,$$

for all values of x in G. Since ϵ can be arbitrarily chosen, the integral $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ has been shewn to converge to zero, as $n \sim \infty$, uniformly for all values of x in G.

An examination of the proof of Theorem I shews that the theorem may be stated more generally. In the first place the point x' may be taken to be a point in a p-dimensional cell $(a^{(1)}, a^{(2)}, \dots a^{(p)};\ b^{(1)}, b^{(2)}, \dots b^{(p)})$ which will replace the linear interval (a, b) of the theorem. The theorem of I, § 430, holds good for a function in a p-dimensional domain, and in the proof, the cell (a, b) will be divided into r parts in each of which the fluctuation of the continuous function $\phi(x')$ is,

$$< \frac{\epsilon}{K(b^{(1)} - a^{(1)})(b^{(2)} - a^{(2)}) \dots (b^{(p)} - a^{(p)})}.$$

Instead of (α, β) a cell $(\alpha^{(1)}, \alpha^{(2)}, \dots \alpha^{(p)};\ \beta^{(1)}, \beta^{(2)}, \dots \beta^{(p)})$ will be employed in the condition (2).

Moreover the set G may be a set of points in any number q, of dimensions. Further the numbers n may be replaced by a set of numbers $n^{(1)}, n^{(2)}, \dots n^{(t)}$, each of which belongs to a sequence with no upper limit. To the number n_ϵ there will correspond a set of numbers $n_\epsilon^{(1)}, n_\epsilon^{(2)}, \dots n_\epsilon^{(t)}$, such that all the integrals

$$\left|\int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'\right| < \frac{\epsilon}{\sum\limits_{s=1}^{s=r} |c_s|},$$

provided $$n^{(1)} \geqq n_\epsilon^{(1)},\ n^{(2)} \geqq n_\epsilon^{(2)}, \dots n^{(t)} \geqq n_\epsilon^{(t)}.$$

The single limit, as $n \sim \infty$, will thus be replaced by a t-ple limit, as all the numbers $n^{(1)}, n^{(2)}, \dots n^{(t)}$ diverge to ∞.

For example, if x' is in a two-dimensional cell, and n is replaced by two numbers $n^{(1)}$, $n^{(2)}$, the theorem states that

$$\lim_{\substack{n^{(1)}\sim\infty \\ n^{(2)}\sim\infty}} \int_{(a^{(1)},\, a^{(2)})}^{(b^{(1)},\, b^{(2)})} f(x^{(1)\prime}, x^{(2)\prime})\, \Phi(x^{(1)\prime}, x^{(2)\prime}, x, n^{(1)}, n^{(2)})\, d(x^{(1)\prime}, x^{(2)\prime}) = 0,$$

and that the convergence is uniform for all points x in the given set G.

It should also be observed that it is possible to extend the theorem so that n may be taken to be a continuous variable which diverges to ∞, such that $A \leqq n$, where A is some fixed number, provided the conditions (1) and (2) of the theorem are satisfied in such a domain of n. Also n may consist of a group $n^{(1)}, n^{(2)}, \ldots n^{(t)}$, of such continuous variables, each diverging to ∞. In this connection the remarks made in I, § 211, on the relation of the two definitions, by Cauchy and Heine, of continuity of a function at a point are relevant.

It is clear that, instead of the interval or cell (a, b), any bounded measurable set may be considered. For, if $f(x')$ is defined in such a set E, by taking an interval or cell (a, b) which contains E, we may assume $f(x') = 0$ in the complement of E with respect to (a, b); then the theorem can be stated for the integral $\int_{(E)} f(x')\, \Phi(x', x, n)\, dx'$.

A generalization of Theorem I may be obtained by supposing that the condition (2) is modified as follows:

(2*) *For each pair of values of α and β, such that $a \leqq \alpha \leqq \beta \leqq b$,*

$$\int_\alpha^\beta \Phi(x', x, n)\, dx'$$

exists as an L-integral, for each pair of values of n and x (in G), and it converges for each value of x, in G, to zero, as $n \sim \infty$.

It will be observed, that, on account of the condition (1), the convergence is necessarily bounded. This condition is that $\left|\int_\alpha^\beta \Phi(x', x, n)\, dx'\right|$ is less than some fixed number independent of x and n, and that for each value of x it converges to zero, as $n \sim \infty$.

The condition (2*) is then less restrictive than the corresponding condition (2), in which uniform convergence is postulated. The result of the theorem when (2*) is introduced instead of (2) will be that

$$\int_a^b f(x')\, \Phi(x', x, n)\, dx'$$

converges boundedly to zero, as $n \sim \infty$, for the values of x in G.

Only a slight modification of the proof is necessary to make the extension. In the first place the proof as it stands may be employed to shew

that, for each single point x, of G, the convergence takes place. To shew that the convergence is bounded, we have only to consider the inequality

$$\left| \int_a^b f(x') \Phi(x', x, n)\, dx' \right| < 2\epsilon + \sum_{s=1}^{s=r} |c_s| \left| \int_{a_{s-1}}^{a_s} \Phi(x', x, n)\, dx' \right|,$$

which shews that, subject to (2*), the condition of boundedness is satisfied.

280. Theorem I is valid when the interval (a, b) is indefinitely great, provided the condition (1) holds in the indefinite interval, and (2) holds for every finite interval, and provided further that $f(x')$ is absolutely summable in $(-\infty, \infty)$; that is, $\lim\limits_{\substack{\beta \sim \infty \\ \alpha \sim -\infty}} \int_\alpha^\beta |f(x')|\, dx'$ exists.

For all values of β' $(> \beta)$ we have

$$\left| \int_\beta^{\beta'} f(x') \Phi(x', x, n)\, dx' \right| < K \int_\beta^{\beta'} |f(x')|\, dx'.$$

Since β can be so chosen that the integral on the right hand is less than ϵ/K, we have, for all values of β' $(> \beta)$, $\left| \int_\beta^{\beta'} f(x') \Phi(x', x, n)\, dx' \right| < \epsilon$, for all values of x and n. Similarly α may be so chosen that

$$\left| \int_{\alpha'}^{\alpha} f(x') \Phi(x', x, n)\, dx' \right| < \epsilon, \text{ for } \alpha' < \alpha,$$

and for all values of x and n. We now have, if $\beta' > \beta > \alpha > \alpha'$,

$$\left| \int_\alpha^{\beta'} f(x') \Phi(x', x, n)\, dx' \right| < 3\epsilon;$$

for all values of n not less than a fixed value n_ϵ, and for all values of x in G, since the Theorem I holds for the interval (α, β).

It follows that $\left| \int_{-\infty}^{\infty} f(x') \Phi(x', x, n)\, dx' \right| < 3\epsilon$, for $n \geqq n_\epsilon$, and x in G. Therefore $\int_{\infty}^{x} f(x') \Phi(x', x, n)\, dx'$ converges uniformly to zero, as $n \sim \infty$, for all values of x in G. The case in which the condition (2*) is employed instead of (2) leads, by a slight modification of the foregoing proof, to the corresponding extension of the theorem.

In case $\Phi(x', x, n)$ is non-negative for all values of x', x, and n, and provided $\int_{\infty}^{\infty} \Phi(x', x, n)\, dx'$ exists and converges to zero uniformly for all x in G, it is not necessary that $f(x')$ should be absolutely summable in $(-\infty, \infty)$. It is sufficient that, outside some finite interval (A, B), $|f(x')|$ be bounded (say $< U$), and $f(x')$ be summable in every finite interval. For

$$\left| \int_\beta^{\beta'} f(x') \Phi(x', x, n)\, dx' \right| < U \int_{-\infty}^{\infty} \Phi(x', x, n)\, dx'$$

provided $\beta' > \beta > B$. Hence, if $n \geqq n_\epsilon^{(1)}$, the expression on the left-hand side is $< \epsilon$. Similarly, if the limits of the integral be α', α, where $\alpha' < \alpha \leqq A$,

the absolute value of the integral is less than ϵ, provided $n \geqq n_\epsilon^{(2)}$. If n_ϵ be the greater of the two numbers $n_\epsilon^{(1)}, n_\epsilon^{(2)}$, both the integrals are numerically less than ϵ, provided $n \geqq n_\epsilon$. As before it follows that

$$\int_{-\infty}^{\infty} f(x')\,\Phi(x', x, n)\,dx' \sim 0,$$

as $n \sim \infty$, uniformly for all values of x in G. In case $\int_{-\infty}^{\infty} \Phi(x', x, n)\,dx'$ converges boundedly to zero, for all values of x, $\left|\int_{\beta}^{\beta'} f(x')\,\Phi(x', x, n)\,dx'\right|$ is less than a fixed number independent of α, α', n and the corresponding extension of the theorem can be made.

It may happen that, for a particular function $\Phi(x', x, n)$, the condition to which $f(x')$ must be subjected is less stringent in character. If $\chi(x')$ be such that the function $\Phi_1(x', x, n) \equiv \chi(x')\,\Phi(x', x, n)$ satisfies the conditions (1) and (2) of Theorem I, it will be sufficient in order that the interval (a, b) can be taken to be the indefinite interval $(-\infty, \infty)$, that $\int_{-\infty}^{\infty} \left|\frac{f(x')}{\chi(x')}\right| dx'$ have a finite value. In case $\chi(x')\,\Phi(x', x, n) \geqq 0$, it will be sufficient that $\frac{f(x')}{\chi(x')}$ be summable over every finite interval, and that it be bounded for all values of x' outside some interval (A, B).

It is clear that, with the necessary slight changes of statement, all these results are applicable when x' denotes a point in a domain of any number of dimensions.

281. There are cases besides the case considered in § 280, in which $f(x')$ is not necessarily absolutely summable in the interval $(-\infty, \infty)$, in which Theorem I holds for the infinite interval.

Let it be assumed that β may be so chosen that the total variation of $f(x)$ in the interval (β, β'), $V_\beta^{\beta'} f(x)$ is finite, for $\beta' > \beta$, and converges to a finite limit, as $\beta' \sim \infty$; in that case the total variation of $f(x)$ in (β, ∞) is said to be bounded. Let $P(x)$, $-N(x)$ denote the total positive, and negative variations of $f(x)$ in (β, x), then $f(x) = f(\beta) + P(x) - N(x)$, where $P(x) + N(x)$ has a finite limit, as $x \sim \infty$; and consequently $P(x)$, $N(x)$ have finite limits p, v; the limit of $f(x)$ being $f(\beta) + p - v$. We may now write $f(x) = f_1(x) - f_2(x)$, where $f_1(x) \equiv f(\beta) + p - N(x)$, $f_2(x) \equiv p - P(x)$; and thus $f_1(x)$, $f_2(x)$ are monotone non-increasing functions, bounded in (β, ∞). In case $f(x) \sim 0$, as $x \sim \infty$, we have $f(\beta) + p - v = 0$; and both the functions $f_1(x)$, $f_2(x)$ are non-increasing functions which converge to zero, as $x \sim \infty$.

Similar considerations apply to the neighbourhood of the point $x = -\infty$.

Let it now be assumed that α and β can be so chosen that $f(x')$ is of bounded variation in the intervals (β, ∞), $(-\infty, \alpha)$. We have

$$\int_\beta^{\beta'} f_1(x')\,\Phi(x', x, n)\,dx' = f_1(\beta)\int_\beta^{\beta''}\Phi(x', x, n)\,dx' + f_1(\beta')\int_{\beta''}^{\beta'}\Phi(x', x, n)\,dx',$$

where β'' is in the interval (β, β'). Let it now be assumed that

$$\left|\int_\beta^{\beta'}\Phi(x', x, n)\,dx'\right|$$

is, for every value of β' ($>\beta$), and of x, less than some number k_n, which converges to zero, as $n \sim \infty$.

We have then

$$\left|\int_\beta^{\beta'} f_1(x')\,\Phi(x', x, n)\,dx'\right| < k_n\,|f_1(\beta)| + 2k_n\,|f_1(\beta')|;$$
$$< 3k_n\,|f_1(\beta)|;$$

thus the expression on the right-hand side is less than a fixed multiple of k_n, for all values of β'; hence $\left|\int_\beta^\infty f_1(x')\,\Phi(x', x, n)\,dx'\right|$ does not exceed a fixed multiple of k_n; and since the same result holds for $f_2(x)$, we have

$$\lim_{n\sim\infty}\int_\beta^\infty f(x')\,\Phi(x', x, n)\,dx' = 0.$$

The case of the integral over $(-\infty, \alpha)$ may be treated in the same manner. It follows that Theorem I is applicable to the case in which a and b are infinite, when $f(x')$, $\Phi(x', x, n)$ satisfy the specified conditions, provided the conditions (1), (2) are satisfied in the interval (α, β).

Next, let it be assumed that $f(x')$ converges to zero, as $x' \sim \infty$, and as $x' \sim -\infty$, and also that α, β can be so chosen that $f(x')$ is of bounded variation in the intervals (β, ∞), $(-\infty, \alpha)$. In (β, ∞) we have

$$f(x) = f_1(x) - f_2(x),$$

where each of the functions $f_1(x)$, $f_2(x)$ is monotone non-increasing, and converges to zero, as $x \sim \infty$.

We have $\displaystyle\int_\beta^{\beta'} f_1(x')\,\Phi(x', x, n)\,dx' = f_1(\beta)\int_\beta^{\beta''}\Phi(x', x, n)\,dx'$,

where β'' is in the interval (β, β'). Let it now be assumed that

$$\left|\int_\beta^{\beta'}\Phi(x', x, n)\,dx'\right| < k,$$

where k is some positive number, independent of n, x, and β'. Then we have, applying the corresponding result for $f_2(x')$,

$$\left|\int_\beta^\infty f(x')\,\Phi(x', x, n)\,dx'\right| \leqq k\,[f_1(\beta) + f_2(\beta)].$$

Since β may be so chosen that $f_1(\beta)$, $f_2(\beta)$ are both arbitrarily small, we have, for a sufficiently large value of β,

$$\left|\int_\beta^\infty f(x')\,\Phi(x' x, n)\,dx'\right| < \epsilon,$$

for all values of n and x. With the corresponding result for the integral taken over $(-\infty, \alpha)$, it is now seen that the Theorem I holds for the interval $(-\infty, \infty)$, provided the conditions (1) and (2) are satisfied in every finite interval.

The following results have now been obtained:

Theorem I holds for the infinite integral $(-\infty, \infty)$, *when the conditions* (1), (2) *hold for each finite interval* (a, b), *provided also one of the following sets of additional conditions holds;*

(*a*) $f(x')$ *is absolutely summable in* $(-\infty, \infty)$, *and the condition* (1) *holds in that interval.*

(*b*) *Outside some finite interval* $f(x')$ *is bounded, also* $\Phi(x', x, n)$ *is non-negative, and* $\int_{-\infty}^{\infty} \Phi(x', x, n)\,dx'$ *exists, and converges uniformly to zero, as* $n \sim \infty$, *for all values of* x *in* G.

(*c*) *Numbers* α, β *can be so chosen that* $f(x')$ *is of bounded variation in* (β, ∞), *and in* $(-\infty, \alpha)$, *and that*

$$\left|\int_\beta^{\beta'} \Phi(x', x, n)\,dx'\right|, \quad \left|\int_{\alpha'}^{\alpha} \Phi(x', x, n)\,dx'\right|$$

are, for every value of β' $(> \beta)$, *and every value of* α' $(< \alpha)$, *less than a positive number* k_n, *independent of* x, *which converges to zero, as* $n \sim \infty$.

(*d*) $f(x')$ *converges to zero, as* $x' \sim \infty$, *and as* $x' \sim -\infty$, *and numbers* α, β *can be so chosen that* $f(x')$ *is of bounded variation in* (β, ∞), *and in* $(-\infty, \alpha)$, *and also* $\left|\int_\beta^{\beta'} \Phi(x', x, n)\,dx'\right|$, $\left|\int_{\alpha'}^{\alpha} \Phi(x', x, n)\,dx'\right|$ *are both less than some positive number* k, *independent of* n, x, α', β'.

There is another case in which, for an infinite interval, the absolute summability of $f(x')$ can be dispensed with. The following theorem will be established:

If $f(x')$ *be summable, but not necessarily absolutely summable in the infinite interval* (a, ∞), *and if* $\Phi(x', x, n)$ *satisfies the condition* (1) *of Theorem I in* (a, ∞), *and the condition* (2) *in every finite interval, and also the further condition that its total variation in the interval* (a, ∞) *is less than some fixed number* L, *independent of* n *and* x, *then* $\int_a^\infty f(x')\,\Phi(x', x, n)\,dx'$ *converges to zero, as* $n \sim \infty$, *uniformly for all values of* x *in* G.

Let $a < A < A'$, then, from I, § 424, we have

$$\left|\int_A^{A'} f(x')\,\Phi(x', x, n)\,dx' - \Phi(A, x, n)\,dx' \int_A^{A'} f(x')\,dx'\right|$$

$$\leq V_A^{A'}\,\Phi(x', x, n) \times \text{the upper boundary of } \left|\int_\alpha^\beta f(x')\,dx'\right|,$$

where $A \leqq \alpha \leqq \beta \leq A'$. If η be a prescribed positive number, A may be so chosen that $\left|\int_\alpha^\beta f(x')\,dx'\right| < \eta$, for all values of α, β that are not less than A. We thus have $\left|\int_A^{A'} f(x')\,\Phi(x', x, n)\,dx'\right| < (K + L)\,\eta$; and the number η can be so chosen that $(K + L)\,\eta < \epsilon$. We then have

$$\left|\int_a^\infty f(x')\,\Phi(x', x, n)\,dx'\right| < \left|\int_a^A f(x')\,\Phi(x', x, n)\,dx'\right| + \epsilon;$$

and if $n \geqq n_\epsilon$, where n_ϵ is some number belonging to the sequence of values of n,

$$\left|\int_a^\infty f(x')\,\Phi(x', x, n)\,dx'\right| < 2\epsilon, \text{ for } n \geqq n_\epsilon,$$

and for all values of x in G. Thus the theorem has been established.

It is easy to see that the result holds also for

$$\int_{-\infty}^\infty f(x')\,\Phi(x', x, n)\,dx',$$

provided similar conditions are satisfied.

282. In case the function $f(x)$ is such that $|f(x)|^q$ is summable in (a, b), for some value of $q > 1$, the condition (1) in the Theorem I may be replaced by the following less stringent condition:

(1 a) *For each pair of numbers x, n, for which $\Phi(x', x, n)$ is defined, that function of x' is such that $|\Phi(x', x, n)|^{\frac{q}{q-1}}$ is summable, and such that $\int_a^b |\Phi(x', x, n)|^{\frac{q}{q-1}}\,dx'$ does not exceed a fixed number $K^{\frac{q}{q-1}}$, independent of the particular values of x and n.*

The condition (2) will be unchanged.

In accordance with a theorem given in § 173, a continuous function $\phi(x')$ can be so determined that

$$\int_a^b |f(x') - \phi(x')|^q\,dx' < \left(\frac{\epsilon}{K}\right)^q.$$

By applying an inequality given in I, § 435, we see that

$$\left|\int_a^b \{f(x') - \phi(x')\}\,\Phi(x', x, n)\,dx'\right| < \epsilon.$$

If, for every continuous function $\phi(x')$, $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'$ converges to zero, as $n \sim \infty$, uniformly, or more generally boundedly, for all values

of x in G, it is clear, since ϵ is arbitrary, that the theorem holds for $f(x')$. Thus it will be sufficient to consider the integral $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'$, where $\phi(x')$ is continuous in (a, b). As in § 279, a function $\psi(x')$ which has only a finite set of values $c_1, c_2, \ldots c_r, 0$ can be so determined that $|\phi(x') - \psi(x')| < \dfrac{\epsilon}{K(b-a)^{\frac{1}{q}}}$, except at points of a finite set. The integral $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'$ may be expressed by

$$\int_a^b \{\phi(x') - \psi(x')\}\,\Phi(x', x, n)\,dx' + \sum_{s=1}^{s=r} c_s \int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'.$$

The first integral does not in numerical value exceed

$$\left\{\int_a^b |\phi(x') - \psi(x')|^q\,dx'\right\}^{\frac{1}{q}} \cdot \left\{\int_a^b |\Phi(x', x, n)|^{\frac{q}{q-1}}\,dx'\right\}^{\frac{q-1}{q}},$$

and this is less than ϵ. As before, if condition (2) be assumed, the expression $\sum_{s=1}^{s=r} c_s \int_{a_{s-1}}^{a_s} \Phi(x', x, n)\,dx'$ is numerically less than ϵ, for $n \geqq n_\epsilon$, and for all values of x in G. If condition (2*) be assumed, the expression is bounded for all values of n.

Thus $\left|\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'\right| < 2\epsilon$, if $n \geqq n_\epsilon$, and the theorem has been proved for the function $\phi(x')$, and therefore for $f(x')$, in case condition (2) is assumed. In case condition (2*) is taken,

$$\left|\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'\right|$$

is shewn as above to converge to zero for each value of x, and the convergence is bounded for all values of x.

283. In case $f(x)$ is bounded and summable in (a, b), the condition (1) of Theorem I may be replaced by the following condition:

(1 b) $\int_a^b |\Phi(x', x, n)|\,dx'$ *exists, and does not exceed a fixed number* K, *for all values of* n *and* x (*in* G); *and also, for each measurable set* e *contained in* (a, b), $\int_{(e)} \Phi(x', x, n)\,dx'$ *converges to zero, as* $n \sim \infty$, *uniformly, or more generally, boundedly, for all values of* x *in* G.

The condition (2), or (2*), of § 279, is contained in the second condition of the theorem.

If η be an assigned positive number, a function $\phi_\eta(x)$ can be so defined that $|f(x') - \phi_\eta(x')| < \eta$, and that $\phi_\eta(x')$ has only a finite set of values $c_1, c_2, \ldots c_m$ which it takes in measurable sets $e_1, e_2, \ldots e_m$ (see I, § 385).

We have

$$\left|\int_a^b \phi_\eta(x')\,\Phi(x', x, n)\,dx'\right| \leqq \sum_{r=1}^{r=m} |c_r| \left|\int_{(e_r)} \Phi(x', x, n)\,dx'\right|;$$

also, provided $n \geqq n_\epsilon$, some number dependent on ϵ,

$$\left|\int_{(e_r)} \Phi(x', x, n)\,dx'\right| < \frac{\epsilon}{\sum\limits_{r=1}^{r=m} |c_r|};$$

and thus we have

$$\left|\int_a^b \phi_\eta(x')\,\Phi(x', x, n)\,dx'\right| < \epsilon, \text{ for } n \geqq n_\epsilon,$$

and for all values of x in G.

Again $\left|\int_a^b \{f(x') - \phi_\eta(x')\}\,\Phi(x', x, n)\,dx'\right| < \eta \int_a^b |\Phi(x', x, n)|\,dx' < \eta K.$

It follows that

$$\left|\int_a^b f(x')\,\Phi(x', x, n)\,dx'\right| < \eta K + \epsilon, \text{ for } n \geqq n_\epsilon,$$

and for all values of x in G, in case condition (2) holds; and accordingly the proposition is established, since η and ϵ are arbitrary.

If condition (2*) holds, it is seen that the convergence is bounded.

284. In case $f(x')$ have only ordinary discontinuities in (a, b), the condition (1) of Theorem I can be replaced by the following:

(1 *c*) $\int_a^b |\Phi(x', x, n)|\,dx'$ *is less than some fixed number* K, *independent of* n *and* x (*in* G).

The condition (2) or (2*) will be unchanged.

If k be a positive number, the set of points of (a, b) at which the saltus of $f(x')$ is $\geqq k$ is finite (see I, § 239). This finite set of points divides (a, b) into a finite number of parts; if (α, β) be one of these parts, it may be divided into a finite number of smaller parts in each of which the fluctuation of the function that has the values of $f(x')$ at all interior points of (α, β), and has the values $f(\alpha + 0)$, $f(\beta - 0)$ at α and β respectively has a value $< k_1$, where k_1 is a number chosen to be $> k$. The whole interval (a, b) can accordingly be divided into a number of parts such that the inner fluctuation of $f(x')$ in each one of these parts is $< k_1$. Let $\phi(x')$ have in all the interior points of each one of these parts the value of $f(x')$ at the centre of the part, then $|f(x') - \phi(x')| < k_1$, except at the points of division of (a, b). In these end-points we may take $\phi(x') = 0$; thus $\phi(x')$ has only a finite set of values.

We have $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx' = \Sigma c_m \int_{a_m}^{b_m} \Phi(x', x, n)\,dx'$,

where c_m is the value of $\phi(x')$ at $x' = \frac{1}{2}(a_m + b_m)$, and m has only a finite set of values.

It follows from condition (2) of Theorem I that $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'$ converges to zero, as $n \sim \infty$, uniformly for all points x in G. From condition (2*) it follows that $\int_a^b \phi(x')\,\Phi(x', x, n)\,dx'$ is bounded for all values of x, and n converges to zero for each value of x.

Again $\int_a^b \{f(x') - \phi(x')\}\,\Phi(x', x, n)\,dx' < k_1 \int_a^b |\,\Phi(x', x, n)\,|\,dx' < k_1 K$ for all values of x and n. Therefore

$$\left|\overline{\lim_{n\sim\infty}} \int_a^b f(x')\,\Phi(x', x, n)\,dx'\right| \leqq k_1 K.$$

Since k and k_1 are arbitrarily small, we see that if condition (2) is assumed to hold $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ converges to zero as $n \sim \infty$, uniformly for all values of x in G.

If condition (2*) holds, the convergence for each value of x is established as above, and it is seen that the convergence is bounded.

285. In case $f(x')$ is of bounded variation in (a, b), the condition (1) of Theorem I may be replaced by the following:

(1 d) $\left|\int_\alpha^\beta \Phi(x', x, n)\,dx'\right|$ *does not exceed a finite number* M, *independent of* α, β, n, *and* x (*in* G); *where* (α, β) *is in* (a, b).

The condition (2) or (2*) will be unchanged.

Since every function of bounded variation is the difference of two monotone functions, it is clearly sufficient to consider the case in which $f(x')$ is monotone in (a, b).

It has been shewn in I, § 249, that $f(x') = \phi(x') + s(x')$, where $\phi(x')$ is continuous and monotone, and $s(x')$ is the limit of a sequence $s_r(x')$, such that the total variation of $s(x') - s_r(x')$ in (a, b) diminishes indefinitely as r increases. Moreover $s_r(x')$ is constant in each interval of a finite set into which (a, b) is divided; also $s(a) = s_r(a) = 0$.

Employing the theorem given in I, § 424, we see that

$$\left|\int_a^b \{s(x') - s_r(x')\}\,\Phi(x', x, n)\,dx'\right| < M V_a^b \{s(x') - s_r(x')\}.$$

Also $\lim_{n\sim\infty} \int_a^b s_r(x')\,\Phi(x', x, n)\,dx' = 0$, the convergence being uniform with respect to n; since the expression is the finite sum of multiples of the integral of $\Phi(x', x, n)$ taken through intervals contained in (a, b).

It follows that $\int_a^b s(x')\,\Phi(x', x, n)\,dx'$ converges uniformly to zero, as $n \sim \infty$. We have accordingly to prove the theorem for the continuous monotone function $\phi(x')$.

First, if $\phi(x')$ is an indefinite integral $\int_a^{x'} \chi(x')\,dx' + A$, we need only consider $\int_a^{x'} \chi(x')\,dx'$. By integration by parts we have

$$\int_a^b \phi(x')\,\Phi(x', x, n)\,dx' = \int_a^b \chi(x')\,dx' \,.\int_a^b \Phi(x', x, n)\,dx'$$
$$- \int_a^b \chi(x') \left[\int_a^{x'} \Phi(\xi, x, n)\,d\xi\right] dx'.$$

The first term on the right-hand side converges to zero, as $n \sim \infty$, uniformly for all the values of x, in case condition (2) holds; and it converges boundedly if condition (2*) holds.

Moreover $\left|\chi(x') \int_a^{x'} \Phi(\xi, x, n)\,d\xi\right|$ is less than the summable function $M\chi(x')$ and it converges to zero, as $n \sim \infty$, hence, by the theorem of § 203, relating to integrable sequences which involve a parameter, we have

$$\lim_{n \sim \infty} \int_a^b \chi(x') \left[\int_a^{x'} \Phi(\xi, x, n)\,d\xi\right] dx' = 0,$$

the convergence being uniform for all values of x in G.

If $\phi(x')$ is not an indefinite integral, a new variable t can be so chosen that $x' = \psi(t)$, $y' = \phi(x') = \phi\{\psi(t)\}$, and the function $\psi(t)$ is monotone non-diminishing; thus $\phi\{\psi(t)\}$ is monotone and non-diminishing as t increases. The variable t denotes the length of the arc of the curve $y' = \phi(x')$, so that $\psi'(t) \leqq 1$. We have then

$$\int_a^b \phi(x')\,\Phi(x', x, n)\,dt = \int_0^l \phi\{\psi(t)\}\,\Phi(\psi(t), x, n)\,\psi'(t)\,dt,$$

where $t = 0$, when $x' = a$; and $t = l$, when $x' = b$.

Denoting $\phi\{\psi(t)\}$ by $\phi_1(t)$, and $\Phi(\psi(t), x, n)\,\psi'(t)$ by $\Phi_1(t, x, n)$, we have to consider the integral $\int_0^l \phi_1(t)\,\Phi_1(t, x, n)\,dt$.

On account of the equality $\int_a^\beta \Phi(x', x, n)\,dx' = \int_{a'}^{\beta'} \Phi_1(t, x, n)\,dt$, where a', β' are the values of t which correspond to a and β respectively, we see that $\left|\int_{a'}^{\beta'} \Phi_1(t, x, n)\,dt\right| < M$; moreover $\Phi_1(t, x, n)$ satisfies the condition (2), or (2*), of Theorem I. Also $\phi_1(t)$ is an indefinite integral; for its total variation in a set of points t, of measure $< \epsilon$, is given by

$$\Sigma\,|\,\Delta\phi_1(t)\,| = \Sigma\,|\,\Delta\phi(x')\,| = \Sigma\,|\,\Delta y'\,| \leqq \Sigma\,(\Delta t) < \epsilon.$$

It follows from what has already been proved that

$$\int_0^l \phi_1(t)\,\Phi_1(t, x, n)\,dx'$$

converges to zero as $n \sim \infty$, uniformly for all values of x in G; and therefore the same statement holds for $\int_a^b \phi(x')\,\Phi(\xi, x, n)\,dx'$. The sufficiency of the conditions (1 d), (2) has now been established.

THE GENERAL CONVERGENCE THEOREM IN THE CASE OF NON-SUMMABLE FUNCTIONS

286. Theorem I can be extended to the case in which $f(x')$ is no longer summable in the linear interval (a, b), but has an HL-integral in that interval, provided $\Phi(x', x, n)$ satisfies the additional condition that its total variation $V_a^b \Phi(x', x, n)$ in the interval (a, b), of x', is less than some positive number A, independent of x and n.

If $f_\Delta(x') = f(x')$ at all points of (a, b) except those of a finite set Δ of intervals which enclose the points of non-summability, and if $f_\Delta(x') = 0$, in the intervals of the set Δ, we have

$$\int_a^b f(x')\,\Phi(x', x, n)\,dx' = \int_a^b \{f(x') - f_\Delta(x')\}\,\Phi(x', x, n)\,dx' + \int_a^b f_\Delta(x')\,\Phi(x', x, n)\,dx'.$$

The limit, as $n \sim \infty$, of the second integral on the right-hand side is zero, since $f_\Delta(x')$ is summable, provided $\Phi(x', x, n)$ satisfies the conditions of Theorem I, the convergence being uniform, or bounded, according as condition (2) or condition (2*) is assumed to hold.

The first integral is, in accordance with the theorem of I, § 424, equal to

$$\Phi(a, x, n)\int_a^b \{f(x') - f_\Delta(x')\}\,dx' + V_a^b \Phi(x', x, n)\,M,$$

where M is the upper boundary of $\left|\int_{a'}^{b'} \{f(x') - f_\Delta(x')\}\,dx'\right|$ for all intervals (a', b') contained in (a, b). In accordance with I, § 453, this is numerically less than $K\epsilon + A\epsilon$, where Δ can be so chosen that ϵ is arbitrarily small.

Thus we have

$$\overline{\lim_{n\sim\infty}} \int_a^b f(x')\,\Phi(x', x, n)\,dx' < (K + A)\,\epsilon.$$

The following theorem has now been established.

If $\Phi(x', x, n)$ satisfies the conditions of Theorem I, either with (2) *or* (2*), *and also the additional condition that $V_a^b \Phi(x', x, n)$, the variation of $\Phi(x', x, n)$ in the finite interval (a, b), of x', is less than some fixed positive number, independent of x and n, then $\lim\limits_{n\sim\infty} \int_a^b f(x')\,\Phi(x', x, n)\,dx'$ converges to zero, as $n \sim \infty$, uniformly, or boundedly, as the case may be, for all points x, in G; where $f(x')$ is any function which has an HL-integral in the linear interval (a, b).*

In accordance with I, § 453, if b' be any number in the interval (a, b), $\int_a^{b'} \{f(x') - f_\Delta(x')\}\, dx'$ is numerically less than ϵ, for all values of b' in (a, b). It is now easily seen that $\int_a^{b'} f(x')\, \Phi(x', x, n)\, dx'$ converges to zero as $n \sim \infty$, uniformly for all values of b' in the interval (a, b).

In order to extend the theorem to the case of an infinite interval (a, ∞), it is necessary to introduce the restriction that $\Phi(x', x, n)$ is, for each pair of values of x and n, a monotone function of x' in the interval (a, ∞). The condition (1) being assumed to hold in (a, ∞), $\Phi(x', x, n)$ is also bounded in the interval (a, ∞). We have then

$$\int_\alpha^{\alpha'} f(x')\, \Phi(x', x, n)\, dx' = \Phi(\alpha, x, n) \int_\alpha^{\bar{\alpha}} f(x')\, dx' + \Phi(\alpha', x, n) \int_{\bar{\alpha}}^{\alpha'} f(x')\, dx'.$$

If $\int_a^{\infty} f(x')\, dx'$ exists as $\lim\limits_{\alpha \sim \infty} \int_a^{\alpha} f(x')\, dx'$, α may be so chosen that $\left| \int_\alpha^{\beta} f(x')\, dx' \right| < \eta$ for all values of $\beta > \alpha$. We then have

$$\left| \int_\alpha^{\bar{\alpha}} f(x')\, dx' \right| < \eta, \quad \left| \int_{\bar{\alpha}}^{\alpha'} f(x')\, dx' \right| < 2\eta,$$

and if the condition (1) holds in the whole interval (a, ∞) we have, for all values of α',

$$|\, \Phi(\alpha, x, n)\,| < K, \quad |\, \Phi(\alpha', x, n)\,| < K;$$

and thus

$$\left| \int_\alpha^{\alpha'} f(x')\, \Phi(x', x, n)\, dx' \right| < 3K\eta$$

for all values of $\alpha' > \alpha$, provided α is sufficiently large, and thus

$$\int_\alpha^{\infty} f(x')\, \Phi(x', x, n)\, dx'$$

exists, and is numerically $\leq 3K\eta$. Since the theorem is applicable to the integral $\int_a^{\alpha} f(x')\, \Phi(x', x, n)\, dx'$, we see that it is also applicable to

$$\int_a^{\infty} f(x')\, \Phi(x', x, n)\, dx'.$$

The following theorem has now been established:

If $f(x')$ has an HL-integral in (a, ∞), and $\Phi(x', x, n)$ is monotone, for each pair of values of x and n in the interval (a, ∞) of x', and satisfies in that interval the conditions (1), *and* (2) *or* (2*) *of Theorem I, then*

$$\int_a^{\infty} f(x')\, \Phi(x', x, n)\, dx'$$

converges, uniformly or boundedly, as the case may be, for all values of x (in G) to zero, as $n \sim \infty$.

It will be observed that, in this case, the condition (1), that

$$|\Phi(x', x, n)| < A,$$

for all values of x, n and x' in (a, ∞), includes the condition that $V_a^\infty \Phi(x', x, n)$ is less than a positive number independent of x and n.

287. Let it be assumed that $f(x')$ has a D-integral in the interval (a, b). Denoting $\int_a^x f(x)\,dx$ by $F(x)$, which is continuous in (a, b), we have, if it be assumed that $\Phi(x', x, n)$ is, for each value of x and n, of bounded variation in (a, b) and has a finite differential coefficient $\frac{\partial \Phi(x', x, n)}{\partial x'}$ at every point of (a, b),

$$\int_a^b f(x')\,\Phi(x', x, n)\,dx' = F(b)\,\Phi(b, x, n) - \int_a^b F(x')\,\frac{\partial \Phi(x', x, n)}{\partial x'}\,dx',$$

since, in accordance with I, § 474, integration by parts is applicable. Let it now be assumed that $\frac{\partial \Phi(x', x, n)}{\partial x'}$ satisfies the condition (1), of Theorem I, that $\left|\frac{\partial \Phi(x', x, n)}{\partial x'}\right| < K$, for all the values of x and n, or more generally that it satisfies the condition (1 c) of § 284, that $\int_a^b \left|\frac{\partial \Phi(x', x, n)}{\partial x'}\right| dx' < K$ for all the values of x and n. Since

$$\int_\alpha^\beta \frac{\partial \Phi(x', x, n)}{\partial x'}\,dx' = \Phi(\beta, x, n) - \Phi(\alpha, x, n),$$

the condition (2) of Theorem I will be satisfied by $\frac{\partial \Phi(x', x, n)}{\partial x'}$ if, for each point x' of (a, b), $\Phi(x', x, n)$ converges uniformly to zero, as $n \sim \infty$ for all values of x in G. If both these conditions are satisfied,

$$\int_a^b F(x')\,\frac{\partial \Phi(x', x, n)}{\partial x'}\,dx'$$

converges to zero, as $n \sim \infty$, uniformly for all values of x in G.

Since $\Phi(b, x, n)$ converges to zero, as $n \sim \infty$, uniformly for all values of x in G, it is now seen that $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ has the same property.

The following theorem has accordingly been established:

Let $f(x')$ have a D-integral in the finite interval (a, b). Let $\Phi(x', x, n)$ be for each pair of values of x and n, where x is any point of the set G, of bounded variation in (a, b), and have at each point a differential coefficient $\frac{\partial \Phi(x', x, n)}{\partial x'}$. If either of the conditions

$$\left|\frac{\partial \Phi(x', x, n)}{\partial x'}\right| < K, \quad \int_a^b \left|\frac{\partial \Phi(x', x, n)}{\partial x'}\right| < K,$$

is satisfied, where K is independent of x and n, and if $\Phi(x', x, n)$ converges to zero, as $n \sim \infty$, for each value of x' in (a, b), uniformly for all points x of G, then $\lim\limits_{n \sim \infty} \int_a^b f(x') \Phi(x', x, n)\, dx' = 0$, *the convergence being uniform for all values of x, in G.*

NECESSITY OF THE CONDITIONS OF THE GENERAL CONVERGENCE THEOREM

288. It has been shewn that the conditions (1), (2), to be satisfied by the function $\Phi(x', x, n)$, are sufficient in order that Theorem I may hold good for every summable function $f(x')$. It will now be shewn that these conditions (1) and (2) are necessary in order that the convergence may take place for every function $f(x')$ that is summable in (a, b). It will in fact be shewn that:

Unless the conditions (1), (2) *of Theorem I are both satisfied, a function $f(x')$ summable in the interval (a, b) exists, such that the corresponding integral does not converge to zero, as $n \sim \infty$, uniformly for all points x, in G.*

The particular case of this theorem which arises when the parameter x is confined to have a single value, and therefore disappears, was given* by Lebesgue.

In order to shew that the condition (2) is necessary, let $f(x')$ be defined to have the value 1 in the interval (α, β), and the value 0 at all other points of (a, b). Unless $\int_\alpha^\beta \Phi(x', x, n)\, dx'$ converges to 0, as $n \sim \infty$, uniformly for all points x, of G, this function $f(x')$ is such as is required. This will be the case whatever be the interval (α, β); hence the condition (2) is necessary.

For each pair of values of x and n, $|\Phi(x', x, n)|$ must be equivalent to a function which has a finite upper boundary in (a, b), $U(x, n)$, which is finite; for otherwise a summable function $f(x')$ could be so determined that $f(x')\Phi(x', x, n)$ is not summable (see I, § 397). For a particular pair of values of x and n, there exists a set E_λ, of points of x', of measure > 0, for which

$$|\Phi(x', x, n)| > U(x, n) - \lambda,$$

where λ is a positive number, provided the smallest possible value of $U(x, n)$ has been taken.

For each pair of values of x and n, the function $\Phi(x', x, n)$ may be replaced, in this manner, by an equivalent function. There is accordingly no loss of generality in assuming that $\Phi(x', x, n)$ is such that, for each

* *Annales de Toulouse* (3), vol. I (1909), p. 53.

positive value of λ, and for each pair of values of x and n, the set of points x', at which

$$|\Phi(x', x, n)| > U(x, n) - \lambda$$

has a measure greater than zero, and such that at no point is

$$|\Phi(x', x, n)| > U(x, n).$$

In order that the integral may converge for each value of x, in G, for such a value of x, $U(x, n)$ must have a finite upper boundary $u(x)$, as $n \sim \infty$. If this is not the case for a particular value of x, there must be a sequence of increasing values of n, say $n_1, n_2, n_3, \ldots$ such that $U(x, n_1)$, $U(x, n_2)$, $U(x, n_3)$, ... forms a divergent sequence of numbers; and for each value of p there is a set of points x' of measure > 0, for which

$$|\Phi(x', x, n_p)| > U(x, n_p) - \epsilon.$$

It will be shewn that it is then possible to construct a function $f(x')$, such that the integral diverges as $n \sim \infty$, for the particular value of x.

If $u(x)$ is finite for each value of x, it may happen that $u(x)$ has no finite upper boundary for all values of x in G, and then the condition (1) is not satisfied. If this is the case there must be a sequence $x_1, x_2, x_3, \ldots$ of values of x, such that the sequence $u(x_1), u(x_2), u(x_3), \ldots$ is divergent. There then exists a sequence $n_1, n_2, \ldots$ of increasing values of n, such that the sequence $U(x_1, n_1)$, $U(x_2, n_2)$, $U(x_3, n_3)$, ... diverges. It then follows that there exists a sequence $k_1, k_2, k_3, \ldots$ of increasing numbers, such that the sets of points x' at which $|\Phi(x', x_p, n_p)| > k_p$ have a measure > 0, for all values of p. It will be shewn in this case that $f(x')$ can be so constructed that the integral does not converge to zero, uniformly for all the values of x. In case x_p is independent of p, we get back to the case first considered for which there is divergence of $U(x, n)$ for one particular value of x, for a sequence of values of n; and this case is accordingly included in the case in which the x_p are not all identical.

We therefore assume that, for a sequence of pairs of values x_p, n_p of x and p, $|\Phi(x', x_p, n_p)| > k_p$, in a set E_p, of points x', such that $m(E_p) > 0$; where $\{k_p\}$ is a sequence of increasing numbers without upper limit.

The set E_p must have a part F_p, of measure greater than zero, so that for all points of F_p, $\Phi(x', x_p, n_p)$ is of the same sign and is numerically $> k_p$; and this is the case for each value of p. Suppose that, for an infinite set of values of p, $\Phi(x', x_p, n_p)$ is positive, and $m(F_p) > 0$. There is no loss of generality in this assumption, because if the sign were negative, it would become positive by changing the sign of $\Phi(x', n, x)$ throughout.

We may suppose all those values of p and k_p removed, for which p does not belong to this infinite set of values of p. It may therefore be assumed that $\Phi(x', x_p, n_p) > k_p$, in F_p, and $m(F_p) > 0$, for all values of p.

The sets $\{F_p\}$ may be replaced by sets $\{e_p\}$, such that e_p is a part of F_p, $m(e_p) > 0$, no two of the sets e_p have a point in common, and

$$m(e_p) > m(e_{p+1}),$$

for all values of p. To see this, we observe that the sets $F_2, F_3, \ldots$ may be so diminished, without making any of their measures zero, that

$$m(F_1) > 2m\{M(F_2, F_3, \ldots)\}.$$

We then obtain e_1 by removing from F_1 the points which it has in common with $M(F_2, F_3, \ldots)$; then $m(e_1) > m\{M(F_2, F_3, \ldots)\}$.

Next, by diminishing $F_3, F_4, \ldots$ we may make

$$m(F_2) > 2m\{M(F_3, F_4, \ldots)\};$$

we obtain e_2 by removing from F_2 all the points that it has in common with $M(F_3, F_4, \ldots)$; thus $m(e_1) > m(e_2) > m\{M(F_3, F_4, \ldots)\}$. Proceeding in this manner we obtain the sets $e_1, e_2, e_3, \ldots$, no two of which have a point in common, all of which have measures > 0, and such that

$$m(e_1) > m(e_2) > m(e_3) \ldots.$$

In e_p, we have $\qquad \Phi(x', x_p, n_p) > k_p.$

Let p_1 be a value of p, and consider $\mu_{p_1}\int_{(e_{p_1})} \Phi(x', x_p, n_p)\,dx'$, where μ_{p_1} is a constant such that $\mu_{p_1} m(e_{p_1}) = \frac{1}{2}$. If the integral does not converge to zero, as $p \sim \infty$, the function $f(x')$ defined by $f(x') = \mu_{p_1}$, in e_{p_1}, and $= 0$, elsewhere, is a function such as is required. If it does converge to zero, we have $\left|\mu_{p_1}\int_{(e_{p_1})} \Phi(x', x_p, n_p)\,dx'\right| < 1$, provided $p \geqq p^{(1)}$.

Let l_p denote the lower boundary of $\Phi(x', x_p, n_p)$ in e_p, and let U_p be the upper boundary of $|\Phi(x', x_p, n_p)|$ in (a, b); then both l_p and U_p increase indefinitely with p.

There exists a smallest integer $p_2 \geqq p^{(1)}$, such that $l_{p_2} > 2^3 U_{p_1}$; let μ_{p_2} be such that $\mu_{p_2} m(e_{p_2}) = \dfrac{1}{2^2 U_{p_1}}$.

If $$\mu_{p_1}\int_{(e_{p_1})} \Phi(x', x_p, n_p)\,dx' + \mu_{p_2}\int_{(e_{p_2})} \Phi(x', x_p, n_p)\,dx'$$

does not converge to zero, as $p \sim \infty$, the function defined as having the value μ_{p_1}, in e_{p_1}; μ_{p_2}, in e_{p_2}; and elsewhere zero, is a function such as is required. If it does converge to zero, its absolute value is < 1, provided p is not less than some number $p^{(2)}$; there exists a smallest integer p_3 $(\geqq p^{(2)})$ such that $l_{p_3} > 2^4 U_{p_2}$; let $\mu_{p_3} m(e_{p_3}) = \dfrac{1}{2^3 U_{p_2}}$.

Proceeding in this manner, we may be able, after a finite number of steps, to define a function $f(x')$ having the values $\mu_{p_1}, \mu_{p_2}, \ldots \mu_{p_r}$ in the sets $e_{p_1}, e_{p_2}, \ldots e_{p_r}$, respectively, and elsewhere equal to zero, which will

be a function such as is required. When this is not the case for any finite value of r, let $f(x')$ have the value μ_{p_m} in e_{p_m}, for every value of m. This function $f(x')$ is summable, for its integral in (a, b) is

$$\sum_{r=1}^{\infty} \mu_{p_r} m(e_{p_r}) = \sum_{r=1}^{\infty} \frac{1}{2^r U_{p_{r-1}}},$$

and this latter series is convergent. Also, we have

$$\int_a^b f(x')\,\Phi(x', x_{p_m}, n_{p_m})\,dx' = \mu_{p_m}\int_{(e_{p_m})} \Phi(x', x_{p_m}, n_{p_m})\,dx'$$
$$+ \sum_{r=1}^{r=m-1} \mu_{p_r}\int_{(e_{p_r})} \Phi(x', x_{p_m}, n_{p_m})\,dx' + \sum_{r=m+1}^{\infty} \mu_{p_r}\int_{(e_{p_r})} \Phi(x', x_{p_m}, n_{p_m})\,dx'.$$

The first term on the right-hand side is $\geqq \mu_{p_m} l_{p_m} m(e_{p_m})$, which is

$$\geqq \frac{l_{p_m}}{2^m U_{p_{m-1}}} \geqq 2.$$

The second term is > -1, and the third term is greater than

$$-U_{p_m} \sum_{r=m+1}^{\infty} \mu_{p_r} m(e_{p_r}),$$

or than
$$-U_{p_m} \sum_{m+1}^{\infty} \frac{1}{2^r U_{p_{r-1}}} > -\sum_{m+1}^{\infty} \frac{1}{2^r} > -\frac{1}{2^m}.$$

It follows that $\int_a^b f(x')\,\Phi(x', x_{p_m}, n_{p_m})\,dx' > \frac{1}{2}$; and since this holds for an infinite set of values of m, the integral $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ cannot converge to zero uniformly for all values of x in G. Hence the condition (1) of Theorem I has been shewn to be necessary, in order that the uniform convergence may take place for every summable function $f(x')$.

289. It can be shewn that the conditions given in §§ 282–285 are necessary in each case, in order that the uniform convergence shall hold good for every function $f(x')$ of the particular type. The proof will here be given that the conditions (1 c), (2) are necessary in the case in which $f(x')$ is restricted to have only ordinary discontinuities.

It is clear that the condition (2) is necessary, for we may take $f(x') = 1$, in the interval (α, β), and equal to zero outside that interval. If the condition (1 c) is not satisfied, it will be possible to determine a sequence (x_1, n_1), (x_2, n_2), ... (x_p, n_p) ... of pairs of values of x and n, such that $\int_a^b |\Phi(x', x_p, n_p)|\,dx'$ increases indefinitely as p and n_p do so. Let us assume that this integral exceeds L_p, where $L_1, L_2, \ldots$ is a divergent sequence of increasing numbers. It may happen that, for all values of p, the values of x_p are identical; but this case will be included in the general case. It will be shewn that a continuous function $f(x')$ can be defined for which the uniform convergence does not hold.

We have $\int_a^b |\Phi(x', x_1, n_1)| \, dx' > L_1$; let $\chi_1(x') = 1$, or -1, according as $\Phi(x', x_1, n_1)$ is $\geqq 0$, or negative. Thus $\chi_1(x') = 1$, in a set E_1, and $\chi_1(x') = -1$, in the set $C(E_1)$. The set E_1 can be enclosed in the intervals of a non-overlapping set, of measure $< m(E_1) + \epsilon$; and a finite set of these intervals of total measure $> m(E_1)$ can be chosen. Let $g_1(x') = 1$ in the intervals of this finite set, and let $g_1(x') = -1$ in the rest of the interval (a, b), except that $g_1(a) = g_1(b) = 0$.

The functions $\chi_1(x')$, $g_1(x')$ differ from one another only at points of a set of which the measure is $< \epsilon$. The function $g_1(x')$ is the limit of a sequence of continuous functions $\{h_1^{(\iota)}(x')\}$, all numerically < 1. Moreover we can take all the functions $h_1^{(\iota)}(x')$ so that they have the value zero at a and at b, since $g_1(x')$ has this property. Since the functions $h_1^{(\iota)}(x')$ are all bounded, we have

$$\lim_{\iota\sim\infty}\int_a^b h_1^{(\iota)}(x')\,\Phi(x', x_1, n_1)\,dx' = \int_a^b g_1(x')\,\Phi(x', x_1, n_1)\,dx'.$$

By a proper choice of ϵ, we can ensure that

$$\int_a^b g_1(x')\,\Phi(x', x_1, n_1)\,dx' > L_1;$$

and by choosing a sufficiently large value of ι, say ι_1, we have

$$\int_a^b h_1^{(\iota_1)}(x')\,\Phi(x', x_1, n_1)\,dx' > L_1.$$

If $\int_a^b h_1^{(\iota_1)}(x')\,\Phi(x', x_p, n_p)\,dx'$ does not converge to zero, as $p \sim \infty$, we have obtained a continuous function $h_1^{(\iota_1)}(x')$ which vanishes for $x' = a$, $x' = b$, and is such that $\int_a^b h_1^{(\iota_1)}(x')\,\Phi(x', x, n)\,dx'$ does not converge to zero, as $n \sim \infty$, uniformly for all the values of x, in G. If it does converge to zero, then, for all sufficiently large values of p, it is numerically < 1. Take p_2 such a value of p that $L_{p_2} > 2^2 L_1$; and let $h_2^{(\iota_2)}(x')$ be the function corresponding to $h_1^{(\iota_1)}(x')$, where

$$\int_a^b |\Phi(x', x_{p_2}, n_{p_2})|\,dx' > L_{p_2}.$$

If $$\int_a^b \left\{h_1^{(\iota_1)}(x') + \frac{h_2^{(\iota_2)}(x')}{2L_1}\right\}\Phi(x', x_p, n_p)\,dx'$$

does not converge to zero, as $p \sim \infty$, then the function $h_1^{(\iota_1)}(x') + \dfrac{h_2^{(\iota_2)}(x')}{2L_1}$ is a function such as is required. If it does converge to zero, it is numerically < 1, provided p is sufficiently large; let $p_3\,(> p_2)$ be such a value that $\int_a^b |\Phi(x', x_{p_3}, n_{p_3})|\,dx' > L_{p_3}$, and also such that $L_{p_3} > 2^3 L_{p_2}$. Let $h_3^{(\iota_3)}(x')$ be the function corresponding to $h_1^{(\iota_1)}(x')$ and $h_2^{(\iota_2)}(x')$.

If $$\int_a^b \left\{h_1^{(\iota_1)}(x') + \frac{h_2^{(\iota_2)}(x')}{2L_{p_1}} + \frac{h_3^{(\iota_3)}(x')}{2^2 L_{p_2}}\right\} \Phi(x', x_p, n_p)\, dx'$$

does not converge to zero, as $p \sim \infty$, the continuous function

$$h_1^{(\iota_1)}(x') + \frac{h_2^{(\iota_2)}(x')}{2L_{p_1}} + \frac{h_3^{(\iota_3)}(x')}{2^2 L_{p_2}}$$

is a function such as is required. Proceeding in this manner, and assuming that $L_{p_r} > 2^r L_{p_{r-1}}$, we may, after a finite number of steps, obtain a function such as is required. But if not, we have a function

$$f(x') = h_1^{(\iota_1)}(x') + \frac{h_2^{(\iota_2)}(x')}{2L_1} + \ldots + \frac{h_m^{(\iota_m)}(x')}{2^{m-1} L_{p_{m-1}}} + \ldots$$

defined by a uniformly convergent series of continuous functions; thus the function $f(x')$ is continuous, and $f(a) = f(b) = 0$.

Moreover

$$\int_a^b f(x') \Phi(x', x_{p_r}, n_{p_r})\, dx' = \frac{1}{2^{r-1} L_{p_{r-1}}} \int_a^b h_r^{(\iota_r)}(x') \Phi(x', x_{p_r}, n_{p_r})\, dx'$$
$$+ \sum_{s=1}^{s=r-1} \frac{1}{2^{s-1} L_{p_{s-1}}} \int_a^b h_s^{(\iota_s)}(x') \Phi(x', x_{p_r}, n_{p_r})\, dx'$$
$$+ \sum_{s=r+1}^{\infty} \frac{1}{2^{s-1} L_{p_{s-1}}} \int_a^b h_s^{(\iota_s)}(x') \Phi(x', x_{p_r}, n_{p_r})\, dx'.$$

The first term on the right-hand side is $> \frac{L_{p_r}}{2^{r-1} L_{p_{r-1}}}$, or > 2. The second term is > -1, and the third term is greater than $-L_{p_r} \sum_{s=r+1}^{\infty} \frac{1}{2^{s-1} L_{p_{s-1}}}$ or $> -\frac{1}{2}$. It follows that $\int_a^b f(x') \Phi(x', x_{p_r}, n_{p_r})\, dx' > \frac{1}{2}$, for every value of r, and thus that $f(x')$ is a continuous function such as is required, for which $\int_a^b f(x') \Phi(x', x, n)\, dx'$ does not converge to zero, uniformly for all points x, in G.

SINGULAR INTEGRALS

290. The following theorem may be deduced from Theorem I.

THEOREM II. *Let $F(x', x, n)$ be defined for each point x in a set G, contained in (a, b), and for each value of n in an integral, or non-integral sequence of increasing numbers without upper limit, and for all values of x' in the interval (a, b). Let μ denote a positive number $(< b - a)$, and let $F(x', x, n)$ satisfy the following conditions:*

(1) *For each pair of values of x and n, and for all the points x' in (a, b) such that $|x' - x| \geqq \mu$, the function $F(x', x, n)$ is equivalent to a function that does not exceed in absolute value a positive number K_μ, independent of the values of x and n.*

(2) *If α, β be any two numbers such that $a \leqq \alpha \leqq \beta \leqq b$, $\int_\alpha^\beta F(x', x, n)\,dx'$ exists as an L-integral, for all values of n, and for all those values of x which belong to G and are not interior to the interval $(\alpha - \mu, \beta + \mu)$; and as n is indefinitely increased, it converges to zero, uniformly for all such values of x.*

Then $\int_a^{x-\mu} f(x')\,F(x', x, n)\,dx'$, $\int_{x+\mu}^b f(x')\,F(x', x, n)\,dx'$ converge to zero, as $n \sim \infty$, uniformly for all values of x, in G; for any function $f(x')$ that is summable in (a, b).

To prove the theorem, let the function $\Phi(x', x, n)$ of Theorem I be defined to have the values of $F(x', x, n)$, for each pair of values of x and n, and for all values of x' in the interval $(a, x - \mu)$; let $\Phi(x', x, n) = 0$ when x' is not in the interval $(a, x - \mu)$. Thus $\Phi(x', x, n)$ satisfies the conditions (1) and (2) of Theorem I.

Also $\int_a^b \Phi(x', x, n)\,dx' = \int_a^{x-\mu} F(x', x, n)\,dx'$; it thus follows that the convergence of $\int_a^{x-\mu} F(x', x, n)\,dx'$ to zero, as $n \sim \infty$, uniformly for all the points x of G, is established. The second part of the theorem is proved in a precisely similar manner. If the conditions of the theorem hold good when $\mu = 0$, it is then identical with Theorem I. In accordance with Theorem II, the question of the nature of $\lim_{n\sim\infty} \int_a^b f(x')\,F(x', x, n)\,dx'$ is made to depend upon that of $\lim_{n\sim\infty} \int_{x-\mu}^{x+\mu} f(x')\,F(x', x, n)\,dx'$; the integral over the neighbourhood $(x - \mu, x + \mu)$ of x. In this matter the character of the summable function $f(x')$ outside this neighbourhood of x is irrelevant.

An integral $\int_a^b f(x')\,F(x', x, n)\,dx'$, for which the conditions of Theorem II are not satisfied when $\mu = 0$, may be termed a *singular integral*. It will be seen that, in the theory of Fourier's series, and of other modes of representation of functions by means of series or integrals, the theory of singular integrals is of fundamental importance.

In the case of an integral $\int_{-\infty}^{\infty} f(x')\,F(x', x, n)\,dx'$, where x is confined to belong to a set of points G, contained in the finite interval (a_1, b_1), we may take an interval (a, b) which contains (a_1, b_1) in its interior, and consider separately the three integrals taken over the intervals $(-\infty, a)$, (a, b), (b, ∞). Theorem II can be applied to the integral over (a, b), and in case the integrals

$$\int_b^\infty f(x')\,F(x', x, n)\,dx', \quad \int_{-\infty}^a f(x')\,F(x', x, n)\,dx'$$

converge to zero, as $n \sim \infty$, uniformly for all values of x, the theorem can be extended to $\int_{-\infty}^{\infty} f(x') F(x', x, n)\, dx'$. Sufficient conditions that these two integrals may so converge are obtained by employing the theorems of § 281.

291. It is easily seen that Theorem II can be extended to the case of a function of two or more variables. For simplicity, the following theorem will be stated for a function of three variables only.

Let $f(x', y', z')$ be summable in a given rectangular parallelopiped Δ, and let $F(x', y', z', x, y, z, n)$ be defined for all points (x', y', z') in Δ, and all points (x, y, z) in a given set G, contained in Δ; and for all values of n in some increasing sequence of numbers with no upper boundary. Let the function F satisfy the conditions (1), *for each set of values of x, y, z, n, and for all points (x', y', z') such that $(x' - x)^2 + (y' - y)^2 + (z' - z)^2 \geqq \mu^2$, the function $F(x', y', z', x, y, z, n)$ is equivalent to a function that does not exceed in absolute value some positive number K_μ, independent of n, and of (x', y', z'), (x, y, z); and* (2), *in every cell Δ_1 contained in Δ, and for all points (x, y, z), of G, which are at a distance $\geqq \mu$ from every point of Δ_1,*

$$F(x', y', z', x, y, z, n)\, d(x', y', z')$$
$${}_1$$

exists as an L-integral, and converges to zero, as $n \sim \infty$, uniformly for all values of (x, y, z). Then

$$\int_{(\Delta - S)} f(x', y', z') F(x', y', z', n)\, d(x', y', z')$$

converges to zero, as $n \sim \infty$, uniformly for all values of x, in G; S denotes, for each point (x, y, z), the set of points (x', y', z') at which

$$(x' - x)^2 + (y' - y)^2 + (z' - z)^2 < \mu^2.$$

In order to prove the theorem, let $\Phi(x', y', z', x, y, z, n)$ have the value $F(x', y', z', x, y, z, n)$ whenever $(x' - x)^2 + (y' - y)^2 + (z' - z)^2 \geqq \mu^2$, and when this is not the case, let $\Phi(x', y', z', x, y, z, n)$ have the value zero. We have then only to apply Theorem I to the function

$$\Phi(x', y', z', x, y, z, n).$$

Another case of Theorem II which is of importance in the theory of double and multiple Fourier's series may be given for the two-dimensional case, and can be immediately extended to the case of any number of dimensions.

If (x, y) be a point in the rectangle Δ, a point (x', y'), of Δ, for which one at least of the numbers $|x' - x|$, $|y' - y|$ is $\leqq \mu$, is said to be in the *cross-neighbourhood* (μ) of the point (x, y); that cross-neighbourhood consisting of the totality of all such points (x', y').

The theorem may be stated as follows:

If $(F\ x', y', x, y, n)$ be defined for each point (x, y) in a set G, contained in the cell Δ, and for each value of n in some increasing divergent sequence, and for all values of (x', y') in Δ; and if $F(x', y', x, y, n)$ satisfy the following conditions:

(1) *For each set of values of (x, y, n), and for all points x' not in the cross-neighbourhood (μ), of (x, y), the function $F(x', y', x, y, n)$ is equivalent to a function that does not exceed in absolute value a positive number K_μ, independent of (x, y) and n.*

(2) *If Δ_1 be any cell contained in Δ, $\int_{(\Delta_1)} F(x', y', x, y, n)\, d(x', y')$ exists as an L-integral, for all values of n, and for all those values of (x, y) which belong to G, and are such that Δ_1 has no point which belongs to the cross-neighbourhood (μ) of (x, y), and it converges to zero, uniformly for all such points (x, y).*

Then $\int_{\Delta - H_\mu(x,y)} f(x')\, F(x', y', x, y, n)\, d(x', y')$ converges to zero, uniformly for all points (x, y) in G, for any function $f(x', y')$ summable in Δ; where $H_\mu(x, y)$ denotes the cross-neighbourhood (μ), of (x, y).

The importance of the theorem arises from the fact that, when it holds for every value of μ (> 0), however small, the limit, as $n \sim \infty$, of

$$\int_{(\Delta)} f(x', y')\, F(x', y', x, y, n)\, d(x', y')$$

depends only upon that of

$$\int_{H_\mu(x,y)} f(x', y')\, F(x', y', x, y, n)\, d(x', y')$$

taken over the arbitrarily small cross-neighbourhood (μ) of the point (x, y).

In order to reduce this theorem to Theorem I, we have only to define $\Phi(x', y', x, y, n)$ as having the value zero in the cross-neighbourhood (μ) of (x, y), and as having the value $F(x', y', x, y, n)$ outside that cross-neighbourhood.

THE CONVERGENCE OF SINGULAR INTEGRALS

292. The most important of the applications of Theorems I and II to the theory of series and integrals arise in the case in which the function $F(x', x, n)$, of Theorem II, has the form $F(x' - x, n)$. It will be assumed that this function satisfies the conditions (1) and (2) of Theorem II, for all positive values of μ. The question of the character of

$$\lim_{n\sim\infty} \int_a^b f(x')\, F(x' - x, n)\, dx'$$

then reduces to the investigation of the limit of

$$\int_{x-\mu}^{x+\mu} f(x')\, F(x'-x, n)\, dx',$$

or of

$$\int_{-\mu}^{\mu} f(x+t)\, F(t, n)\, dt; \text{ where } x' = x + t.$$

It thus appears that the character of the limit of the interval at a point x depends only on the properties of the function $f(x')$ in the neighbourhood of the point x.

Let it be assumed that the function $F(t, n)$ satisfies the two conditions,

(*a*) $$\lim_{n\sim\infty} \int_{-\mu}^{\mu} F(t, n)\, dt = 1,$$

(*b*) $$\int_{-\mu}^{\mu} | F(t, n) |\, dt \leqq A,$$

where A is a positive number independent of n and μ.

We have

$$\int_{-\mu}^{\mu} f(x+t)\, F(t, n)\, dt = f(x) \int_{-\mu}^{\mu} F(t, n)\, dt + \int_{-\mu}^{\mu} \{f(x+t) - f(x)\}\, F(t, n)\, dt.$$

Let it be first assumed that the function $f(x)$ is continuous at the point x, then μ can be so chosen that $| f(x+t) - f(x) | < \eta$, for all values of t in the interval $(-\mu, \mu)$, where η is a prescribed positive number.

It then follows that

$$\left| \overline{\underline{\lim}}_{n\sim\infty} \int_{-\mu}^{\mu} f(x+t)\, F(t, n)\, dt - f(x) \right| < \eta A;$$

and since η can be taken to be arbitrarily small, by proper choice of μ, we have

$$\lim_{n\sim\infty} \int_a^b f(x')\, F(x'-x, n)\, dx' = f(x).$$

If the set G consists of all the points of an interval (α, β) in which $f(x')$ is continuous, the continuity at α and β being on both sides, μ may be so determined that the condition $| f(x+t) - f(x) | < \eta$ is satisfied for all the points x of the interval (α, β). In that case the convergence of the integral to the value $f(x)$ is uniform in the interval (α, β).

The following theorem has been now established:

If the function $F(x'-x, n)$ satisfies the conditions (1) *and* (2) *of Theorem II, and satisfies the conditions*

(*a*) $$\lim_{n\sim\infty} \int_{-\mu}^{\mu} F(t, n)\, dt = 1,$$

(*b*) $$\int_{-\mu}^{\mu} | F(t, n) |\, dt \leqq A,$$

where A is independent of μ and n, for all sufficiently small values of μ (>0), then $\int_a^b f(x') F(x'-x, n)\,dx'$ *converges, as $n \sim \infty$, to the value $f(x)$ at a point x at which $f(x')$ is continuous; and it converges uniformly to $f(x)$ in any interval (α, β) in which $f(x')$ is continuous, the continuity at α and β being on both sides. The function $f(x')$ is any function that is summable in (a, b).*

In case $f(x')$ is a function of one of the types considered in §§ 282–285 the conditions (1 a), (1 b), (1 c) and (1 d) may be substituted for the condition (1). It should be observed that, in case $F(t, n)$ is never negative, the condition (b) is contained in the condition (a), since, if necessary, a finite set of values of n may be disregarded.

293. In the case in which μ can be so chosen that the function $f(x')$ is of bounded variation in the interval $(x-\mu, x+\mu)$, it is sufficient, instead of the condition (b), to assume that the condition

$$(b') \qquad \left| \int_{\lambda_1}^{\lambda_2} F(t, n)\,dx \right| \leqq A$$

is satisfied, for every interval (λ_1, λ_2) contained in $(-\mu, \mu)$. For in that case, in accordance with the theorem of I, § 424,

$$\left| \int_{-\mu}^{\mu} \{f(x+t) - f(x)\} F(t, n)\,dt \right|$$

cannot exceed A multiplied by the total variation of $f(x+t)$ in the interval $(-\mu, \mu)$ of t. This is seen by dividing the interval of integration into the two parts $(0, \mu)$ and $(-\mu, 0)$. In any interval in which $f(x')$ is continuous and of bounded variation, it is expressible as the difference $P(x') - N(x')$ of two continuous monotone functions. In the interval $(-\mu, \mu)$ of t, the total variation of $f(x+t)$ cannot exceed the sum of the variations of $P(x+t)$ and $N(x+t)$, which is

$$|P(x+\mu) - P(x-\mu)| + |N(x+\mu) - N(x-\mu)|.$$

For a point x of the interval, these are both arbitrarily small, by proper choice of μ. Moreover, if x be confined to an interval interior to the interval in which $f(x')$ is continuous and of bounded variation, μ can be so chosen that $|P(x+\mu) - P(x-\mu)|$, $|N(x+\mu) - N(x-\mu)|$ are both less than an arbitrarily prescribed number, the same for all the values of x. It then follows, as before, that the integral converges to $f(x)$, at a point in the neighbourhood of which $f(x')$ is of bounded variation, and uniformly in a whole interval interior to another interval in which $f(x')$ is of bounded variation and continuous.

The following theorem has been established:

If the function $F(x'-x, n)$ satisfies the conditions (1) *and* (2) *of Theorem II, or one of the conditions* (1 a), (1 b), (1 c), (1 d) *instead of* (1), *in case $f(x')$ belongs to one of the corresponding classes of functions, and if*

further $F(x'-x,n)$ *satisfies the conditions* (*a*) *and* (*b'*), *then at a point* x *in a neighbourhood of which* $f(x')$ *is continuous and of bounded variation,* $\int_a^b f(x')F(x'-x,n)\,dx'$ *converges to* $f(x)$, *as* $n\sim\infty$. *Also for an interval which is interior to an interval in which* $f(x')$ *is continuous and of bounded variation, the convergence of the integral to the limit* $f(x)$ *is uniform in the interval.*

294. Let us now consider the convergence of the integral at a point x at which the function $f(x')$ has an ordinary discontinuity, so that $f(x+0)$ and $f(x-0)$ have definite values.

The following theorem will be established:

If the condition (*a*) *of the theorem of* § 292 *be replaced by the conditions* (*a'*)

$$\lim_{n\sim\infty}\int_0^\mu F(t,n)\,dt=\lim_{n\sim\infty}\int_{-\mu}^0 F(t,n)\,dt=\tfrac{1}{2},$$

the condition (*b*) *remaining unaltered, then at any point* x *of ordinary discontinuity of the function* $f(x')$,

$$\lim_{n\sim\infty}\int_a^b f(x')F(x'-x,n)\,dx'=\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

We have

$$\begin{aligned}\int_{-\mu}^{\mu} & f(x+t)F(t,n)\,dt\\ &=\int_0^\mu\{f(x+t)-f(x+0)\}F(t,n)\,dt+f(x+0)\int_0^\mu F(t,n)\,dt\\ &\quad+\int_{-\mu}^0\{f(x+t)-f(x-0)\}F(t,n)\,dt+f(x-0)\int_{-\mu}^0 F(t,n)\,dt.\end{aligned}$$

It can be assumed that μ is taken so small that

$$|f(x+t)-f(x+0)|,\quad |f(x+t)-f(x-0)|,$$

for t in $(0,\mu)$ and in $(-\mu,0)$, are both less than η.

It follows that

$$\left|\overline{\lim_{n\sim\infty}}\int_{-\mu}^{\mu} f(x+t)F(t,n)\,dt-\tfrac{1}{2}\{f(x+0)+f(x-0)\}\right|$$
$$<\eta\int_{-\mu}^{\mu}|F(t,n)|\,dt<A\eta.$$

Since η is arbitrarily small, we have

$$\lim_{n\sim\infty}\int_a^b f(x')F(x'-x,n)\,dt'=\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

In case the point x is a point in a neighbourhood of which $f(x')$ has bounded variation, the condition (*b*) of the theorem may be replaced by the condition (*b'*), that $\left|\int_{\lambda_1}^{\lambda_2}F(t,n)\,dt\right|\leqq A$ for every interval (λ_1,λ_2)

contained in $(-\mu, \mu)$, and for all values of n. The proof is precisely similar to that in § 293. Thus:

If the conditions (a'), (b') are satisfied by $F(t, n)$, then at any point x, in a neighbourhood of which $f(x')$ is of bounded variation,

$$\lim_{n\sim\infty}\int_a^b f(x')\,F(x'-x, n)\,dx' = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

295. It has been shewn in § 294 that, subject to the conditions of Theorem II, and the conditions (a), (b) of § 292, the singular integral converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point of continuity or of ordinary discontinuity. If we assume that the value of $f(x)$ at any point of ordinary discontinuity is taken to be $\frac{1}{2}\{f(x+0)+f(x-0)\}$, the integral then converges at such a point to the value $f(x)$. It can, however, be shown that, provided $F(t, n)$ satisfies certain conditions, the convergence of the integral to $f(x)$ holds good at all points of a set which includes points at which $f(x)$ has a discontinuity of the second kind.

It will be assumed that $F(t, n)$ is an even function of t, and that it possesses a continuous partial differential coefficient $F_1(t, n)$, with respect to t, for each value of n. It will further be assumed that $F(t, n)$ converges to zero, as $n \sim \infty$, for each value of t that is $\neq 0$.

We have

$$\int_{-\mu}^{\mu} f(x+t)\,F(t, n)\,dt = \int_0^{\mu}\phi(t)\,F(t, n)\,dt + 2f(x)\int_0^{\mu} F(t, n)\,dt,$$

where $\phi(t)$ denotes $f(x+t)+f(x-t)-2f(x)$. The second term on the right-hand side converges to $f(x)$, as $n \sim \infty$, in accordance with the condition (a) of § 292.

There then remains for consideration the integral $\int_0^{\mu}\phi(t)\,F(t, n)\,dt$, which may be expressed as $\phi_1(\mu)\,F(\mu, n) - \int_0^{\mu}\phi_1(t)\,F_1(t, n)\,dt$, where $\phi_1(t)$ denotes $\int_0^t \phi(t)\,dt$; since $F(0, n)$ is finite, and $\phi_1(0) = 0$.

Let us assume that $\int_0^t \phi(t)\,dt$ has a differential coefficient equal to zero at the point $t = 0$; we have then $\phi_1(t) = t\chi(t)$, where $\chi(t)$ is continuous, and $\chi(0) = 0$. It is known (see I, § 432) that this condition is satisfied for almost all points x, in (a, b). We have then to consider the limit of

$$\int_0^{\mu} t\chi(t)\,F_1(t, n)\,dt;$$

the term $\phi_1(\mu)\,F(\mu, n)$, or $\mu\chi(\mu)\,F(\mu, n)$, converges to zero, as $n \sim \infty$.

The integral $\int_0^{\mu}\chi(t)\,[tF_1(t, n)]\,dt$ converges to zero, as $n \sim \infty$, if the

function $tF_1(t, n)$ satisfies the conditions (1 c), (2) of § 284, for the interval (μ', μ), and the conditions (a), (b) of § 292, for $(0, \mu')$, where $\mu' < \mu$; since $\chi(t)$ is a continuous function of bounded variation in (μ', μ).

If α', β' be such that $0 < \mu' \leqq \alpha' < \beta' \leqq \mu$, where μ' is a fixed number $(< \mu)$, we have

$$\int_{\alpha'}^{\beta'} tF_1(t, n)\, dt = \Big[tF(t, n)\Big]_{\alpha'}^{\beta'} - \int_{\alpha'}^{\beta'} F(t, n)\, dt.$$

Since $F(t, n)$ converges to zero, for $t = \alpha'$, and $t = \beta'$, and since $F(t, n)$ satisfies the condition (2) of § 284, it follows that the integral on the left-hand side converges to zero, as $n \sim \infty$; and thus that $tF_1(t, n)$ satisfies the condition (2) of § 284.

The condition (1 c) that $\int_{\mu'}^{\mu} | tF_1(t, n) |\, dt$ should be bounded with respect to n is satisfied if the condition that $\int_0^{\mu} | tF_1(t, n) |\, dt$ is bounded, and is included in the latter condition.

Since

$$| tF_1(t, n) | = \left| \frac{\partial}{\partial t} \{tF(t, n) - F(t, n)\} \right| \leqq \left| \frac{\partial}{\partial t} \{tF(t, n)\} \right| + | F(t, n) |,$$

we see that $\int_0^{\mu} | tF_1(t, n) |\, dt$ is bounded with respect to n, if

$$\int_0^{\mu} \left| \frac{\partial}{\partial t} \{tF(t, n)\} \right| dt$$

is so, and if $\int_0^{\mu} | F(t, n) |\, dt$ satisfies the condition (b), of § 292, which we assume to be the case. Now $\int_0^{\mu} \left| \frac{\partial}{\partial t} \{tF(t, n)\} \right| dt$ is the total variation of $\int_0^{t} \frac{\partial}{\partial t} \{tF(t, n)\}\, dt$ (see I, § 415), or of $tF(t, n)$, in the interval $(0, \mu)$. Therefore the condition (1 c) of § 284 is satisfied if $tF(t, n)$ has a total variation in the interval $(0, \mu)$, less than some fixed number independent of n. Moreover

$$\lim_{n \sim \infty} \int_0^{\mu} tF_1(t, n)\, dt = - \lim_{n \sim \infty} \int_0^{\mu} F(t, n)\, dt = - \tfrac{1}{2};$$

it being assumed that the condition (a) is satisfied. Hence it is seen that $\lim\limits_{n \sim \infty} \int_0^{\mu} \chi(t)\, [tF_1(t, n)]\, dt = 0$, since $\chi(0) = 0$.

The following theorem has now been established:

If $F(x' - x, n)$ *satisfies the conditions* (1), (2) *of Theorem II, for all values of* μ (> 0), *and also the conditions* (a), (b) *of* § 292, *and if*

$$\lim_{n \sim \infty} F(x' - x, n) = 0$$

when $x' \neq x$, and $F(t, n)$ is an even function of t; then, provided $tF(t, n)$ has a total variation in $(0, \mu)$ less than a fixed number independent of n,

$$\int_a^b f(x') F(x' - x, n)\, dx'$$

converges to $f(x)$, as $n \sim \infty$, for every point x in (a, b) for which

$$\int_0^t \{f(x + t) + f(x - t) - 2f(x)\}\, dt$$

has a differential coefficient at $t = 0$, equal to zero; this is the case for almost all values of x.

In case $F(t, n)$ is not an even function of t, we may define $\phi_1(t)$ to denote $f(x + t) - f(x)$, or $f(x - t) - f(x)$; and thus

$$\int_0^\mu f(x + t) F(t, n)\, dt = \int_0^\mu \phi(t) F(t, n)\, dt + f(x) \int_0^\mu F(t, n)\, dt,$$

and by proceeding as before, it can be shewn that, subject to similar conditions, the convergence holds good at every point x at which

$$\int_a^b \{f(x + t) - f(x)\}\, dt, \text{ and } \int_a^b \{f(x - t) - f(x)\}\, dt$$

have differential coefficients equal to zero; and this is the case for almost all values of x.

296. Making, as in § 295, the assumption that $\lim\limits_{n \sim \infty} F(t, n) = 0$, for each value of t, except zero, we have

$$\int_0^\mu \chi(t) . tF_1(t, n)\, dt = \left\{\int_0^{\alpha_n} + \int_{\alpha_n}^\mu\right\} \chi(t) [tF_1(t, n)]\, dt.$$

Let it be now assumed that $\int_0^t |\phi(t)|\, dt$ has a differential coefficient for $t = 0$, equal to zero; this is the case (see I, § 432) for almost every value of x. We have then $\int_0^t |\phi(t)|\, dt = t\chi_1(t)$, where $\chi_1(t)$ is continuous, and $\chi_1(0) = 0$.

We have now

$$\left|\int_0^{\alpha_n} \chi(t)\, tF_1(t, n)\, dt\right| \leqq \int_0^{\alpha_n} |\chi(t) . tF_1(t, n)|\, dt < M(\alpha_n) \int_0^{\alpha_n} |\chi(t)|\, dt;$$

where $M(\alpha_n)$ is the maximum of $|tF_1(t, n)|$ in the interval $(0, \alpha_n)$; and since $\int_0^{\alpha_n} |\chi(t)|\, dt = \alpha_n \chi(\alpha_n')$, where $0 < \alpha_n' \leqq \alpha_n$, the absolute value of the integral is $< \alpha_n M(\alpha_n) \chi(\alpha_n')$.

If it be assumed that $M(\alpha_n)\alpha_n$ has a finite upper boundary with respect to n, and that α_n converges to zero, as $n \sim \infty$, we have

$$\left|\int_0^{\alpha_n} \chi(t)\, tF_1(t, n)\, dt\right| < \epsilon,$$

for all sufficiently large values of n.

Next, consider $\int_{\alpha_n}^{\mu} \chi(t) . tF_1(t, n)\, dt$, or $\int_{\alpha_n}^{\mu} \frac{\chi(t)}{t} [t^2 F_1(t, n)]\, dt$. Since $\frac{\chi(t)}{t}$ has bounded variation in the interval (α_n, μ), the integral is numerically less than $V_{\alpha_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\}$ multiplied by the maximum of $\left| \int t^2 F_1(t, n)\, dt \right|$ for all intervals interior to (α_n, μ), where $V_{\alpha_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\}$ denotes the total variation of $\frac{\chi(t)}{t}$ in the interval (α_n, μ). Hence

$$\left| \int_{\alpha_n}^{\mu} \chi(t) . tF_1(t, n)\, dt \right| < N(\alpha_n)\, V_{\alpha_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\},$$

where $N(\alpha_n)$ denotes the absolute value of the maximum of $\int t^2 F_1(t, n)\, dt$, for all intervals interior to (α_n, μ). We have also

$$\int_{\alpha_n}^{\mu} \frac{\phi(t)}{t^2}\, dt = \left[\frac{1}{t^2} \int_0^t \phi(t)\, dt\right]_{\alpha_n}^{\mu} + 2 \int_{\alpha_n}^{\mu} \frac{1}{t^3} \left\{ \int_0^t \phi(t)\, dt \right\} dt$$

$$= \frac{1}{\mu} \chi(\mu) - \frac{1}{\alpha_n} \chi(\alpha_n) + 2 \int_{\alpha_n}^{\mu} \frac{1}{t^2} \chi(t)\, dt.$$

Thus
$$\frac{\chi(\mu)}{\mu} - \frac{\chi(\alpha_n)}{\alpha_n} = \int_{\alpha_n}^{\mu} \left\{ \frac{\phi(t)}{t^2} - \frac{2}{t^3} \int_0^t \phi(t)\, dt \right\} dt;$$

and this holds good when μ is replaced by any number t in the interval (α_n, μ).

It follows, employing the theorem in I, § 415, that

$$V_{\alpha_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\} \leqq \int_{\alpha_n}^{\mu} \left| \frac{\phi(t)}{t^2} \right| dt + 2 \int_{\alpha_n}^{\mu} \frac{1}{t^3} \left| \int_0^t \phi(t)\, dt \right| dt$$

$$\leqq \int_{\alpha_n}^{\mu} \left| \frac{\phi(t)}{t^2} \right| dt + 2 \int_{\alpha_n}^{\mu} \frac{1}{t^2} \chi_1(t)\, dt$$

$$< \frac{\chi_1(\mu)}{\mu} - \frac{\chi_1(\alpha_n)}{\alpha_n} + 4 \int_{\alpha_n}^{\mu} \frac{1}{t^2} \chi_1(t)\, dt$$

$$\leqq \frac{\chi_1(\mu)}{\mu} - \frac{\chi_1(\alpha_n)}{\alpha_n} + 4 \left(\frac{1}{\alpha_n} - \frac{1}{\mu} \right) \bar{\chi}_1,$$

when $\bar{\chi}_1$ is the maximum of $\chi_1(t)$ in the interval (α_n, μ).

It follows that $\alpha_n V_{\alpha_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\} \leqq \chi_1(\mu) + 4\bar{\chi}_1 < \epsilon$, provided μ be chosen sufficiently small. We have now

$$\left| \int_{\alpha_n}^{\mu} \chi(t) . tF_1(t, n)\, dt \right| < \frac{N(\alpha_n)}{\alpha_n} \epsilon;$$

and thus, provided $\frac{N(\alpha_n)}{\alpha_n}$ is bounded with respect to n, the integral on the left-hand side is less than an arbitrarily chosen number. If then also $|\alpha_n M(\alpha_n)|$ is bounded, we see that $\int_0^{\mu} \phi(t) F(t, n)\, dt$ converges to zero, as $n \sim \infty$.

The following theorem has been established:

If $F(x'-x, n)$ satisfies the conditions (1), (2) *of* § 290, *for every value of* μ (>0), *and a sequence* $\{\alpha_n\}$ *of positive numbers converging to zero, as* $n \sim \infty$, *can be so determined that, for a sufficiently small fixed number* μ, $\alpha_n M(\alpha_n)$ *and* $\frac{N(\alpha_n)}{\alpha_n}$ *are both less than fixed positive numbers independent of* n, *then* $\int_a^b f(x') F(x'-x, n)\, dx'$ *converges to* $f(x)$, *for all points* x, *interior to* (a, b), *at which* $\int_0^t |f(x+t)+f(x-t)-2f(x)|\, dt$ *has a differential coefficient at* $t=0$, *equal to zero. The number* $M(\alpha_n)$ *denotes the maximum of* $\left| t\frac{\partial F(t,n)}{\partial t}\right|$ *in the interval* $(0, \alpha_n)$, *and* $N(\alpha_n)$ *denotes the absolute value of the maximum of* $\int t^2 \frac{\partial F(t,n)}{\partial t}\, dt$, *for all intervals interior to* (α_n, μ).

It is clear that at any point x, at which $\int_0^t |f(x+t)-f(x)|\, dt$, $\int_0^t |f(x-t)-f(x)|\, dt$ both have, at $t=0$, differential coefficients of which the value is zero, then $\int_0^t |f(x+t)+f(x-t)-2f(x)|\, dt$ has at x the same property. It follows from the theorem given in I, § 432, that this property holds for almost all values of x in the interval (a, b).

It is clear that, in the proof of the above theorem, there may be substituted for $F_1(t, n)$ any function $\psi(t, n)$ which satisfies the same conditions as $F_1(t, n)$ does in the theorem.

Thus we obtain the following theorem, due to Lebesgue (*loc. cit.*):

For any point x *at which* $\int_0^t |\phi(t)|\, dt$ *has a differential coefficient for* $t=0$, *equal to zero,*

$$\int_0^\mu \left[\int_0^t \phi(t)\, dt\right] \psi_1(t, n)\, dt$$

converges to zero as $n \sim \infty$, *provided* $\psi(t, n)$ *satisfies the conditions that, for some sequence* $\{\alpha_n\}$ *of numbers converging to zero,* $\alpha_n M(\alpha_n)$ *and* $\frac{N(\alpha_n)}{\alpha_n}$ *are bounded for all values of* n; *where* $M(\alpha_n)$ *denotes the maximum of* $|t\psi(t, n)|$ *in the interval* $(0, \alpha_n)$ *and* $N(\alpha_n)$ *denotes the absolute value of the maximum of* $\int t^2 \psi(t, n)\, dt$ *for all intervals interior to* (α_n, μ).

297. Let $u(t)$ denote $\frac{1}{t}\int_0^t \{f(x+t)+f(x-t)\}\, dt$, and let it be assumed that $u(t)$ has bounded variation in the interval $(0, \mu)$. For $t=0$, we may take $u(0)=u(+0)$; then $u(t)$ is continuous in the interval $(0, \mu)$. We denote $f(x+t)+f(x-t)$ by $\phi(t)$.

We have

$$\int_0^\mu \phi(t)\, F(t, n)\, dt = \mu u(\mu)\, F(\mu, n) - \int_0^\mu u(t) . tF_1(t, n)\, dt;$$

if it be assumed that $|tF(t, n)| < K$, for all the values of t and n, then since

$$\mu u(\mu)\, F(\mu, n) = \{u(\mu) - u(+0)\}\, \mu F(\mu, n) + u(+0)\, \mu F(\mu, n),$$

we have $$\overline{\lim_{n\sim\infty}} \{\mu u(\mu)\, F(\mu, n) - u(+0)\, \mu F(\mu, n)\} < \eta_\mu,$$

where $\eta_\mu \sim 0$, as $\mu \sim 0$.

Again, considering $\int_0^\mu u(t)\{tF_1(t, n)\}\, dt$, we have

$$\left|\int_0^\mu \{u(t) - u(+0)\}\, tF_1(t, n)\, dt\right| = V_0^\mu \{u(t)\}\, L(\mu),$$

where $L(\mu)$ is the absolute value of the maximum of $\int tF_1(t, n)\, dt$ in intervals contained in the interval $(0, \mu)$.

Since $$\int tF_1(t, \mu)\, dt = [tF(t, \mu)] - \int F(t, n)\, dt,$$

we see that $L(\mu)$ is less than a fixed number independent of n, provided the absolute maximum of $\int F(t, n)\, dt$ for all intervals contained in $(0, \mu)$ is so. Since $\lim\limits_{\mu\sim\infty} V_0^\mu \{u(t)\} \sim 0$, as $\mu \sim 0$, it follows that

$$\overline{\lim_{n\sim\infty}} \left|\int_0^\mu \{u(t) - u(+0)\}\, tF_1(t, n)\, dt\right| < \zeta_\mu,$$

where $\zeta_\mu \sim 0$, as $\mu \sim 0$.

It is now seen that

$$\overline{\lim_{n\sim\infty}} \left|\left\{\int_0 \phi(t)\, F(t, n)\, dt - u(+0)\int_0^\mu F(t, n)\, dt\right\}\right| < \eta_\mu + \zeta_\mu;$$

and assuming that $F(t, n)$, an even function of t, satisfies the conditions of Theorem I, for every interval (μ', μ) when $0 < \mu' < \mu$, the above limit is zero when μ', instead of 0, is the lower limit in the integrals.

We then have $\lim\limits_{n\sim\infty} \int_0^\mu \phi(t)\, F(t, n)\, dt = u(+0) \lim\limits_{n\sim\infty} \int_0^\mu F(t, n)\, dt.$

The following theorem has now been established:

If $F(t, n)$ be an even function of t, and satisfies the conditions of Theorem I, in every interval (μ', μ), $(\mu' > 0)$, and if $|tF(t, n)| < K$, for all values of n, and all values of t in the interval $(0, \mu)$, then $\int_a^b f(x')\, F(x' - x, n)\, dx'$ converges to $u(+0) \lim\limits_{n\sim\infty} \int_0^\mu F(t, n)\, dt$, at any point at which the function

$$u(t) \equiv \frac{1}{t}\int_0^t \{f(x+t) + f(x-t)\}\, dt$$

has bounded variation in some interval $(0, \mu)$, of t.

This theorem is a generalization of a theorem given by de la Vallée Poussin for the case of Fourier's series (see § 345).

THE OSCILLATION OF A SINGULAR INTEGRAL

298. Let it be assumed that $F(t, n) \geqq 0$, for all values of t and n, and that the condition $\lim\limits_{n\sim\infty}\int_{-\mu}^{\mu} F(t,n)\,dt = 1$ is satisfied, together with the conditions (1) and (2) of Theorem II.

If M_μ, m_μ denote the upper and lower boundaries of $f(x')$ in the interval $(x-\mu, x+\mu)$, and U, L are the lower and upper boundaries of M_μ, m_μ as $\mu \sim 0$, the number μ can be so chosen that $M_\mu < M + \eta$, $m_\mu > m - \eta$.

We have

$$\int_{-\mu}^{\mu} f(x+t)\,F(t,n)\,dt < (M+\eta)\int_{-\mu}^{\mu} F(t,n)\,dt > (m-\eta)\int_{-\mu}^{\mu} F(t,n)\,dt.$$

It follows that

$$\overline{\lim_{n\sim\infty}}\int_a^b f(x')\,F(x'-x,n)\,dx' < M+\eta,$$

and

$$\underline{\lim_{n\sim\infty}}\int_a^b f(x')\,F(x'-x,n)\,dx' > m-\eta.$$

Since η becomes arbitrarily small, by choosing μ small enough, we have the following theorem:

If $F(t, n) \geqq 0$, for all values of t, n, and satisfies the condition

$$\lim_{n\sim\infty}\int_{-\mu}^{\mu} F(t,n)\,dt = 1,$$

then provided the function $F(x'-x, n)$ satisfies the conditions (1), (2) *of Theorem II, we have*

$$M \geqq \overline{\lim_{n\sim\infty}}\int_a^b f(x')\,F(x'-x,n)\,dx' \geqq \underline{\lim_{n\sim\infty}}\int_a^b f(x')\,F(x'-x,n)\,dx \geqq m,$$

where M, m are the maximum and minimum of $f(x')$ at the point x.

THE FAILURE OF CONVERGENCE OR OF UNIFORM CONVERGENCE OF THE SINGULAR INTEGRAL

299. *When the function $F(t, n)$ is such that the condition* (*b*) *is not satisfied, so that $\int_{-\mu}^{\mu} |F(t,n)|\,dt$ increases indefinitely as $n \sim \infty$, it is possible to define a function $f(x')$, continuous in (a, b), and such that, at a particular point x, $\int_a^b f(x')\,F(x'-x,n)\,dx'$ does not converge to $f(x)$, as $n \sim \infty$.*

One at least of the two integrals $\int_0^{\mu} |F(t,n)|\,dt$, $\int_{-\mu}^{0} |F(t,n)|\,dt$ is

unbounded; let us assume that the first of these is unbounded. It has been shewn in § 289, that a continuous function $\chi(t)$, such that $\chi(0) = \chi(\mu) = 0$, exists, such that $\int_0^\mu \chi(t) F(t, n)\,dt$ does not converge to zero, as $n \sim \infty$.

Let $f(x') = 0$, in the interval (a, x) and in the interval $(x + \mu, b)$; and let $f(x') = \chi(x' - x)$, in the interval $(x, x + \mu)$.

We have then

$$\int_a^b f(x') F(x' - x, n)\,dx' = \int_x^{x+\mu} f(x') F(x' - x, n)\,dx' = \int_0^\mu \chi(t) F(t, n)\,dt.$$

It follows that $\int_a^b f(x') F(x' - x, n)\,dx'$ does not converge to zero, which is the value of the continuous function $f(x')$ at the point x.

The following theorem will now be established:

If $\int_{-\mu}^{\mu} |F(t, n)|\,dt$ increases indefinitely as $n \sim \infty$, it is possible to define a continuous function $f(x')$ such that $\int_a^b f(x') F(x' - x, n)\,dx'$ converges to $f(x)$, as $n \sim \infty$, at a prescribed point x, but does not converge uniformly in any neighbourhood of x.

It has been shewn in § 289, that it is possible to define a continuous function $\phi(x')$, of which the numerical maximum is M, such that, for a given point x, $\int^b \phi(x') F(x' - x, n)\,dx'$ has a value which exceeds $M\int_a^b |F(x' - x, n)|\,dx' - \epsilon$, where ϵ is arbitrarily assigned. Moreover this function $\phi(x)$ can be so chosen as to be of bounded variation; because it is clear that a function which is constant in each interval of a finite set, and is elsewhere zero, can be taken to be the limit of a sequence of continuous functions of bounded variation.

Also, if $\psi(x')$ be a function which has the value 0 in the interval $(x - h, x + h)$, and is numerically not greater than M, we have

$$\left|\int_a^b \psi(x') F(x' - x, n)\,dx'\right| \leqq MR(h),$$

where $R(h)$ is the maximum value of

$$\int_a^{x-h} |F(x' - x, n)|\,dx' + \int_{x+h}^b |F(x' - x, n)|\,dx'$$

which is finite, on account of the condition (1) of Theorem II. Consider a sequence of intervals no two of which overlap or abut on one another, and such that their end-points have the point x for limiting point. Let their lengths be $4h_1, 4h_2, \ldots 4h_p, \ldots$; and let $x_1, x_2, \ldots x_p, \ldots$ be their middle points. Let k_p be the distance from x of the nearer end of the interval of which the length is $4h_p$.

A function $f_p(x')$, continuous and of bounded variation, can be so determined, that it is numerically $< l_p < 1$, and such that, for $n = n_p$,

$$\left|\int_a^b f_p(x')\, F(x' - x_p, n_p)\, dx'\right| > p + 2R(h_p);$$

the number n_p may be so chosen as to exceed any prescribed integer.

Let $\phi_p(x') = f_p(x')$, in the interval $(x_p - h_p, x_p + h_p)$, and let $\phi_p(x') = 0$ outside the interval $(x_p - 2h_p, x_p + 2h_p)$. This function $\phi_p(x')$ may be so determined as to be continuous and of bounded variation, and such as to satisfy the conditions $|\phi_p(x')| < l_p < 1$, $|\phi_p(x') - f_p(x')| < 1$, in the whole interval $(x_p - 2h_p, x_p + 2h_p)$.

We have then

$$\left|\int_a^b \{\phi_p(x') - f_p(x')\}\, F(x' - x_p, n_p)\right| \leq R(h_p),$$

and therefore

$$\left|\int_a^b \phi_p(x')\, F(x' - x_p, n_p)\, dx'\right| > p + R(h_p).$$

Now let $f(x') = \phi_p(x')$ in each interval $(x_p - 2h_p, x_p + 2h_p)$; and outside all these intervals let $f(x') = 0$. Then

$$\left|\int_a^b f(x')\, F(x' - x_p, n_p)\, dx'\right| > p + R(h_p) - R(2h_p) \geqq p:$$

it follows that $\int_a^b f(x')\, F(x' - x, n_p)\, dx'$ cannot converge uniformly to $f(x)$ in any neighbourhood of x; since the numbers n_p, x_p can be so chosen that the integral increase indefinitely with p. That $f(x')$ may be so defined that $\int_a^b f(x')\, F(x' - x, n)\, dx'$ converges at the point x, to the value zero, may be seen as follows.

The integral is equivalent to $\sum\limits_{p=1}^{\infty} \int_{x_p - 2h_p}^{x_p + 2h_p} \phi_p(x')\, F(x' - x, n)\, dx'$; and the terms of this series are numerically less than those of the series $\sum\limits_{p=1}^{\infty} l_p R(k_p)$. If we take l_p equal to the smaller of the two numbers $\frac{1}{p^2}$, $\frac{1}{p^2 R(k_p)}$, this series is convergent. The series which represents $\int_a^b f(x')\, F(x' - x, n)\, dx'$ therefore converges uniformly with respect to n, and since each term converges to zero, as $n \sim \infty$, it follows that

$$\int_a^b f(x')\, F(x' - x, n)\, dx'$$

converges to zero, the value of $f(x')$ at the point x. It is clear that the point x is a point of continuity of the function $f(x')$; it is an isolated point of non-uniform convergence of the integral.

The above constructions of a series which is non-convergent at a single point, and of a series which although convergent, converges non-uniformly in every neighbourhood of a particular point, are due to Lebesgue (*loc. cit.*).

APPLICATIONS OF THE THEORY

300. As a first application of the preceding theory, let

$$F(x', x, n) = \frac{\{1-(x'-x)^2\}^n}{2\int_0^1 (1-t^2)^n\,dt},$$

when $0 \leqq x' \leqq 1$, and the set G consists of the points of the interval $(0, 1)$.

To show that the conditions of Theorem II are satisfied, we see that if $|x'-x| \geqq \mu$,

$$F(x', x, n) \leq \frac{(1-\mu^2)^n}{2\int_0^1 (1-t^2)^n\,dt} < \frac{\int_0^\mu (1-t^2)^n\,dt}{2\mu\int_0^1 (1-t^2)^n\,dt} < \frac{1}{2\mu};$$

thus the condition (1) is satisfied; we can take $K_\mu = \dfrac{1}{2\mu}$.

Also

$$\int_{\alpha_1}^{\beta_1} F(x', x, n)\,dx' = \frac{\int_{\alpha_1}^{\beta_1}\{1-(x'-x)^2\}^n\,dx'}{2\int_0^1 (1-t^2)^n\,dt} < \frac{(\beta_1-\alpha_1)(1-\mu^2)^n}{2\int_0^1 (1-t^2)^n\,dt} < \frac{(1-\mu^2)^n}{2\dfrac{1}{\sqrt{n}}\left(1-\dfrac{1}{n}\right)^n};$$

since $$\int_0^1 (1-t^2)^n\,dt > \int_0^{\frac{1}{\sqrt{n}}} (1-t^2)^n\,dt > \frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n;$$

it has here been assumed that x is not interior to the interval $(\alpha_1-\mu, \beta_1+\mu)$.

We have $$\lim_{n\sim\infty} \frac{\sqrt{n}\,(1-\mu^2)^n}{2\left(1-\dfrac{1}{n}\right)^n} = \frac{e}{2}\lim_{n\sim\infty}\frac{\sqrt{n}}{(1+\lambda)^n} = 0;$$

where $1+\lambda = \dfrac{1}{1-\mu^2}$. Hence the condition (2) of Theorem II is satisfied. Again, we have

$$\int_0^\mu F(t, n)\,dt = \frac{\int_0^\mu (1-t^2)^n\,dt}{2\int_0^1 (1-t^2)^n\,dt} < \tfrac{1}{2};$$

and writing the integral in the form $\tfrac{1}{2} - \dfrac{\int_\mu^1 (1-t^2)^n\,dt}{2\int_0^1 (1-t^2)^n\,dt}$; we see that the limit of the integral is $\tfrac{1}{2}$.

It now follows from the theorems in §§ 292–294, that

$$\lim_{n\sim\infty}\frac{\int_0^1 [1-(x'-x)^2]^n\,dx'}{2\int_0^1 (1-t^2)^n\,dt}=\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

at any point x interior to $(0, 1)$, at which $f(x)$ is ordinarily discontinuous. It follows also that, in any interval in which $f(x)$ is continuous, the continuity at the end-points being on both sides, the convergence to $f(x)$ is uniform. The function $f(x)$ is in general subject only to the condition that it is summable in $(0, 1)$.

The asymptotic value of $\int_{-1}^1 (1-t^2)^n\,dt$ is $\sqrt{\frac{\pi}{n}}$; hence the limit

$$\lim_{n\sim\infty}\sqrt{\frac{n}{\pi}}\int_0^1 [1-(x'-x)^2]^n\,dx'$$

has the same values as the above limit.

This singular integral was studied by Landau*, in the case in which $f(x)$ is a continuous function, who applied it to obtain a proof of Weierstrass' theorem (§ 159) that a function that is continuous in a given interval can be uniformly approximated to by a sequence of finite polynomials. Since $\int_0^1 [1-(x'-x)^2]^n\,dx'$ is a polynomial of degree $2n$ in x, if $f(x)$ is continuous in the interval $(0, 1)$, then in any interval (a, b), interior to $(0, 1)$, the sequence of polynomials obtained by giving n the values $1, 2, 3, \ldots$ in the expression $\sqrt{\frac{n}{\pi}}\int_0^1 f(x')\,[1-(x'-x)^2]^n\,dx'$ converges uniformly in (a, b) to the value of the continuous function $f(x)$.

The theorem of § 295 may be applied to the function

$$F(t, n)=\sqrt{\frac{n}{\pi}}(1-t^2)^n.$$

We have $$\frac{\partial}{\partial t}[tF(t, n)]=\sqrt{\frac{n}{\pi}}(1-t^2)^{n-1}\{1-(2n+1)t^2\},$$

and thus $tF(t, n)$ increases steadily from $t=0$ to $t=\frac{1}{\sqrt{2n+1}}$ and then steadily diminishes. The total variation of $tF(t, n)$ in the interval $(0, \mu)$ is therefore

$$2\sqrt{\frac{n}{\pi}}\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n-\sqrt{\frac{n}{\pi}}\mu(1-\mu^2)^n,$$

* *Rend. di circ. mat. di Palermo*, vol. xxv (1908), p. 337. The above theory for any summable function was given by Hobson, *Proc. Lond. Math. Soc.* (2), vol. vi (1908), p. 364.

and this is less than $\sqrt{\frac{2}{\pi}}$, whatever value n may have. Thus the conditions of the theorem of § 295, being satisfied, it follows that

$$\sqrt{\frac{n}{\pi}}\int_0^1 f(x')\,[1-(x'-x)^2]^n\,dx'$$

converges to $f(x)$ at every point at which

$$\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt$$

has a differential coefficient at $t=0$, of the value zero, and this condition is satisfied at almost all points of the interval $(0, 1)$.

It has thus been shewn that:

If $f(x)$ be summable in the interval $(0, 1)$, the limit, as $n \sim \infty$, of the sequence of polynomials $\left(\frac{n}{\pi}\right)^{\frac{1}{2}}\int_0^1 f(x')\,[1-(x'-x)^2]^n\,dx'$ is $f(x)$, at any interior point of the interval at which

$$\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt = o(t);$$

and which is the case almost everywhere. The convergence to $f(x)$ is uniform in any interval of continuity of the function, the continuity on both sides at the ends of the interval being presupposed.

301. The limit $\displaystyle\lim_{n\sim\infty}\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\,e^{-n^2(x'-x)^2}\,dx'$

was considered by Weierstrass, and was employed by him to prove his fundamental theorem relating to continuous functions. It will here be assumed that $f(x)$ is summable in every finite interval, and that, outside a certain finite interval $(-A, A)$, it is bounded.

Taking $F(t, n) = \frac{n}{\sqrt{\pi}}e^{-n^2t^2}$, we have, if $t \geqq \mu$, $F(t, n) \leqq \frac{n}{\sqrt{\pi}}e^{-n^2\mu^2}$, and since $\frac{n}{\sqrt{\pi}}e^{-n^2\mu^2}$ has the single maximum $\frac{1}{\mu\sqrt{2\pi}}e^{-\frac{1}{2}}$, we have

$$F(t, n) \leqq \frac{1}{\mu\sqrt{2\pi}}e^{-\frac{1}{2}}, \text{ for } t \geqq \mu.$$

Also, when x is not in the interval $(\alpha'-\mu, \beta'-\mu)$, we have

$$\int_{\alpha'}^{\beta'}\frac{n}{\sqrt{\pi}}e^{-n^2(x'-x)^2}\,dx' < \frac{n}{\sqrt{\pi}}(\beta'-\alpha')\,e^{-n^2\mu^2},$$

and this converges to zero, as $n \sim \infty$, uniformly for all values of x in any finite interval (α, β).

We have

$$\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-n^2(x'-x)^2}\, dx' = \frac{n}{\sqrt{\pi}}\left\{\int_A^{\infty} + \int_{-\infty}^{-A}\right\} f(x')\, e^{-n^2(x'-x)^2}\, dx'$$

$$+ \frac{n}{\sqrt{\pi}}\left\{\int_{-A}^{x-\mu} + \int_{x+\mu}^{A}\right\} f(x')\, e^{-n^2(x'-x)^2}\, dx' + \frac{n}{\sqrt{\pi}}\int_{x-\mu}^{x+\mu} f(x')\, e^{-n^2(x'-x)^2}\, dx';$$

and the first part of the expression is less than a fixed multiple of

$$\left\{\int_{n(A-x)}^{\infty} + \int_{-\infty}^{-n(A+x)}\right\} e^{-t^2}\, dt.$$

For all points x in a fixed interval interior to $(-A, A)$, this converges uniformly to zero, as $n \sim \infty$.

The second part of the above expression converges uniformly to zero, since the conditions of Theorem I are satisfied. Further, we have

$$\int_x^{x+\mu} \frac{n}{\sqrt{\pi}}\, e^{-n^2(x'-x)^2}\, dx' = \frac{1}{\sqrt{\pi}}\int_0^{\mu} n e^{-n^2t^2}\, dt = \frac{1}{\sqrt{\pi}}\int_0^{n\mu} e^{-t'^2}\, dt',$$

and the limit, as $n \sim \infty$, is $\frac{1}{2}$. Similarly the limit of $\int_{x-\mu}^{x} \frac{n}{\sqrt{\pi}}\, e^{-n^2(x'-x)^2}\, dx'$ is $\frac{1}{2}$.

We have now established the following theorem:

If $f(x')$ is summable in every finite interval, and is bounded outside some fixed finite interval, then $\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-n^2(x'-x)^2}\, dx'$ *converges to*

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\}$$

at any point at which $f(x)$ has an ordinary discontinuity, or is continuous. Moreover, in any interval in which $f(x)$ is continuous, the continuity being, at the ends of the interval, on both sides, the convergence is uniform.

Since $\frac{d}{dt}[te^{-n^2t^2}] = e^{-n^2t^2}(1 - 2n^2t^2)$, we see that $te^{-n^2t^2}$ increases steadily up to a maximum at $t = \frac{1}{\sqrt{2}n}$, and then steadily decreases. The total variation of $\frac{n}{\sqrt{\pi}}\, te^{-n^2t^2}$ in the interval $(0, \mu)$ is accordingly

$$\sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}} - \frac{n}{\sqrt{\pi}}\, \mu e^{-n^2\mu^2},$$

which is less than $\sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}}$. Hence, in accordance with the theorem of § 295, we have the theorem that:

$\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-n^2(x'-x)^2}\, dx'$ *converges to $f(x)$ at any point at which*

$$\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\, dt$$

has a differential coefficient at $t = 0$, which has the value zero: and this is the case at almost every point of any finite interval.

If $f(x')$ have an HL-integral in $(-A, B)$, and is bounded outside that interval, since the total variation of $te^{-n^2t^2}$ in an interval (μ, α) is $n(e^{-\mu^2n^2} - e^{-\mu^2\alpha^2})$, for all sufficiently large values of n, and this is less than $\frac{1}{\mu\sqrt{2}}e^{-\frac{1}{2}}$, which is independent of n, it is seen that the condition of the theorem of § 286 is satisfied; therefore:

$\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-n^2(x'-x)^2}\, dx'$ *converges to* $\frac{1}{2}\{f(x+0) + f(x+0)\}$ *at any ordinary point of discontinuity, where $f(x')$ has an HL-integral in $(-A, B)$ and is bounded outside that interval. The convergence to $f(x)$ takes place at all those points of the intervals complementary to the set of points of non-summability of $f(x)$ at which* $\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\, dt$ *has, for $t = 0$, the differential coefficient zero.*

The condition in the above theorems, that $f(x')$ should be bounded, outside some finite interval, may be replaced by a less stringent condition. It can in fact be shewn that it is sufficient that for $|x| > A$, the condition $|f(x)| < e^{qx^2}$ should be satisfied, where A, q are fixed positive numbers.

We have only to consider the part

$$\frac{n}{\sqrt{\pi}}\left\{\int_A^{\infty} + \int_{-\infty}^{-A}\right\} f(x')\, e^{-n^2(x'-x)^2}\, dx'$$

of $\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-n^2(x'-x)^2}\, dx'$. This is less than a fixed multiple of

$$\int_{n(A-x)}^{\infty} e^{\frac{q}{n^2}t^2 - t^2}\, dt + \int_{-\infty}^{-n(A+x)} e^{\frac{q}{n^2}t^2 - t^2}\, dt.$$

For $n > \sqrt{q}$, this has a definite meaning, and it converges to zero, uniformly for all points x in an interval interior to $(-A, A)$, as $n \sim \infty$

Other examples of singular integrals, the convergence of which may be investigated in accordance with the methods here given, are

$$\frac{1}{2}\sqrt{\frac{n}{\pi}}\int_{-\pi}^{\pi} f(x')\left[\cos\frac{x'-x}{2}\right]^n dx',$$

$$\sqrt{\frac{n}{\pi}}\int_0^{\pi} f(x')\left[\sin\frac{x'}{2} - \sin\frac{x}{2}\right]^n dx'.$$

The first of these has been investigated by de la Vallée Poussin*. Other applications of the theory given in the present chapter will be given in later chapters, in connection with the theory of Fourier's series and integrals.

* *Bull. de l'acad. roy. de Belgique* (1908), p. 193.

THE CONVERGENCE OF THE INTEGRALS OF PRODUCTS OF FUNCTIONS

302. The theorems given in §§ 201–213, relating to the conditions that a sequence $\{s_n(x)\}$ should be integrable, have been extended, especially by W. H. Young*, to obtain conditions that an integral $\int_a^x f(x)\, s_n(x)\, dx$ should converge to $\int_a^x f(x)\, s(x)\, dx$, in a finite or infinite interval or cell (a, b). The function $f(x)$ is in general taken to be summable in (a, b), but in some cases it may be less restricted. Theorems of this kind will here be deduced from the general convergence Theorem I, of § 279, and its modifications and extensions.

Theorem I, and its modifications, may be applied to determine sufficient conditions that, if $\{s_n(x', x)\}$ is a sequence of functions all summable in the interval or cell (a, b) of x', for all values of x in a given set of points G, in a domain of any number of dimensions, and if $s(x', x)$ be another such function, then $\int_a^b f(x')\, s_n(x', x)\, dx'$ converges, as $n \sim \infty$, to $\int_a^b f(x')\, s(x', x)\, dx'$, uniformly for all values of x in G, for all functions $f(x')$ which are summable in (a, b), or which belong to one or other of the more restricted classes of functions that have been considered in the modifications of Theorem I.

Let $\Phi(x', n, x) \equiv s(x', x) - s_n(x', x)$; we have then, from Theorem I, the following result:

It is sufficient in order that, for every summable function $f(x')$,

$$\int_a^b f(x')\, s_n(x', x)\, dx'$$

should converge to $\int_a^b f(x')\, s(x', x)\, dx'$, uniformly for all values of x in G, (1), *that, for each pair of values of x and n, $|s(x', x) - s_n(x', x)|$ should, for almost all values of x', not exceed a number K independent of the particular values of x and n; and* (2), *that, for each pair of values of α, β, such that $a \leqq \alpha \leqq \beta \leqq b$, $\int_\alpha^\beta \{s(x', x) - s_n(x', x)\}\, dx'$ shall converge to zero, as $n \sim \infty$, uniformly for all values of x in G.*

It is sufficient for bounded convergence for all values of x in G, if condition (1) is satisfied and condition (2) is replaced by (2′), that the integral converges to zero for each value of x. This theorem holds good when x' is a point in a cell of any number of dimensions, integration over the cell replacing integration over the linear interval (a, b); (α, β) will be replaced by a cell contained in the cell (a, b).

* *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 463.

In case G consists of a single point, we have as sufficient conditions that

$$\lim_{n\sim\infty}\int_a^b f(x')\, s_n(x')\, dx' = \int_a^b f(x')\, s(x')\, dx',$$

the conditions (1), that $|s(x') - s_n(x')| < K$, for all values of n and x', and (2), that $\lim_{n\sim\infty}\int_a^\beta s_n(x')\, dx' = \int_a^\beta s(x')\, dx'$, for each interval (or cell) contained in the interval (or cell) (a, b). These conditions are both satisfied, in particular, if $s_n(x')$ converges boundedly to $s(x')$.

In case $s_n(x', x)$ converges uniformly to $s(x', x)$, in G, the condition (1) of the above theorem being assumed to hold, it has been shewn in § 203 that the condition (2) must be satisfied.

We thus obtain the following theorem:

If, in the interval or cell (a, b), the sequence $\{s_n(x', x)\}$ converges to $s(x', x)$, uniformly for all values of x in a given set G, of one or more dimensions, and if the condition is satisfied that $|s(x', x) - s_n(x', x)| < K$, a number independent of x and n, for almost all values of x', then $\int_a^b f(x')\, s_n(x', x)\, dx'$ converges uniformly in G to $\int_a^b f(x')\, s(x', x)\, dx'$, where $f(x')$ is any function summable in (a, b).

In particular $\lim_{n\sim\infty}\int_a^b f(x')\, s_n(x')\, dx' = \int_a^b f(x')\, s(x')\, dx'$, where $f(x')$ is any function summable in (a, b), provided $\{s_n(x')\}$ converges almost everywhere to $s(x')$, and $|s(x') - s_n(x')|$ is less than a fixed number K, independent of n and x'.

This theorem also follows directly from the theorem in § 203, since $|f(x)\{s(x', x) - s_n(x', x)\}|$ is less than the summable function $K\,|f(x)|$.

These theorems hold good when the interval or cell (a, b) is infinite, provided $|f(x)|$ is summable in (a, b).

303. Considering the case in which $\{f(x')\}^q$ is summable in (a, b) for some value of $q > 1$, and $\{s(x', x)\}^{\frac{q}{q-1}}$, $\{s_n(x', x)\}^{\frac{q}{q-1}}$ are summable in (a, b) for each value of x, we obtain from § 282 the following result:

It is sufficient, in order that $\int_a^b f(x')\, s_n(x', x)\, dx'$ should converge to $\int_a^b f(x')\, s(x', x)\, dx'$, uniformly, or boundedly, for all values of x in G, for all functions $f(x')$ such that $|f(x')|^q$, for some value of q (> 1), is summable in (a, b), that

(1 *a*) $\int_a^b |s(x', x) - s_n(x', x)|^{\frac{q}{q-1}} dx'$ *should not exceed a fixed number $K^{\frac{q}{q-1}}$, independent of n and x', and also that the condition (2) or (2*) be satisfied as regards the convergence of $\int_a^b s_n(x', x)\, dx'$ to $\int_a^b s(x', x)\, dx'$.*

The interval (a, b) may in this case also be replaced by a cell of any number of dimensions:

If we employ the results obtained in § 212, we obtain the following theorem:

If $s_n(x)$ converges almost everywhere in (a, b) to $s(x)$, and the condition is satisfied that $\int_a^b |s_n(x)|^p dx < K$, where K is independent of n, for some value of $p > 1$, then $\int_a^b f(x) s_n(x) dx$ converges to $\int_a^b f(x) s(x) dx$, where $f(x)$ is any function such that $|f(x)|^{\frac{p}{p-1}}$ is summable in (a, b).

304. We find also the following results, by employing the theorems in §§ 283–285:

It is sufficient, in order that $\int_a^b f(x') s_n(x', x) dx'$ should converge to $\int_a^b f(x') s(x', x) dx'$, uniformly, or boundedly, for all values of x in G, for all bounded and summable functions $f(x')$, that

(1 *b*) *$\int_a^b |s(x', x) - s_n(x', x)| dx'$ does not exceed a fixed number K, independent of n and x, and that, for every measurable set e contained in (a, b) $\int_{(e)} \{s(x', x) - s_n(x', x)\} dx'$ should converge to zero, as $n \sim \infty$, uniformly, or boundedly, for all values of x in G.*

It is sufficient, in order that $\int_a^b f(x') s_n(x', x) dx'$ should converge to $\int_a^b f(x') s(x', x) dx'$, uniformly, or boundedly, for all values of x in G, for every function $f(x')$ which has only ordinary discontinuities, that

(1 *c*) *$\int_a^b | s(x', x) - s_n(x', x) | dx$ should not exceed a number K, independent of n and x, and further that the condition* (2) *or* (2*), *of* § 279, *be satisfied, as the case may be.*

It is here assumed that (a, b) is essentially a linear interval.

It is sufficient, in order that $\int_a^b f(x') s_n(x', x) dx'$ should converge to $\int_a^b f(x') s(x', x) dx'$, uniformly, or boundedly, for all values of x in G, for all functions $f(x')$ of bounded variation in the linear interval (a, b), that

(1 *d*) *$\left| \int_\alpha^\beta \{s(x', x) - s_n(x', x)\} dx' \right|$ does not exceed a fixed number M, independent of α, β, n, and x, when $a \leq \alpha \leq \beta \leq b$; and further that the condition* (2) *or* (2*) *be satisfied.*

It will be observed that, in case $s_n(x', x)$ converges to $s(x', x)$ uniformly for all the values of x' and x, both conditions of the theorem are satisfied.

305. Taking the function $\Phi(x', x, n)$ of Theorem 1, let it now be assumed that the set G consists of the points of the interval (a, b), for which $\int_a^b \Phi(x', x, n) f(x')\, dx'$ is considered. Let $\Phi(x', x, n)$ be defined by

$$\Phi(x', x, n) = s(x') - s_n(x'), \text{ for } x' \leqq x, \text{ and } \Phi(x', x, n) = 0, \text{ for } x' > x;$$

and let the Theorem 1 be applied to this function; there may be a set of points x', of measure zero at which the definition of $\Phi(x', x, n)$ does not apply.

The function $\Phi(x', x, n)$ is taken to satisfy the conditions

$$|s(x') - s_n(x')| < K,$$

for all values of n, and all (or almost all) values of x' in (a, b); and further that $\int_a^x \{s(x') - s_n(x')\}\, dx'$ converges to zero as $n \sim \infty$, uniformly (or more generally boundedly) for all values of x in (a, b).

We obtain thus the following theorem:

If $f(x)$ be summable in (a, b), and $|s(x) - s_n(x)|$ is bounded for all the values of n and x (a set of points of measure zero being possibly excepted), and if $\int_a^x s_n(x')\, dx'$ converges uniformly, or boundedly, in (a, b) to $\int_a^x s(x')\, dx'$, then $\int_a^x f(x') s_n(x')\, dx$ converges uniformly, or boundedly, as the case may be, in (a, b), to $\int_a^x f(x') s(x')\, dx'$. If the interval is (a, ∞) the theorem holds provided $f(x)$ is absolutely summable in (a, ∞), the convergence of $\int_a^x f(x) s_n(x)\, dx$ to $\int_a^x f(x) s(x)\, dx$ being then uniform, or bounded, in any finite interval.

In case $s_n(x)$ converges to $s(x)$ almost everywhere, and so that, at the points of convergence, either (1), $|s(x) - s_n(x)| < K$, at all the points of convergence, or (2), $|s_n(x)|$ is bounded, it is known (see § 204) that $\int_a^x s_n(x)\, dx$ converges uniformly in any finite interval to $\int_a^x s(x)\, dx$. We thus obtain the following theorem:

If $s_n(x)$ converges boundedly to $s(x)$ (with the possible exception of points of a set of measure zero which may be disregarded), and $f(x)$ be absolutely summable in a finite or infinite interval, then $\int_a^x f(x) s_n(x)\, dx$ converges uniformly to $\int_a^x f(x) s(x)\, dx$ in any finite interval. The same result holds if $s_n(x)$ converges to $s(x)$ so that $|s(x) - s_n(x)|$ is bounded for all values of n, and almost all values of x; provided $f(x) s(x)$ is absolutely summable. If, in either case, $s(x) - s_n(x) \geqq 0$, for all values of n and x, it is sufficient that $f(x)$ should be summable in every finite interval, and bounded outside some finite interval (a, A).

306. By applying the modification of Theorem I, given in § 284, we obtain the following theorem:

If $f(x)$ have only ordinary discontinuities, and be absolutely summable in the finite or infinite interval (a, b), and if $\int_a^b |s(x) - s_n(x)|\, dx$ is bounded as n varies, and $\int_a^x s_n(x)\, dx$ converges to $\int_a^x s(x)\, dx$ uniformly in (a, b), or in case $b = \infty$, in each finite interval (a, A), then $\int_a^x f(x)\, s_n(x)\, dx$ converges uniformly to $\int_a^x f(x)\, s(x)\, dx$, in (a, b), or when $b = \infty$, in a finite interval (a, A); it being assumed that $f(x)\, s(x)$ is summable in (a, b).

Since $\int_a^b |s(x) - s_n(x)|\, dx \leqq \int_a^b |s(x)|\, dx + \int_a^b |s_n(x)|\, dx$ it follows that, if $\int_a^b |s_n(x)|\, dx$ is bounded in (a, b), and $|s(x)|$ is summable, then

$$\int_a^b |s(x) - s_n(x)|\, dx$$

is bounded. We have therefore the following theorem:

If $f(x)$ have only ordinary discontinuities in (a, b), and if $s(x)$, $f(x)\, s(x)$, $f(x)\, s_n(x)$ be absolutely summable in the finite or infinite interval, then if $\int_a^b |s_n(x)|\, dx$ is bounded and $\int_a^x s_n(x)\, dx$ converges either uniformly, or not, to $\int_a^x s(x)\, dx$, in each finite interval (a, A), $\int_a^x f(x)\, s_n(x)\, dx$ converges uniformly, or boundedly, as the case may be, to $\int_a^x f(x)\, s(x)\, dx$ in any finite interval (a, A) contained in (a, b).

A very similar theorem has been given by W. H. Young (*loc. cit.*) in which $f(x)$ is taken to be bounded as well as to have only ordinary discontinuities.

307. Next, let the Theorem I (a) of § 282 be employed, in the case in which the set G consists of the points of the finite, or infinite, interval (a, b). Let $\Phi(x', x, n)$ have the value $s(x') - s_n(x')$, when $x' \leqq x$, and let it have the value 0, when $x' > x$. It then follows that, $|f(x)|^q$, for some value of $q > 1$, being summable, $\int_a^x f(x')\, s_n(x')\, dx'$ converges uniformly in any finite interval (a, A) to $\int_a^x f(x')\, s(x')\, dx'$, provided the conditions are satisfied that $\int_a^b |s(x') - s_n(x')|^{\frac{q}{q-1}}\, dx'$ exists and is less than a fixed number independent of n, and further provided that $\int_a^x s_n(x')\, dx'$

converges uniformly to $\int_a^x s(x')\,dx'$ in (a, b), or in case b is infinite, in each finite interval (a, A).

It has been shewn in § 212 that this last condition is satisfied if either (1), $|s(x)|^{\frac{q}{q-1}}$ is summable in (a, b) and $\int_a^b |s(x') - s_n(x')|^{\frac{q}{q-1}}\,dx$ exists, and is bounded; where $s_n(x')$ converges almost everywhere to $s(x')$; or (2), if $\int_a^b \{s_n(x')\}^{\frac{q}{q-1}}\,dx$ exists and is bounded as n varies, $s_n(x')$ converging almost everywhere to $s(x')$. On changing x' into x, we obtain the following theorem:

If $s_n(x)$ converges almost everywhere in the interval (a, b), where b may be ∞, to $s(x)$, then, if $f(x)$ be any function such that $|f(x)|^q$, where $q > 1$, is summable in (a, b), $\int_a^x f(x)\,s_n(x)\,dx$ converges to $\int_a^x f(x)\,s(x)\,dx$, uniformly in (a, b), or if b is infinite, in any interval (a, A), provided either (1), $\{s(x)\}^{\frac{q}{q-1}}$ is summable in (a, b), and $\int_a^b \{s(x) - s_n(x)\}^{\frac{q}{q-1}}\,dx$ exists, and is bounded as n varies, or (2), if $\int_a^b \{s_n(x)\}^{\frac{q}{q-1}}\,dx$ exists, and is bounded.

The theorem holds also when it is not assumed that $s_n(x)$ converges to $s(x)$, provided $|f(x)|^q$, $|s(x)|^{\frac{q}{q-1}}$, $|s_n(x)|^{\frac{q}{q-1}}$ are summable, $\int_a^b |s(x) - s_n(x)|^{\frac{q}{q-1}}\,dx$ is bounded as n varies, and that $\int_a^x s_n(x)\,dx$ converges to $\int_a^x s(x)\,dx$, uniformly in (a, b), or in case $b = \infty$, in each finite interval (a, A).

The first part of this theorem was given in different forms by Lebesgue* and W. H. Young†, for the case $q = 2$.

308. In the theorem of § 285, let $\Phi(x', x, n)$ denote $s(x') - s_n(x')$, or zero, according as $x \leqq x'$, or $x > x'$, when the set G, the field of x, consists of the interval (a, b). The function $f(x')$ being of bounded variation in (a, b), the conditions to be satisfied are that $\left|\int_a^x \{s(x') - s_n(x')\}\,dx'\right|$ is bounded for all values of x in (a, b), and for all values of n, and that for each value of x it converges to zero uniformly, or boundedly. These conditions will be satisfied if $\int_a^x s_n(x')\,dx'$ converges boundedly, or uniformly, to $\int_a^x s(x')\,dx'$. We have thus the following theorem:

If $\int_a^x s_n(x)\,dx$ converges uniformly, or boundedly, to $\int_a^x s(x)\,dx$ in the

* *Annales de Toulouse* (3), vol. I (1909), p. 50.

† *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 469.

interval (a, b), *and* $f(x)$ *be any function of bounded variation in* (a, b), *then* $\int_a^x s_n(x) f(x)\, dx$ *converges uniformly, or boundedly, to* $\int_a^x s(x) f(x)\, dx$, *in the interval* (a, b).

In the case of an infinite interval (a, ∞), it must be assumed that the total variation of $f(x)$ in (a, A) has a finite upper limit, as A is increased indefinitely.

We have

$$\left|\int_A^{A'} \{s(x) - s_n(x)\} f(x)\, dx - f(A)\int_A^{A'} \{s(x) - s_n(x)\}\, dx\right| < M \,.\, V_A^{A'} f(x),$$

where M denotes the upper boundary of $\left|\int_\alpha^\beta \{s(x) - s_n(x)\}\, dx\right|$ for all intervals (α, β) in (A, A'). By choosing A large enough $V_A^{A'} f(x) < \epsilon$, for all vlaues of A'; and if $f(x)$ converges to zero, as $x \sim \infty$, A may be chosen so small that $|f(A)| < \epsilon$. In this case $\left|\int_A^{A'} \{s(x) - s_n(x)\} f(x)\, dx\right|$ is less than a fixed multiple of ϵ; it being assumed that $\left|\int_a^x \{s(x') - s_n(x')\}\, dx'\right|$ is bounded with respect to (n, x) in the whole interval (a, ∞). Since ϵ is arbitrary, the theorem holds for the case of the infinite interval.

If instead of the condition that $f(x)$ converges to zero, as $x \sim \infty$, it be assumed that $\int_a^\infty s(x)\, dx$ exists, and that the convergence of $\int_a^x s_n(x)\, dx$ to $\int_a^x s(x)\, dx$ is uniform in (a, ∞), the result will also follow. Thus:

The above theorem holds for an infinite interval (a, ∞) *provided either* (1), $f(x)$ *converges to zero as* $x \sim \infty$, *or* (2), $\int_a^x s_n(x)\, dx$ *converges uniformly to* $\int_a^x s(x)\, dx$, *in* (a, ∞). *The convergence of* $\int_a^x s_n(x) f(x)\, dx$, *in* (a, ∞), *to* $\int_a^x s(x) f(x)\, dx$ *is bounded, and is uniform in each finite interval, in case* $\int_a^x s_n(x)\, dx$ *converges uniformly in each finite interval to* $\int_a^x s(x)\, dx$.

309. Let α denote a parameter which is confined to have values in some set G, of points in one or more dimensions. Let $s_n(x, \alpha)$ be positive and steadily diminishing, as x increases in (a, ∞), for each value of α and each value of n, and let $|s_n(a, \alpha)|$ be less than a fixed number A, independent of n and α. Let λ_n be a divergent sequence of positive numbers, and let $f(x)$ be summable in the infinite interval (a, ∞). Further, let it be assumed that, in any fixed finite interval, $s_n(x, \alpha)$ converges to $s(x, \alpha)$ for each value of x in the interval, uniformly with respect to α.

Taking the theorem in § 279, let $\Phi(x, a, n)$ have the value

$$s_n(x, a) - s(x, a),$$

when $a \leqq x \leqq \lambda_n$, and let it have the value zero when $\lambda_n < x$.

In accordance with the hypotheses $|\Phi(x, a, n)|$ is bounded with respect to (x, a, n), and thus the condition (1) of the general theorem in § 279 is satisfied. Again $\int_{a_1}^{b_1} \{s_n(x, a) - s(x, a)\}\, dx$, for each pair of values of a_1 and b_1 in (a, ∞), converges to zero, as $n \sim \infty$, uniformly for all the values of a; since $|s_n(x, a)| < |s_n(d, a)| < K$ (see § 203), thus the condition (2) is satisfied in any finite interval.

The total variation of $\Phi(x, a, n)$ in the interval (a, ∞) is

$$s_n(a, a) - s(a, a),$$

which is less than a fixed number independent of n and a. It thus appears that all the conditions of the last theorem in § 281 are satisfied.

The following theorem has been established:

If (1), $s_n(x, a)$ *is positive for all values of* n, x, a, *and steadily decreases as* x *increases in the interval* (a, ∞), *for each value of* n, *and each value of the parameter* a *in some set of points of one or more dimensions, and* (2), *if* $\int_a^\infty f(x)\, dx$ *is convergent, and* (3), $s_n(x, a)$ *converges to* $s(x, a)$ *for each value of* x, *uniformly for all the values of the parameter, and if* $\{\lambda_n\}$ *be a divergent sequence of positive numbers, then* $\int_a^{\lambda_n} f(x)\, s_n(x, a)\, dx$ *converges to*

$$\int_0^\infty f(x)\, s(x, a)\, dx,$$

as $n \sim \infty$, *uniformly with respect to* a.

In case there is no parameter, which is equivalent to taking the set of points to which a belongs to be a single point, we have* the following theorem:

If $s_n(x)$ *is positive in* (a, ∞), *for all values of* n, *and decreases steadily as* x *increases, for each fixed value of* n, *and if* $\int_0^\infty f(x)\, dx$ *exists, then*

$$\lim_{n \sim \infty} \int_a^{\lambda_n} f(x)\, s_n(x)\, dx = \int_0^\infty f(x)\, s(x)\, dx,$$

where $s_n(x)$ *converges to* $s(x)$ *for each value of* x, *and* $s_n(a)$ *is less than a fixed number independent of* n, *and* $\{\lambda_n\}$ *is a divergent sequence.*

* See Bromwich's *Theory of Infinite Series*, p. 443. In Bromwich's statement it is postulated that the convergence of $s_n(x)$ to $s(x)$ is uniform in any fixed interval. This assumption is unnecessarily restricted, since $\{s_n(x)\}$ is monotone for each value of x.

EXAMPLES

(1) Consider $\int_0^c x^p (1+x)^{-2} \log x \, dx$, where $p + 1 > 0$. If $0 < c < 1$, the series

$$1 - 2x + 3x^2 - \ldots$$

converges uniformly to $(1+x)^{-2}$, also $x^p \log x$ is bounded in the interval $(0, c)$. Thus $\int_0^c x^p (1+x)^{-2} \log x \, dx$ may be obtained by substituting the expansion and integrating term by term.

Next consider $\int_c^1 x^p (1+x)^{-2} \log x \, dx = \int_0^{1-c} (1-x')^p (2-x')^{-2} \log (1-x') \, dx'$.

(2) Consider $$\int_0^1 (1+x)^{-1+q} x^p (\log x)^q \, dx.$$

310. In the theorem of § 286, let $\Phi(x', x, n) = s(x') - s_n(x')$, for $x' \leqq x$, and $\Phi(x', x, n) = 0$, for $x' > x$, where the set G, the field of x, is taken to be the interval (a, b). In accordance with the condition (2) or (2*), $\int_a^x s_n(x') \, dx'$ converges uniformly, or boundedly, to $\int_a^x s(x') \, dx'$; also in accordance with condition (1), $| s(x') - s_n(x') |$ is bounded for all values of n and x' (in the interval (a, b)). If it be assumed that $s_n(x')$ converges everywhere to $s(x')$, and that $V_a^b \, s_n(x')$ is finite, and bounded for all values of n, then it can easily be shewn that $V_a^b \, s(x')$ is finite, and consequently $V_a^b \{s(x') - s_n(x')\}$ is bounded for all values of n. We have accordingly the following theorem:

If in a finite interval (a, b), a sequence $\{s_n(x)\}$ converges to $s(x)$, and $| s(x) - s_n(x) |$ is bounded for all n and x, and consequently $\int_a^x s_n(x) \, dx$ converges uniformly to $\int_a^x s(x) \, dx$, and if $V_a^b s_n(x)$ is finite, and bounded for all values of n, then if $f(x)$ be any function which has an HL-integral in (a, b), $\int_a^x f(x) \, s_n(x) \, dx$ converges uniformly to $\int_a^x f(x) \, s(x) \, dx$.

In particular, if the functions $s_n(x)$ are all monotone (increasing or diminishing) in the interval (a, b), $s_n(a)$, $s_n(b)$ are bounded, $| s_n(x) |$ is then bounded for all values of n and x, and it then follows that $\int_a^x s_n(x) \, dx$ converges uniformly to $\int_a^x s(x) \, dx$.

We therefore have the following theorem:

If in any finite interval (a, b), $s_n(x)$ is monotone in the interval (a, b) (increasing or diminishing) for all values of n, and $s_n(a)$, $s_n(b)$ are numerically less than fixed numbers independent of n, and $s_n(x)$ converges everywhere to $s(x)$, then, if $f(x)$ be any function which has an HL-integral in (a, b),

$\int_a^x f(x)\, s_n(x)\, dx$ *converges uniformly to* $\int_a^x f(x)\, s(x)\, dx$. *The theorem also holds for an infinite interval* (a, ∞), *it being assumed that* $\lim\limits_{x \sim \infty} s_n(x)$ *is bounded for all values of* n.

The extension to the case of an infinite interval is made by an application of the mean value theorem.

If we apply the theorem of § 287, to the case in which G consists of the points of the interval (a, b), and $\Phi(x', x, n) = s(x') - s_n(x')$, for $x' \leqq x$, and $\Phi(x', x, n) = 0$ when $x' > x$, we obtain the following theorem:

Let $f(x')$ *have a D-integral in the finite interval* (a, b), *and let it be assumed that* $s_n(x')$ *converges to* $s(x')$ *everywhere in* (a, b), *and that*

$$\int_a^b \left| \frac{d}{dx'} \{s(x') - s_n(x')\} \right| dx'$$

exists and is less than some number K, *independent of* n, *and that* $s_n(x')$, $s(x')$ *are, for each value of* n, *of bounded variation in* (a, b), *then* $\int_a^x f(x')\, s_n(x')\, dx'$ *converges, uniformly in* (a, b), *to* $\int_a^x f(x')\, s(x')\, dx'$.

311. Instead of Theorem I, of § 279, the following theorem is sometimes useful for application:

If $\Phi(x', x, n)$, $f(x')$ *are such that* $f(x')\, \Phi(x', x, n)$ *is summable for each value of* n, *and for each value of* x, *in* G, *and if*, (1), $\left| \int_{(E)} f(x')\, \Phi(x', x, n)\, dx' \right| < \epsilon$, *when* ϵ *is arbitrarily chosen, provided* $m(E) < \eta_\epsilon$, $n > N_\epsilon$, *where* η_ϵ *converges to zero with* ϵ, *whatever value* x *may have, in* G, *and if*, (2), $\int_\alpha^\beta \Phi(x', x, n)\, dx'$ *converges to zero, as* $n \sim \infty$, *uniformly for all values of* x, *in* G, *whatever values* α, β *may have, such that* $a \leqq \alpha < \beta \leqq b$, *and if*, (3), $\left| \int_{(E)} \Phi(x', x, n)\, dx' \right| < \epsilon$, *provided* $n > N_\epsilon'$ *and* $m(E) < \eta_\epsilon'$; *then* $\int_a^b f(x')\, \Phi(x', x, n)\, dx'$ *converges to zero, as* $n \sim \infty$, *uniformly for all values of* x *in* G. *Further the integral may be taken over any measurable set* H, *in* (a, b), *instead of over the whole interval.*

Let N be a fixed positive number, and let $f(x') = f_N(x') + \phi_N(x')$, where $f_N(x') = f(x')$, $\phi_N(x') = 0$, when $|f(x')| \leqq N$, and $\phi_N(x') = f(x')$, $f_N(x') = 0$, when $|f(x')| > N$. A function $\psi_N(x')$ having only a finite set of values, all in the interval $(-N, N)$, can be so defined that

$$0 \leqq f_N(x') - \psi_N(x') < \eta,$$

and $\psi_N(x') = 0$, when $f_N(x') = 0$; where η is an arbitrarily chosen positive number.

We have now

$$\int_a^b f(x')\,\Phi(x', x, n)\,dx' = \int_a^b \psi_N(x')\,\Phi(x', x, n)\,dx'$$
$$+ \int_a^b \{f_N(x') - \psi_N(x')\}\,\Phi(x', x, n)\,dx' + \int_{(E_N)} f(x')\,\Phi(x', x, n)\,dx',$$

where E_N is the set of points at which $\phi_N(x') \neq 0$. The number N may be so chosen that the absolute value of the third integral on the right-hand side is $< \zeta$, for $n > n_\zeta$, and for all values of x in G.

We have $\displaystyle\int_a^b \psi_N(x')\,\Phi(x', x, n)\,dx' = \Sigma c \int_{(e_c)} \Phi(x', x, n)\,dx'$,

where the numbers c are the values, finite in number, of $\psi_N(x')$, and e_c is the set of points at which $\psi_N(x') = c$. Each set e_c can be enclosed in a set of intervals of which the total measure is $< m(e_c) + \eta_c'$, and a finite set Δ_c of these intervals can be so chosen that the measure of the remainder of them is arbitrarily small. The set e_c consists of a set $e_c^{(1)}$, contained in Δ_c, and of a set $e_c^{(2)}$ in the remaining intervals; also $m(\Delta - e_c) < \eta_c'$.

Thus $\displaystyle\int_{(e_c)} \Phi(x', x, n)\,dx' = \left\{\int_{(\Delta_c)} - \int_{(\Delta_c - e_c^{(1)})} + \int_{(e_c^{(2)})}\right\} \Phi(x', x, n)\,dx'$.

Since $\displaystyle\int_{(\Delta_c)} \Phi(x', x, n)\,dx'$ converges to 0, as $n \sim \infty$, uniformly for all x in G, and since $m(\Delta_c - e_c^{(1)})$, $m(e_c^{(2)})$ are arbitrarily small, it follows that $\displaystyle\left|\int_{(\Delta)} \Phi(x', x, n)\,dx'\right| < \frac{\eta}{rc}$, provided n is greater than some number n_c, where r denotes the number of values of $\psi_N(x')$.

Since this holds for each value of c, we have

$$\left|\int_a^b \psi_N(x')\,\Phi(x', x, n)\,dx'\right| < \eta,$$

provided n is greater than $\bar{n}$, the greatest of all the numbers n_c. We have further

$$\left|\int_a^b \{f_N(x') - \psi_N(x')\}\,dx'\right| < \eta \int_a^b |\,\Phi(x', x, n)\,|\,dx'.$$

It will be shewn that it follows from the conditions (2), (3), of the theorem that $\displaystyle\int_a^b |\,\Phi(x', x, n)\,|\,dx'$ is less than a fixed finite number A, for all values of x and n.

Since $\displaystyle\left|\int_{(E)} \Phi(x', x, n)\,dx'\right| < \epsilon$, for all sets E such that $m(E) < \eta_\epsilon'$, and for $n > N_\epsilon'$, we have

$$\left|\int_{(E)} \Phi^+(x', x, n)\,dx'\right| = \left|\int_{(E_1)} \Phi^+(x', x, n)\,dx'\right| < \epsilon,$$

for $n > N_\epsilon'$, where $\Phi^+(x', x, n)$ is the function which is equal to $\Phi(x', x, n)$ when this latter function is $\geqq 0$, and is otherwise zero; E_1 denotes that

part of E in which $\Phi^+(x', x, n)\,dx' > 0$; this set E_1 depends on x but its measure cannot exceed η_ϵ'.

Now divide the interval (a, b) into s parts, each of length $< \eta_\epsilon'$; we see that $\int_a^b \Phi^+(x', x, n)\,dx' < r\epsilon$, provided $n > N_\epsilon'$, and for all values of x in G. Therefore $\int_a^b \Phi^+(x', x, n)\,dx'$ is, for all values of n and x, less than some fixed number, when, if necessary, a finite set of values of n is rejected. The similar property can be shewn to hold for the corresponding function $\Phi^-(x', x, n)$. Therefore $\int_a^b |\Phi(x', x, n)|\,dx'$ cannot exceed a fixed number A, and hence

$$\left|\int_a^b \{f_N(x') \quad \psi_N(x')\}\,\Phi(x', x, n)\,dx'\right| < A\eta.$$

Lastly, we have $\left|\int_{(e_c^{(2)})} \Phi(x', x, n)\,dx'\right| < \epsilon, < \frac{\eta}{rc}$, for all sufficiently large values of n, whatever value x has, in G. The same holds for $(\Delta - e_c^{(1)})$.

It has now been shewn that $\int_a^b f(x')\,\Phi(x', x, n)\,dx'$ is in absolute value less than an arbitrarily chosen number, provided n exceeds some value dependent on that number, whatever value x may have, in G.

Let $\Phi(x', x, n) = s(x', x) - s_n(x', x)$; we have then the following theorem:

If $f(x')\,s(x', x)$ and $f(x')\,s_n(x', x)$ are summable in (a, b), for all values of n, and for all values of the parameter x, in G, and if $\int_\alpha^\beta s_n(x', x)\,dx'$ converges to $\int_\alpha^\beta s(x', x)\,dx'$, for each pair of values of (α, β) in (a, b), uniformly for all points x in G, and if

$$\lim_{\substack{n\sim\infty \\ m(E)\sim 0}} \int_{(E)} f(x')\,\{s_n(x', x) - s(x', x)\}\,dx' = 0,$$

$$\lim_{\substack{n\sim\infty \\ m(E)\sim 0}} \int_{(E)} \{s_n(x', x) - s(x', x)\}\,dx' = 0,$$

uniformly for all x in G: then

$$\lim_{n\sim\infty} \int_a^b f(x')\,s_n(x', x)\,dx' = \int_a^b f(x')\,s(x', x)\,dx',$$

and the interval (a, b) may be replaced by any measurable set of points in (a, b).

The particular case of this theorem when G consists of a single point, so that the parameter x may be omitted, was established otherwise by W. H. Young. In that case the conditions are simplified, because $\lim_{m(E)\sim 0} \int_{(E)} f(x')\,s(x')\,dx'$ and $\lim_{m(E)\sim 0} \int_{(E)} s(x')\,dx'$ are both zero.

CHAPTER VIII

TRIGONOMETRICAL SERIES

312. The theory of the representation of functions of a real variable by means of series of cosines and sines of multiples of the variable is of the highest importance, not only on account of the fact that such mode of representation is at present an indispensable tool in the various branches of Mathematical Physics, but also because this theory has exercised the most far-reaching influence upon the development of modern Mathematical Analysis. Historically, the questions which have arisen in connection with this theory have influenced the development of the theory of functions of a real variable to an extent which is comparable with the degree in which the theory of functions in general has been affected by the theory of power series. The theory of sets of points, which led later to the abstract theory of aggregates, arose directly from questions connected with trigonometrical series. The precise formulation by Riemann of the conception of the definite integral, and the gradual development of the modern notion of a function as existent independently of any special mode of representation by an analytical expression, are further examples of the results of the study of the properties of these series upon Mathematical Analysis.

It is a significant fact that the theory of this mode of representation of a function had its origin in the attempt to investigate the form of a stretched string in a state of vibration. The problem of the expansion of the reciprocal of the distance between two planets in a series of cosines of multiples of the angle between their radii vectores led to an independent development* of the theory of trigonometrical series. The discussions which arose in connection with the first of these problems were, however, of much greater importance in the history of the development of the theory of functions; they form the first stage in the development of what is known as the theory of Fourier's series, in intimate connection with which the modern theory of functions of real variables had its origin.

THE PROBLEM OF VIBRATING STRINGS

313. The first general solution of the differential equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, which determines the form of a string vibrating transversely, was given by d'Alembert† in the form $y = f(x + at) + \phi(x - at)$. He further shewed

* The importance of this fact has been emphasized by H. Burkhardt in his work "Entwickelungen nach oscillirenden Functionen," published as a *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. x (1901), and later.

† *Memoirs of the Berlin Academy*, 1747, p. 214.

that, if $x = 0$, $x = l$, represent the fixed ends of the string, the form of the string at any time t is representable by $y = f(at + x) - f(at - x)$, where the function $f(z)$ is subject to the condition $f(z) = f(2l + z)$. D'Alembert was thus led to the search for analytical expressions which remain unaltered when $2l$ is added to the argument. In a second memoir, d'Alembert observed that the motion is determinate if the values of y and $\frac{\partial y}{\partial t}$ be assigned at some fixed time. Thus, in modern notation, if $y = f_1(x)$, $\frac{\partial y}{\partial t} = f_2(x)$, for $t = 0$, then, for all values of x between 0 and l,

$$f(x) - f(-x) = f_1(x),$$
$$f(x) + f(-x) = \frac{1}{a}\int f_2(x)\,dx;$$

it follows that $f(x)$ is determined for all values of x between l and $-l$, and thence, by means of the condition $f(z) = f(2l + z)$, for all values of x.

The treatment of the same problem which was shortly afterwards given by Euler* was in form of a similar character to that of d'Alembert, but the difference of meaning assigned by these writers to the word "function" was of fundamental importance in the controversy which afterwards arose between the two mathematicians in relation to this problem. D'Alembert understood by a function $y = f(x)$, a single analytical expression, whereas Euler employed the same expression and notation to denote an arbitrarily given graph. Both, however, held the view that two analytical expressions which are equal for values of the variable in a given interval must also be equal for values of the variable outside that interval. D'Alembert argued that Euler's mode of determination of the function in the solution of the problem presupposes that y can be expressed in terms of x and t by means of a single analytical expression, and that thus an undue restriction is imposed upon the modes of vibration of the string. For example, in the case in which the initial figure of the string is polygonal, d'Alembert regarded the solution of the problem as impossible. The general effect of the controversy is to exhibit on the one hand the narrowness of the restriction of the conception of a function as held by d'Alembert, to functions possessing at every point differential coefficients of all orders, and on the other hand the looseness of the conception of Euler that the ordinary methods of the Calculus are applicable without restriction to quite arbitrary functions.

314. The formal solution of the problem by means of trigonometrical series was given by Daniel Bernoulli† in a memoir in which he shewed that the differential equation and also the boundary conditions of the problem

* *Memoirs of the Berlin Academy*, 1748, p. 69.

† *Ibid.* 1753.

of the vibrating string, for the case in which there are no initial velocities, are formally satisfied by assuming

$$y = a_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + a_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l} + a_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \ldots.$$

He asserted that this represents the most general solution of the problem, and that the solutions of d'Alembert and Euler must therefore be contained in it. In a later memoir, he considered the case of a massless string loaded with n masses vibrating transversely, and indicated an indefinite increase in the number n. A criticism of Bernoulli's theory was published immediately afterwards by Euler, who pointed out that a consequence of Bernoulli's formula was that every arbitrarily assigned function of a variable x could be represented by a series of sines $a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots$. This appeared to Euler to be a *reductio ad absurdum*, since such a series could represent only a function which is odd and periodic; the notion that a function could be capable of representation by a certain analytical expression only in a limited interval being contrary to established opinion at that time. Bernoulli's solution was consequently regarded by Euler as lacking in generality. A considerable controversy* took place on the subject between Bernoulli and d'Alembert.

This problem, together with the related problem of the propagation of plane waves in air, was next taken up by Lagrange†, who obtained Euler's results by the method of starting with a finite number of masses fixed at intervals on a massless string, and then proceeding to the limit when the number of masses becomes indefinitely great. In the course of his analysis Lagrange came near to the determination of the form of the coefficients in the expansion of a function in a series of sines of multiples of the argument. The defect of Lagrange's method lies in the lack of any investigation of the validity of the process of passing to the limit; no restrictions upon the nature of the arbitrary functions were recognized by him as necessary. The remarks made by Euler, d'Alembert, and Bernoulli in the course of the discussion of Lagrange's work failed to elucidate the difficulties connected with this point, and no generally accepted theoretical views emerged from the lengthy controversies, the general course of which has been indicated.

The difficulties felt by the mathematicians of this period in regard to

* For a detailed history of these controversies, see Burkhardt's *Bericht*, vol. I. The early history of the theory of trigonometrical series is given by Riemann in his memoir, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," *Math. Werke*, p. 227. For the general history of the theory of these series see Sachs, "Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variabeln durch trigonometrische Reihen," *Schlömilch's Zeitschrift*, vol. XXV, supplement (1880), p. 231, and *Bulletin des sc. math.* (2), vol. IV, 1880; also Gibson, "On the History of the Fourier Series," *Proceedings of the Edinburgh Math. Soc.* vol. VI, p. 137.

† *Miscellanea Taurinensia*, vols. I, II, III.

the generality of the representation of a function by a trigonometrical series arose in large measure from their restricted conception of the nature of a function. To them it was conceivable that a function given by a continuous curve might be so representable, but since they regarded a function obtained by piecing two or more such curves together, not as one function, but as several different functions, it seemed to them impossible that such a broken curve could be represented by one trigonometrical series; a separate series seemed to be required for each separate portion of the given composite curve. Moreover, the idea was unfamiliar that a particular mode of representation of a function need only be valid for some restricted range of values of the abscissa; and thus a portion of a non-periodic curve was regarded as incapable of being represented by means of a periodic series.

SPECIAL CASES OF TRIGONOMETRICAL SERIES

315. Independently of the discussions of the problem of vibrating strings and of other physical problems, a number of trigonometrical series representing special functions of a simple character were obtained by Euler, d'Alembert and Bernoulli. The methods employed by these writers for this purpose are of a character which fails to satisfy the requirements now regarded as necessary for the establishment of such results; moreover, in many cases the ranges of values of the variable for which the representations of the functions by the series are valid were not assigned.

For example, the series

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \ldots,$$

$$\cos x - \tfrac{1}{4}\cos 2x + \tfrac{1}{9}\cos 3x - \tfrac{1}{16}\cos 4x + \ldots,$$

were obtained by Euler*, as representing $\frac{1}{2}x$, $\frac{1}{12}\pi^2 - \frac{1}{4}x^2$ respectively; the range of values of x $(-\pi, \pi)$ for which these representations are valid was however not given by Euler, who appeared to regard them as valid for all values of x. These series were obtained by integration of the series $\cos x + \cos 2x + \cos 3x + \ldots$, the sum of which was maintained by Euler to be $-\frac{1}{2}$.

By D. Bernoulli† the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$ was obtained as a representation of $\frac{1}{2}(\pi - x)$, and the range of values of x $(0, 2\pi)$ for which this representation is valid was assigned. It was also observed that the sum of the series is discontinuous for $x = 0, 2\pi, 4\pi, \ldots$. The following series were also

* *Petrop. N. Comm.* 1754–55, and *Petrop. N. Acta,* 1789.

† *Petrop. N. Comm.* 1772.

obtained by Bernoulli, and the ranges of the validity of the equations were assigned:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx = \frac{1}{6}\pi^2 - \frac{1}{2}\pi x + \frac{1}{4}x^2,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx = \frac{1}{6}\pi^2 x - \frac{1}{4}\pi x^2 + \frac{1}{12}x^3,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \cos nx = \frac{1}{90}\pi^4 - \frac{1}{12}\pi^2 x^2 + \frac{1}{12}\pi x^3 - \frac{1}{48}x^4,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \sin nx = \frac{1}{90}\pi^4 x - \frac{1}{36}\pi^2 x^3 + \frac{1}{48}\pi x^4 - \frac{1}{240}x^5,$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx = \frac{1}{2}\log \frac{1}{2(1-\cos x)}.$$

The following results among others obtained by Euler may here be mentioned:

$$\frac{1}{4}\pi = \sum_{r=0}^{\infty} (-1)^r \frac{\cos(2r+1)x}{2r+1},$$

$$\frac{1}{4}\pi x = \sum_{r=0}^{\infty} (-1)^r \frac{\sin(2r+1)x}{(2r+1)^2},$$

$$\frac{1}{8}\pi\left(\frac{1}{4}\pi^2 - x^2\right) = \sum_{r=0}^{\infty} (-1)^r \frac{\cos(2r+1)x}{(2r+1)^3}.$$

The true range of validity of these equations will appear later.

LATER HISTORY OF THE THEORY

316. No further advance was made in the subject until 1807, when Fourier, in a memoir on the Theory of Heat, presented* to the French Academy, laid down the proposition that an arbitrary function given graphically by means of a curve, which may be broken by (ordinary) discontinuities, is capable of representation by means of a single trigonometrical series. This theorem is said to have been received by Lagrange with astonishment and incredulity.

Fourier shewed, in a variety of special cases, that a function $f(x)$ is representable for values of x between $-\pi$ and π, by the series

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

where $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

Fourier's results in connection with this subject are best studied in the collected form in which they appear in his *Théorie de la Chaleur*, published

* *Bulletin des sciences de la soc. philomathique*, vol. I, p. 122.

in 1822. Trigonometrical series of the above form, in which the coefficients are determined as above, are known as Fourier's series. It should, however, be remarked that Fourier also studied other trigonometrical series, in which the cosines and sines do not proceed by integral multiples of the argument.

Although Fourier attained to correct views as to the nature of the convergence of the infinite series he employed, he did not give any complete general proof that the series in the general case actually converges to the value of the function; he indicates* however a process of verification of such convergence which was not actually carried out until Dirichlet took up the subject.

317. An attempt to prove Fourier's theorem was made by Poisson, who started with the formula†

$$\int_{-\pi}^{\pi} f(x') \frac{1-h^2}{1-2h\cos(x-x')+h^2}\, dx'$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\, dx' + \frac{1}{\pi}\sum_{n=1}^{\infty} h^n \int_{-\pi}^{\pi} f(x') \cos n(x-x')\, dx',$$

which holds provided $-1 < h < 1$.

Poisson proceeded to shew that, as h approaches the limit 1, the integral on the left-hand side of the equation approaches the limit $f(x)$, and argued that $f(x)$ is represented by the series obtained by putting $h = 1$, on the right-hand side. Apart from the questions connected with the limit of the integral on the left-hand side, the conclusion is invalid unless it is shewn that the series obtained by putting $h = 1$ is convergent. In accordance with a known theorem, given by Abel, for power series (see § 126), in case the power series is convergent for $h = 1$, it converges to the limit of the sum of the series for values of h which are < 1, as h approaches the value 1; but no conclusion can be made immediately as to whether the series is really convergent, or not, when $h = 1$. A direct investigation of its convergence would be required to make the proof a valid one. It will however be shewn later that, by the employment of a theorem due to Littlewood, Poisson's proof may be made complete in the case when $f(x)$ is of bounded variation in the interval $(-\pi, \pi)$.

Two proofs of the validity of the representation were given by Cauchy; one at least of these is certainly invalid in its original form. Both of them depend upon the theory of functions of a complex variable, and will consequently not be discussed here. An example of an invalid proof of a similar character to one of Cauchy's, and also to Poisson's, is the proof given in Thomson and Tait's *Natural Philosophy*.

* See the *Théorie de la chaleur*, chap. ix, especially sect. 423.

† *Journ. de l'école polyt.* cah. 19, 1823, p. 404. See also his *Théorie analytique de la chaleur*.

In 1829, Dirichlet* gave a proof that, in an extensive class of cases, Fourier's series actually converges to the value of the function. His proof, the first rigorous one, was based upon a recognition of the distinction between absolutely convergent and conditionally convergent series. Since a Fourier's series, when convergent, is not necessarily absolutely convergent, it is impossible to obtain a proof of the convergence from the law according to which the terms diminish, as Cauchy had attempted to do. As Dirichlet's proof, apart from its historical interest, still repays a careful study, on account of the light it throws upon the mode of convergence of the series, it will be given below, with some modifications and extensions which arise from later advances in the Theory of Functions.

THE FORMAL EXPRESSION OF FOURIER'S SERIES

318. Let $f(x)$ denote a bounded function, defined for the interval $(0, l)$ of the variable x. A finite trigonometrical series of the form

$$a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \dots + a_s \sin \frac{s\pi x}{l} + \dots + a_{n-1} \sin \frac{(n-1)\pi x}{l}$$

can be so determined that its value is equal to that of the function $f(x)$ at each of the points $x = \frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}, \dots \frac{\overline{n-1}\,l}{n}$. It must be shewn that the coefficients $a_1, a_2, \dots a_{n-1}$ can be determined by means of the linear equations

$$f\left(\frac{l}{n}\right) = a_1 \sin \frac{\pi}{n} + a_2 \sin \frac{2\pi}{n} + \dots + a_{n-1} \sin \frac{(n-1)\pi}{n},$$

$$f\left(\frac{2l}{n}\right) = a_1 \sin \frac{2\pi}{n} + a_2 \sin \frac{2.2\pi}{n} + \dots + a_{n-1} \sin \frac{2(n-1)\pi}{n},$$

....

....

$$f\left(\frac{rl}{n}\right) = a_1 \sin \frac{r\pi}{n} + a_2 \sin \frac{2r\pi}{n} + \dots + a_{n-1} \sin \frac{r(n-1)\pi}{n},$$

....

$$f\left(\frac{\overline{n-1}\,l}{n}\right) = a_1 \sin \frac{(n-1)\pi}{n} + a_2 \sin \frac{2(n-1)\pi}{n} + \dots + a_{n-1} \sin \frac{(n-1)(n-1)\pi}{n}.$$

Multiply the expressions on the two sides of these equations by

$$\sin \frac{s\pi}{n}, \ \sin \frac{2s\pi}{n}, \dots \sin \frac{(n-1)s\pi}{n}$$

* *Crelle's Journal*, vol. IV (1829), p. 157, "Sur la convergence des séries trigonométriques, qui servent à représenter une fonction arbitraire entre des limites données." See also his memoir in Dove and Moser's *Repertorium für Physik*, vol. I, 1837. Memoirs by Dircksen, *Crelle's Journal*, vol. IV (1829), p. 170, and by Bessel, *Astron. Nachrichten*, vol. XVI (1839), No. 361, are on similar lines to those of Dirichlet, but of inferior importance.

respectively, and add the expressions on each side together. It can easily be verified that

$$\sin\frac{r\pi}{n}\sin\frac{s\pi}{n}+\sin\frac{2r\pi}{n}\sin\frac{2s\pi}{n}+\ldots+\sin\frac{(n-1)\,r\pi}{n}\sin\frac{(n-1)\,s\pi}{n}=0,$$

provided r and s are unequal integers not greater than $n-1$; and also it can be shewn that

$$\sin^2\frac{s\pi}{n}+\sin^2\frac{2s\pi}{n}+\ldots+\sin^2\frac{(n-1)\,s\pi}{n}=\frac{1}{2}n.$$

Using these two identities, we have at once

$$a_s=\frac{2}{n}\left[f\left(\frac{l}{n}\right)\sin\frac{s\pi}{n}+f\left(\frac{2l}{n}\right)\sin\frac{2s\pi}{n}+\ldots+f\left(\frac{\overline{n-1}\,\pi}{n}\right)\sin\frac{s\,(n-1)\,\pi}{n}\right];$$

and thus the coefficients in the series have been determined so that the series satisfies the prescribed condition. Let us now assume that the function $f(x)$ is integrable in accordance with Riemann's definition, and let the number n be indefinitely increased. The limit of the expression for a_s is then seen to be $\frac{2}{l}\int_0^l f(x')\sin\frac{s\pi x'}{l}\,dx'$. This process suggests the possibility that the function $f(x)$ may be represented by the infinite series

$$a_1\sin\frac{\pi x}{l}+a_2\sin\frac{2\pi x}{l}+\ldots+a_s\sin\frac{s\pi x}{l}+\ldots,$$

where the coefficients a_s are given by

$$a_s=\frac{2}{l}\int_0^l f(x')\sin\frac{s\pi x'}{l}\,dx,$$

for points x within the interval $(0, l)$. It will be observed that the series cannot possibly represent the function at the point $x=0$, unless $f(0)=0$; nor at the point $x=l$, unless $f(l)=0$. This limiting process is entirely insufficient to shew either that the infinite series converges at all, or that, when it does converge, its limiting sum is at any point equal to the value of the function $f(x)$ at that point.

It will later be shewn by various methods that, for extensive classes of functions, the series

$$\frac{2}{l}\sum_{s=1}^{\infty}\sin\frac{s\pi x}{l}\int_0^l f(x')\sin\frac{s\pi x'}{l}\,dx' \qquad \ldots\ldots(1)$$

actually converges to the value $f(x)$, for values of x within the interval $(0, l)$, at which $f(x)$ is continuous. This series is known as *Fourier's sine series*.

Let us now assume that the function $f(x)\sin\frac{\pi x}{l}$ is represented within the interval $(0, l)$ by the Fourier's sine series.

This series is, in the present case, of the form

$$\frac{2}{l}\sum_{s=1}^{\infty}\sin\frac{s\pi x}{l}\int_0^l f(x')\sin\frac{\pi x'}{l}\sin\frac{s\pi x'}{l}\,dx',$$

which is equivalent to

$$\frac{1}{l}\sum_{s=1}^{\infty}\sin\frac{s\pi x}{l}\int_0^l f(x')\left[\cos\frac{\overline{s-1}\,\pi x'}{l}-\cos\frac{\overline{s+1}\,\pi x'}{l}\right]dx',$$

or to

$$\frac{1}{l}\sin\frac{\pi x}{l}\int_0^l f(x')\,dx'+\frac{1}{l}\sum_{s=1}^{\infty}\left\{\sin\frac{\overline{s+1}\,\pi x}{l}-\sin\frac{\overline{s-1}\,\pi x}{l}\right\}\int_0^l f(x')\cos\frac{s\pi x'}{l}\,dx';$$

and this by hypothesis represents the function $f(x)\sin\frac{\pi x}{l}$.

It thus appears that, on the assumptions made, the function $f(x)$ is represented by the series

$$\frac{1}{l}\int_0^l f(x')\,dx'+\frac{2}{l}\sum_{s=1}^{\infty}\cos\frac{s\pi x}{l}\int_0^l f(x')\cos\frac{s\pi x'}{l}\,dx' \qquad \ldots\ldots(2).$$

This series (2) is of the form

$$\beta_0+\beta_1\cos\frac{\pi x}{l}+\beta_2\cos\frac{2\pi x}{l}+\ldots+\beta_s\cos\frac{s\pi x}{l}+\ldots,$$

and is known as *Fourier's cosine series*.

The cosine series, unlike the sine series, may possibly converge to the values $f(0)$, $f(1)$, for $x=0, l$ respectively, when these functional values are not necessarily zero.

319. Assuming for the present that the function $f(x)$ may be represented for the points of the interval $(0, l)$ by either of these series (1) and (2), we proceed to consider some obvious properties of the series themselves. The sum of the sine series (1) has, for the point $-x$, the same value, with the opposite sign, as for the point x. If then we suppose that the function $f(x)$ is defined not only for the interval $(0, l)$, but for the interval $(-l, l)$, it appears that the series can represent the function for the whole interval $(-l, l)$, only in case $f(-x)=-f(x)$; that is, in case the function $f(x)$ be odd. Further, the series (1) is unaltered by adding to x any multiple of $2l$, and thus the series, considered as existent for all values of x, defines a periodic function, of period $2l$. If $f(x)$ be defined for all values of x, it can only be represented by the series, for all such values of x, provided $f(x)$ is periodic and of period $2l$, and also $f(x)=-f(-x)$; otherwise the representation of the function by the series is valid only for the interval $(0, l)$.

The cosine series (2) is unaltered by changing x into $-x$; therefore the series represents the function $f(x)$ for the interval $(-l, l)$, only when $f(-x)=f(x)$, i.e. when $f(x)$ is an even function. The cosine series, like

the sine series, considered as existent for all values of x, is periodic, and of period $2l$; therefore the series can represent a function $f(x)$, defined for all values of x, only when $f(x)$ is periodic with period $2l$, and also

$$f(x) = f(-x).$$

It is thus seen that, if the function $f(x)$ be defined for the interval $(-l, l)$, it is in general not represented by either the sine or the cosine series for the whole of that interval, although it may be represented by both the series for the interval $(0, l)$. For the part of the function $f(x)$ in the interval $(-l, 0)$ is in general independent of the part in the interval $(0, l)$; neither of the relations $f(-x) = -f(x)$, $f(-x) = f(x)$ being in general satisfied. In fact there is in general no relation between the values of a function, defined for the interval $(-l, l)$, at the two points $-x, x$.

It is however possible to obtain, from the series (1) and (2), a series containing both sines and cosines, such as to represent the function $f(x)$ for the whole interval $(-l, l)$. The function $\frac{1}{2}\{f(x) + f(-x)\}$ is an even function, defined for the whole interval $(-l, l)$, and in accordance with the assumptions, representable for that interval by the series

$$\frac{1}{2l}\int_0^l \{f(x') + f(-x')\}\, dx' + \frac{1}{l}\sum_{s=1}^{\infty} \cos\frac{s\pi x}{l}\int_0^l [f(x') + f(-x')]\cos\frac{s\pi x'}{l}\, dx'.$$

Again, the function $\frac{1}{2}\{f(x) - f(-x)\}$ is an odd function, defined for the whole interval $(-l, l)$, and is accordingly representable by

$$\frac{1}{l}\sum_{s=1}^{\infty} \sin\frac{s\pi x}{l}\int_0^l [f(x') - f(-x')]\sin\frac{s\pi x'}{l}\, dx'.$$

By addition of the two series, we find the series

$$\frac{1}{2l}\int_{-l}^{l} f(x')\, dx' + \frac{1}{l}\sum_{s=1}^{\infty}\int_{-l}^{l} \cos\frac{s\pi}{l}(x - x') f(x')\, dx' \quad \text{......(3)},$$

which is of the form

$$\tfrac{1}{2}\alpha_0 + \left(\alpha_1 \cos\frac{\pi x}{l} + \beta_1 \sin\frac{\pi x}{l}\right) + \left(\alpha_2 \cos\frac{2\pi x}{l} + \beta_2 \sin\frac{2\pi x}{l}\right) + \dots,$$

as representing the function $f(x)$ for the interval $(-l, l)$. This series (3) is known as *Fourier's series*, the sine and cosine series being regarded as the particular cases of it which arise when $f(-x) = -f(x)$, or $f(-x) = f(x)$ respectively.

320. With certain assumptions, the form of the series (3) may be obtained directly. Let it be assumed that a function $f(x)$, defined for the interval $(-l, l)$, can be represented by the series

$$\tfrac{1}{2}a_0 + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \dots + \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) + \dots,$$

in the sense that, for each point x in the interval, the series converges uniformly to the value $f(x)$ of the function at the point x. It then follows that $f(x)$ is continuous in the interval $(-l, l)$, and that $f(l) = f(-l)$.

The fundamental property of the functions

$$1,\ \cos\frac{\pi x}{l},\ \sin\frac{\pi x}{l},\ \cos\frac{2\pi x}{l},\ \sin\frac{2\pi x}{l},\ \dots,\ \cos\frac{n\pi x}{l},\ \sin\frac{n\pi x}{l},\ \dots$$

is that the integral of the product of any pair of them taken over the interval $(-l, l)$ has the value zero. On account of this property, the set of functions is said to be *an orthogonal set of functions for the interval* $(-l, l)$.

On account of the uniform convergence of the series to the value $f(x)$ through the interval $(-\pi, \pi)$, it is legitimate to submit the series to term by term integration, even when it is multiplied by $\cos\frac{n\pi x}{l}$, or by $\sin\frac{n\pi x}{l}$. Observing that $\int_{-l}^{l} dx = 2l$, $\int_{-l}^{l}\cos^2\frac{n\pi x}{l}\,dx = \int_{-l}^{l}\sin^2\frac{n\pi x}{l}\,dx = l$, and employing the property of orthogonality, we obtain in this manner,

$$\frac{1}{l}\int_{-l}^{l} f(x')\cos\frac{n\pi x'}{l}\,dx' = a_n, \text{ for } n = 0, 1, 2, 3, \dots;$$

and
$$\frac{1}{l}\int_{-l}^{l} f(x')\sin\frac{n\pi x'}{l}\,dx' = b_n, \text{ for } n = 1, 2, 3, \dots.$$

Therefore we have, for the interval $(-l, l)$, as the series representing $f(x)$,

$$\frac{1}{2l}\int_{-l}^{l} f(x')\,dx' + \sum_{n=1}^{\infty}\left\{\frac{1}{l}\cos\frac{n\pi x}{l}\int_{-l}^{l} f(x')\cos\frac{n\pi x'}{l}\,dx' + \frac{1}{l}\sin\frac{n\pi x}{l}\int_{-l}^{l} f(x')\sin\frac{n\pi x'}{l}\,dx'\right\},$$

or
$$\frac{1}{2l}\int_{-l}^{l} f(x')\,dx' + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(x')\cos\frac{n\pi}{l}(x'-x)\,dx'.$$

If we replace $\frac{\pi x}{l}$ by x, no essential change will be made in the formula; thus there is no loss of generality in taking $(-\pi, \pi)$ to be the interval in which $f(x)$ is defined, and for which it is represented by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos n(x'-x)\,dx' \quad \dots\dots(4).$$

This expression (4) will be taken to the standard form of Fourier's series.

321. In the above process, by which the form of the series has been obtained, it would have been sufficient to have assumed that the convergence of the series to the value of the function is simply uniform in the interval $(-\pi, \pi)$. In that case the convergence becomes uniform if a suitable system of bracketing the terms of the series is carried out (see § 67). More generally, it is sufficient to assume that, whether the series is convergent or not, when a suitable system of brackets is introduced, the new series, in which the terms of the original series that are in a single bracket are regarded as a single term, converges uniformly in the

interval $(-\pi, \pi)$. The uniformly convergent series would then take the form

$$\left[\tfrac{1}{2}a_0 + \sum_{n=1}^{n=n_1}(a_n \cos nx + b_n \sin nx)\right] + \sum_{n=n_1+1}^{n=n_2}(a_n \cos nx + b_n \sin nx)$$
$$+ \sum_{n=n_2+1}^{n=n_3}(a_n \cos nx + b_n \sin nx) + \ldots,$$

which is assumed to converge uniformly in $(-\pi, \pi)$. The original unbracketed series does not necessarily converge for the values of x in the interval. We find, as before, that

$$\int_{-\pi}^{\pi} f(x') \cos mx'\, dx' = \int_{-\pi}^{\pi} \cos mx' \cdot \sum_{n=n_p+1}^{n=n_{p+1}} (a_n \cos nx + b_n \sin nx)\, dx$$
$$= \int_{-\pi}^{\pi} a_m \cos^2 mx'\, dx' = \pi a_m,$$

where m is one of the numbers $n_p + 1, n_p + 2, \ldots n_{p+1}$. In a similar manner we find that $\int_{-\pi}^{\pi} f(x') \sin mx'\, dx' = \pi b_m$; hence the form of the series has been obtained.

THE GENERAL DEFINITION OF A FOURIER'S SERIES

322. We now take the series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\, dx' + \sum_{n=1} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x') \cos n(x' - x)\, dx'$$

as the starting point, independently of any assumption as to its convergence. In order that the series may be said to exist, whether it converge anywhere, or not, it is necessary that the coefficients

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\, dx', \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x') \cos nx'\, dx', \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x') \sin nx'\, dx'$$

should have definite meanings, whatever value n may have.

Until the last few decades it has been assumed that $f(x)$ is either bounded in the interval $(-\pi, \pi)$, and integrable (R) in that interval, or else that $f(x)$ is unbounded in that interval, but possesses in it an integral in accordance with one of the earlier definitions which were employed to meet such cases. The recent extension of the definition of integration, due to Lebesgue, to the case of functions which, whether bounded or not, are not integrable (R), has led to a corresponding extension of the range of Fourier's series. It has been proposed by Lebesgue* to assign to the series (4) the name *Fourier's series*, in every case in which $f(x)$ is summable, and consequently also $f(x) \cos nx$, $f(x) \sin nx$ are summable, in the interval

* Lebesgue's treatment of the series is contained in a memoir, "Sur les séries trigonométriques," *Annales sc. de l'école normale, supérieure* (3), vol. XX (1903), p. 453, in a memoir, "Recherches sur la convergence des Séries de Fourier," *Math. Annalen*, vol. LXI (1905), p. 251; and in the *Leçons sur les séries trigonométriques*, 1906.

$(-\pi, \pi)$, whether the function be bounded or not. This terminology will be here adopted. The two series

$$\frac{1}{\pi}\int_0^\pi f(x')\,dx' + \sum_{n=1}^{\infty}\frac{2}{\pi}\cos nx\int_0^\pi f(x')\cos nx'\,dx',$$

$$\sum_{n=1}^{\infty}\frac{2}{\pi}\sin nx\int_0^\pi f(x')\sin nx'\,dx',$$

in which $f(x)$ is taken to be summable in the interval $(0, \pi)$, will be termed *Fourier's cosine series*, and *Fourier's sine series* respectively. The first of these series is the Fourier's series corresponding to $f(x)$, provided $f(x)$ is an even function of x, so that it exists in the interval $(-\pi, \pi)$, and the coefficients

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx'$$

then all exist, and have the value zero. The Fourier's sine series is the Fourier's series corresponding to $f(x)$, in case $f(x)$ is an odd function, defined for the interval $(-\pi, \pi)$, in which case the coefficients

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx',\quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'\,dx'$$

all exist, and have the value zero.

Each extension of the definition of an integral, beyond that of Lebesgue, leads to an extension of the scope of the series. Thus cases may be considered in which the coefficients exist as HL-integrals, as D-integrals, as DKY-integrals, or as Y-integrals, or as integrals existing in accordance with other definitions which have been suggested. All series of these kinds may be termed *generalized Fourier's series*, but the only kind which will be considered in this work will be those in which $f(x)$ and consequently $f(x)\cos nx$, $f(x)\sin nx$, have D-integrals, or in particular, HL-integrals, in the interval $(-\pi, \pi)$. Such series will be termed *Fourier's D-series* or *Fourier's (HL) series*. There may exist also Fourier's D-cosine-series, and Fourier's D-sine-series, which as explained above are Fourier's D-series in case the absent coefficients exist and have the value zero.

EXAMPLE

Let us consider the function $f(x) = \dfrac{\phi(x)}{x}$, where $\phi(x)$ is summable in the interval $(-\pi, \pi)$, then the coefficients $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{\phi(x)}{x}\,dx$, $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\frac{\phi(x)}{x}\cos nx$ will not in general exist, either as L-integrals or as D-integrals. Thus the function $\dfrac{\phi(x)}{x}$ will, in accordance with the definition given above, have no Fourier's series, or Fourier's D-series, corresponding to it, although the coefficients $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\frac{\phi(x)}{x}\sin nx\,dx$ will exist as L-integrals. The series

$$\frac{1}{\pi}\sum_{n=1}\sin nx\int_0^\pi\frac{\phi(x')}{x'}\sin nx'\,dx'$$

will, however, exist, but it is not a Fourier's sine-series, because $\dfrac{\phi(x)}{x}$ is not summable in

$(0, \pi)$. For example*, let $\phi(x) = \frac{1}{2}x \cot \frac{1}{2}x$, $\phi(0) = 1$, in the interval $(0, \pi)$. It is easily found that $\frac{1}{\pi}\int_0^\pi \frac{1}{2}\cot\frac{1}{2}x \sin nx\, dx = 1$, but $\frac{1}{\pi}\int_0^\pi \frac{1}{2}\cot\frac{1}{2}x \cos nx\, dx$ does not exist. It follows that $\sin x + \sin 2x + \sin 3x + \dots$ is a generalized Fourier's sine-series corresponding to the function $\frac{1}{2}\cot\frac{1}{2}x$, non-summable in the interval $(0, \pi)$, but it is not a Fourier's D-series. Discussions of such series have been given by Titchmarsh† and by Perron‡.

It should be observed that all summable functions which are equivalent to one another correspond to one and the same Fourier's series. Conditions have been investigated by Carathéodory§ that, among the functions that are equivalent to a function $f(x)$ to which correspond the Fourier's coefficients $a_0, a_1, b_1, a_2, b_2 \dots$, there should exist one which is integrable (R), so that the Fourier's series defined by means of these coefficients should be a Fourier's R-series.

A series of the form $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ is not necessarily a Fourier's series, even assuming that $a_n = o(1)$, $b_n = o(1)$. An example is the series $\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$.

THE PARTIAL SUMS OF A FOURIER'S SERIES

323. It being assumed that $f(x)$, as defined for the interval $(-\pi, \pi)$, is such that the coefficients in the series (4) exist, either as L-integrals or as D-integrals. We denote by $s_{2n+1}(x)$ the finite sum

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \sum_{r=1}^{r=n}\left\{\frac{1}{\pi}\cos rx \int_{-\pi}^{\pi} f(x')\cos rx'\,dx' + \frac{1}{\pi}\sin rx\int_{-\pi}^{\pi} f(x')\sin rx'\,dx'\right\},$$

or $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\left[\frac{1}{2} + \cos(x'-x) + \cos 2(x'-x) + \dots + \cos n(x'-x)\right]dx.$

Since

$$\frac{1}{2} + \cos(x'-x) + \cos 2(x'-x) + \dots + \cos n(x'-x) = \frac{\sin(2n+1)\frac{x'-x}{2}}{2\sin\frac{x'-x}{2}},$$

we have
$$s_{2n+1}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\frac{\sin(2n+1)\frac{x'-x}{2}}{\sin\frac{x'-x}{2}}\,dx'.$$

If we change the variable x' by taking $x' = x + 2z$, and write $2n+1 = m$, the expression takes the form,

$$s_m(x) = \frac{1}{\pi}\int_{-\frac{1}{2}(\pi+x)}^{\frac{1}{2}(\pi-x)} f(x+2z)\frac{\sin mz}{\sin z}\,dz,$$

where $m = 2n+1$.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 431.

† *Proc. Lond. Math. Soc.* (2), vol. XXIII (1924). *Records*, p. XII.

‡ *Math. Annalen*, vol. LXXXVII (1922), p. 84.

§ *Math. Zeitschr.* vol. I (1918), p. 309.

It is convenient to extend the definition of $f(x)$ so that it may apply to all values of x outside the interval $(-\pi, \pi)$. We assume that $f(x)$ is so defined as to be periodic, of period 2π; thus $f(x) = f(x \pm 2r\pi)$, for all integral values of r. In case, in the original definition of $f(x)$ for the interval $(-\pi, \pi)$, the values of $f(\pi)$ and $f(-\pi)$ are unequal, it will be necessary to alter the value of the function at one of the points $\pi, -\pi$, in order that the function, in accordance with the extended definition, may be periodic. This can be done without affecting the values of the coefficients of the series, or the value of $s_m(x)$. Taking the function $f(x)$ then to be periodic, so that it is defined for all real values of x, it is clear that in the expression for $s_m(x)$ the limits of the definite integral may be altered to any two values which differ from one another by π, without altering the value of the integral. We have thus

$$\begin{aligned} s_m(x) &= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2z)\frac{\sin mz}{\sin z}\,dz \\ &= \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\{f(x+2z)+f(x-2z)\}\frac{\sin mz}{\sin z}\,dz. \end{aligned}$$

The integral of the form $\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz$ is known as *Dirichlet's integral*, the term being, however, generally applied to the more general form $\int_0^{a} F(z)\frac{\sin mz}{\sin z}\,dz$, and also to $\int_0^{a} F(z)\frac{\sin mz}{z}\,dz$, where a is such that $0 < a \leq \frac{1}{2}\pi$.

If we take $\frac{1}{2} + \cos(x'-x) + \cos 2(x'-x) + \ldots + \frac{1}{2}\cos n(x'-x)$ which is equal to $\dfrac{\sin(2n+1)\frac{x'-x}{2}}{2\sin\frac{x'-x}{2}} - \frac{1}{2}\cos n(x'-x)$, or to $\dfrac{1}{2}\dfrac{\sin n(x'-x)}{\tan\frac{1}{2}(x'-x)}$, we see that

$$\begin{aligned} s_{2n+1}(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\cos n(x'-x)\,dx' &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\frac{\sin n(x'-x)}{\tan\frac{1}{2}(x'-x)}\,dx' \\ &= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2z)\frac{\sin 2nz}{\tan z}\,dz. \end{aligned}$$

Thus $s_{2n+1}(x)$ has its value* dependent upon an integral

$$\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin 2nz}{\tan z}\,dz,$$

of a form very similar to that of Dirichlet.

* See Neder, *Math. Annalen*, vol. LXXXIV (1921), p. 120, where it is pointed out that this form for $s_{2n+1}(x)$ can sometimes be conveniently employed.

THE CONVERGENCE OF FOURIER'S SERIES

324. If the function $f(x)$ be summable in the interval $(-\pi, \pi)$, the coefficients in the Fourier's series corresponding to $f(x)$ all exist, and are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx;$$

and this independently of any assumption as to the convergence or non-convergence of the series at points of the interval. Thus, corresponding to any summable function $f(x)$, there exist the numbers $a_0, a_1, b_1, a_2, b_2 \ldots$, which may be termed the *Fourier's coefficients*, or *Fourier's constants*, for the summable function $f(x)$. The relation of the constants to the function may be expressed by*

$$f(x) \sim \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots$$

which does not involve any implication as regards the convergence of the series. It will be seen that all equivalent functions have the same set of Fourier's constants.

A similar definition will apply to the Fourier's (D) constants corresponding to a function $f(x)$ which has a D-integral in the interval $(-\pi, \pi)$.

It will be seen later that the Fourier's constants of summable functions, and of particular classes of such functions, possess important properties which do not depend upon the convergence of the Fourier's series.

The question as to the convergence of the series in the whole, or in a part of the interval $(-\pi, \pi)$, or at assigned points of that interval has been fundamental in the history of the subject, and the earlier investigations, from the time of Dirichlet's investigations onwards, were almost exclusively concerned with this question. In considering this question, two lines of investigation may be pursued, according as the function itself, or the series as defined by its coefficients, is taken as the starting point. In the first of these lines of investigation, the question takes the form—what properties must the function have, in order that the Fourier's series may converge at a particular point, or in the whole or a part of the interval? In the second of these modes of approach it is not usually assumed that the series is a Fourier's series, and the question takes the form—what can be inferred as to the convergence of the series from the existence of special restrictive properties of the coefficients? An account will be given of investigations of both these classes; in the earlier investigations the first of these modes of investigation was alone employed. In the first instance an account will be given of the investigations, by various writers, which have as their object the determination of sufficient conditions to be satisfied by the summable function $f(x)$ in order that the series may con-

* See Hurwitz, *Math. Annalen*, vol. LVII (1903), p. 427.

verge either through a whole interval, or at particular points of the interval $(-\pi, \pi)$. It will appear that, for a summable function $f(x)$, the convergence or non-convergence of the series, at a particular point, depends only upon the nature of the function in an arbitrarily small neighbourhood of that point; and is independent of the general character of the function throughout the interval $(-\pi, \pi)$; this general character being limited only by the necessity that the function shall be summable in the whole interval. These investigations have resulted in the discovery of sufficient conditions, of considerable width, which suffice to ensure the convergence of the series at particular points, or generally through the whole or a part of the interval for which the function is defined. The necessary and sufficient conditions for the convergence of the series at a point of the interval, or throughout any particular portion of the interval, have not been obtained. This is not surprising, in view of the very general character of the problem; and indeed it may be the case that no such necessary and sufficient conditions may be obtainable. It is possible that the mere fact of the convergence of the series at a particular point characterizes the nature of the function in the neighbourhood of that point in a manner incapable of reduction to any simpler form; so that, although the characteristics of various sub-classes of the functions which satisfy this condition may be obtained, as has in fact been done, yet the whole class of such functions has no property capable of being stated in any form essentially different from, or simpler than, the mere statement of the fact of the convergence of the series at the point. It will appear that there exist functions, and even continuous functions, for which the series fails to converge at every point belonging to an everywhere-dense set of points. The question whether a Fourier's series, corresponding to a continuous function, can be so determined that it fails to converge at all points of a set of measure greater than zero, or in particular almost everywhere, has not yet been answered.

In order that the Fourier's series, corresponding to a summable function $f(x)$, may converge at a point x, it is necessary that $s_m(x)$, or

$$\frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2z)\frac{\sin mz}{\sin z}\,dz$$

should converge to a definite limit, as the odd integer m is indefinitely increased.

It will appear (§ 434) that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos n(x'-x)\,dx'$ converges to zero, as $n \sim \infty$, consequently it is necessary, for the convergence of the series at the point x, that the integral

$$\frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2z)\frac{\sin 2nz}{\tan z}\,dz$$

should converge to a definite limit, as $n \sim \infty$. Either of these expressions

may be used in the investigations, but Dirichlet's form will be here employed.

It was first shewn by Dirichlet that, for an important class of functions $f(x)$, $s_m(x)$ converges to the value $f(x)$ at every point x interior to the interval $(-\pi, \pi)$, at which $f(x)$ is continuous; that, at a point of ordinary discontinuity of $f(x)$, the series converges to the value

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

which is, of course, not necessarily equal to $f(x)$; and that at the points π and $-\pi$, the series converges to the value $\frac{1}{2}\{f(\pi-0)+f(-\pi+0)\}$. Although Dirichlet's investigation has now been superseded by the employment of methods applicable to a wider class of cases than was considered by him, his investigation has still an interest not exclusively historical. It will therefore be given in § 328, in a form in which certain modifications and simplifications will be employed.

More recent investigations, an account of which will be given, shew that the Fourier's constants have important properties which are related to the functional values, independently of whether the series converges or not. It will appear that, in important classes of cases, Fourier's series may be employed, independently of whether they are known to converge, for the representation of functions, and that such series may be validly subjected to many of the ordinary processes of Analysis, such as substitution for the function in a definite integral and subsequent term by term integration. Much of the recent progress in the Theory of Fourier's series is due to the employment of the conventional sums of the series, especially those of Riemann, Cesàro, and Poisson. By this means a representation of a function by means of a convergent sequence can be obtained when the Fourier's series corresponding to the function is not convergent, or is not known to be convergent.

PARTICULAR CASES OF FOURIER'S SERIES

325. Before proceeding to the theoretical investigations relating to the convergence and the properties of Fourier's series, it will be instructive to consider some simple cases of the use of the series. It will be assumed that, for the functions employed, the series corresponding to a function $f(x)$ converges at every point to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

If we employ the sine series to represent the function defined, for the interval $(0, \pi)$, by $y = \frac{1}{2}(\pi - x)$, we find on evaluation that

$$\frac{2}{\pi}\int_0^\pi \tfrac{1}{2}(\pi - x)\sin nx\,dx = \frac{1}{n};$$

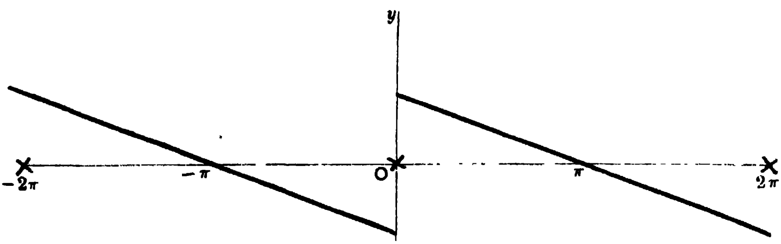
and thus the series is of the form

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots + \frac{1}{n}\sin nx + \ldots.$$

The function defined for all values of x by

$$y = \sin x + \tfrac{1}{2}\sin 2x + \ldots + \frac{1}{n}\sin nx$$

is represented graphically in the figure. The function is discontinuous at the points $0,\ 2\pi,\ 4\pi,\ \ldots\ -2\pi,\ -4\pi,\ \ldots$; the functional value being zero at all those points. It is seen that the series represents the function $\frac{1}{2}(\pi - x)$, not only for the interval $(0, \pi)$, but for the interval $(0, 2\pi)$, except at the points $x = 0$, $x = 2\pi$, where the sum of the series is zero. For the interval $(-2\pi, 0)$ the function represented by the series is $-\frac{1}{2}(\pi + x)$, except at the ends of the interval.



This series may be employed to illustrate some important points connected with the convergence of the series in the neighbourhood of the point $x = 0$, at which the function represented by the series is discontinuous. To this end we shall examine the series by a method employed by Fourier*, and further developed by Kneser†.

Denoting $\sin x + \frac{1}{2}\sin 2x + \ldots + \frac{1}{n}\sin nx$, by $s_n(x)$, we have

$$\frac{ds_n(x)}{dx} = \cos x + \cos 2x + \ldots + \cos nx = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{1}{2}x} - \tfrac{1}{2};$$

therefore

$$\begin{aligned} s_n(x) &= \tfrac{1}{2}\int_0^x \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,dx - \tfrac{1}{2}x \\ &= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)x}{x}\,dx - \tfrac{1}{2}x + \int_0^x \sin\left(n+\tfrac{1}{2}\right)x\,\frac{(x - 2\sin\frac{1}{2}x)}{2x\sin\frac{1}{2}x}\,dx \\ &= \int_0^{(n+\frac{1}{2})x} \frac{\sin z}{z}\,dz - \tfrac{1}{2}x + I(x). \end{aligned}$$

On integrating by parts, we find that

$$I(x) = -\frac{x - 2\sin\frac{1}{2}x}{2x\sin\frac{1}{2}x}\,\frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \int_0^x \frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}}\cdot\frac{4\sin^2\frac{1}{2}x - x^2\cos\frac{1}{2}x}{4x^2\sin^2\frac{1}{2}x}\,dx.$$

* *Théorie de la chaleur*, chap. III, sect. 3

† *Sitzungsber. of the Berlin Math. Soc.* (1904), p. 28. See also Bôcher's "Introduction to the theory of Fourier's series," *Annals of Mathematics* (2), vol. VII (1906), p. 81, where numerical details are worked out.

The expressions $\dfrac{x - 2\sin\frac{1}{2}x}{2x\sin\frac{1}{2}x}$, $\dfrac{4\sin^2\frac{1}{2}x - x^2\cos\frac{1}{2}x}{4x^2\sin^2\frac{1}{2}x}$ both become indefinitely great, as x increases up to 2π; but if x be confined to the interval $(0, b)$, where $0 < b < 2\pi$, they are both bounded functions. It follows, since

$$|\cos(n + \tfrac{1}{2}x)| \leq 1,$$

that a positive number A can be determined, independent of n and x, such that $|I(x)| < A/(n + \frac{1}{2})$, provided x is in the interval $(0, b)$. Hence it appears that $I(x)$ has the limit zero, when n is indefinitely increased, whether x varies with n or not; in fact $|I(x)|$ is arbitrarily small for sufficiently great values of n.

We have now

$$s_n(x) - s(x) = \int_0^{(n+\frac{1}{2})x} \frac{\sin z}{z}\,dz - \tfrac{1}{2}\pi + \frac{\theta A}{n + \frac{1}{2}},$$

provided $0 \leq x \leq b$; where θ is such that $-1 < \theta < 1$.

Also $$\frac{d}{dx}\{s_n(x) - s(x)\} = \frac{1}{2}\frac{\sin(n + \frac{1}{2})x}{\sin\frac{1}{2}x}, \text{ if } 0 < x \leq b;$$

and therefore $s_n(x) - s(x)$ has maxima and minima at the points $x = \dfrac{\lambda\pi}{n + \frac{1}{2}}$, where $\lambda = 1, 2, 3, \dots$.

It can now be shewn that, for sufficiently large values of n, at least,

$$s_n\left(\frac{2\pi}{2n+1}\right) - s\left(\frac{2\pi}{2n+1}\right), s_n\left(\frac{4\pi}{2n+1}\right) - s\left(\frac{4\pi}{2n+1}\right), s_n\left(\frac{6\pi}{2n+1}\right) - s\left(\frac{6\pi}{2n+1}\right), \dots$$

are alternately positive and negative, the first of these differences being positive.

We have

$$\int_0^{\lambda\pi} \frac{\sin z}{z}\,dz = \int_0^{\pi} \sin z\left(\frac{1}{z} - \frac{1}{z+\pi} + \frac{1}{z+2\pi} - \dots + \frac{(-1)^{\lambda+1}}{z + (\lambda - 1)\pi}\right)dz$$
$$= u_1 - u_2 + u_3 - \dots + (-1)^{\lambda+1}u_\lambda,$$

where $u_1, u_2, \dots u_\lambda$ are all positive, and $u_1 > u_2 > u_3 \dots > u_\lambda$. Also

$$u_\lambda < \frac{1}{(\lambda - 1)\pi}\int_0^{\pi} \sin z\,dz < \frac{2}{(\lambda - 1)\pi};$$

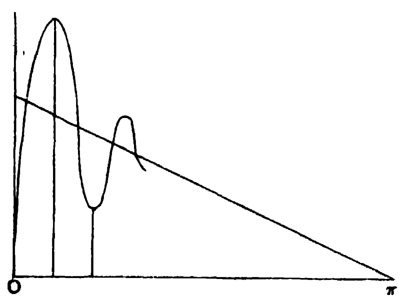
hence $\lim\limits_{\lambda=\infty} u_\lambda = 0$.

Further, it is well known that $\lim\limits_{\lambda=\infty}\displaystyle\int_0^{\lambda\pi}\frac{\sin z}{z}\,dz$, which is the improper integral $\displaystyle\int_0^{\infty}\frac{\sin z}{z}\,dz$, is equal to $\frac{1}{2}\pi$; it follows that $u_1, u_1 - u_2, u_1 - u_2 + u_3, \dots$ are alternately greater and less than $\frac{1}{2}\pi$. Since $\dfrac{2\theta A}{2n+1}$ is arbitrarily small, for sufficiently great values of n, it thus appears that the differences

$$s_n\left(\frac{2\lambda\pi}{2n+1}\right) - s\left(\frac{\overline{2\lambda+1}\pi}{2n+1}\right)$$

are alternately positive and negative for $\lambda = 1, 2, 3, \ldots$; and that for $\lambda = 1$, the difference is positive.

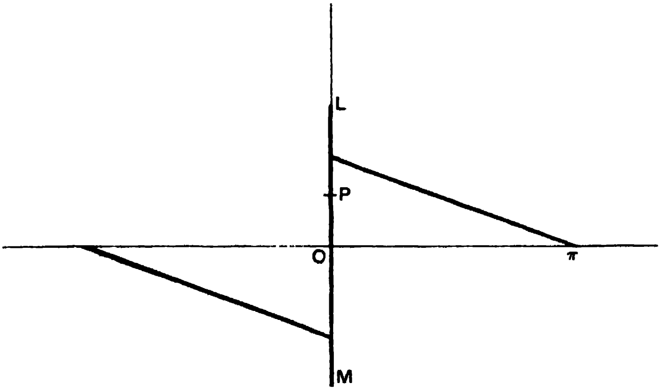
It thus appears that, for large values of n, the form of the curve $y = s_n(x)$ in the neighbourhood of the origin is as in the figure; consisting of a wave-form passing above and below the straight lines which represent $y = s(x)$. The first maximum on the right of the point $x = 0$ has as its abscissa $x = \dfrac{2\pi}{2n+1}$, and its height above the point whose coordinates are $\dfrac{2\pi}{2n+1}$, $s\left(\dfrac{2\pi}{2n+1}\right)$ is nearly $\displaystyle\int_0^\pi \frac{\sin z}{z}\,dz - \tfrac{1}{2}\pi$, which is independent of the value of n. The first minimum on the right of the point $x = 0$ has for its abscissa $x = \dfrac{4\pi}{2n+1}$, and is at a depth approximately $\tfrac{1}{2}\pi - \displaystyle\int_0^{2\pi} \frac{\sin z}{z}\,dz$ below the corresponding point of the locus $y = s(x)$.



As n is continually increased, the abscissae of the maxima and minima of $s_n(x) - s(x)$ become indefinitely small, the magnitudes of these maxima and minima remaining however nearly unaltered. If a particular value of x can be chosen, n can be so determined that $|\, s_n(x) - s(x)\,|$ is arbitrarily small, for such value of n, and for all greater values; but if a particular value of n be chosen, there is always a value of x, viz. $\dfrac{2\pi}{2n+1}$, for which $(x) - s(x)$ is nearly equal to $\displaystyle\int_0^\pi \frac{\sin z}{z}\,dz - \tfrac{1}{2}\pi$.

The graphs $y = s_n(x)$, as n becomes indefinitely great, tend to the form given in the figure, which consists of the continuous curve formed by the straight lines of length $2\displaystyle\int_0^\pi \frac{\sin z}{z}\,dz\ (> \pi)$, through the points $x = 0, 2\pi, -2\pi, \ldots$, and of the series of oblique straight lines which belong to the curve $y = s(x)$. The graph of the curve $y = s(x) = \lim\limits_{n=\infty} s_n(x)$ has been already given. The limit of the graphs of the curves $y = s_n(x)$, and the graph of the limit of $s_n(x)$ differ in the respect that, for the abscissae $x = 0, 2\pi, -2\pi, \ldots$, the former contains the continuous straight lines of length $2\displaystyle\int_0^\pi \frac{\sin z}{z}\,dz$, whereas the latter contains only the single points on the

x-axis. Corresponding to any point P on the straight line LM through the origin, it is possible to determine an indefinite number of pairs of values of x and n, such that the distance of P from the point whose coordinates are x, $s_n(x)$, is less than an arbitrarily prescribed positive number ϵ. Thus the double limit $\lim\limits_{n=\infty, x=0} s_n(x)$ is indeterminate between the limits of indeterminacy $\int_0^\pi \frac{\sin z}{z}\,dz$, $-\int_0^\pi \frac{\sin z}{z}\,dz$.



By letting n increase indefinitely, and x at the same time diminish to zero, in such a manner that nx has α as its limit, where α is any fixed positive number not exceeding π, we have as the particular value of $\lim\limits_{n=\infty, x=0} s_n(x)$, or $\lim\limits_{n=\infty} s_n\left(\frac{\alpha}{n}\right)$, the number $\int_0^\alpha \frac{\sin z}{z}\,dz$. It will be observed that the repeated limit $\lim\limits_{x=0}\lim\limits_{n=\infty} s_n(x)$ has the value $\frac{1}{2}\pi$, or $-\frac{1}{2}\pi$, according as x approaches its limit from the positive, or from the negative side. The repeated limit $\lim\limits_{n=\infty}\lim\limits_{x=0} s_n(x)$ has the value zero.

The distinction between the graph $y = s(x)$, which represents the series, and the limit to which the graphs $y = s_n(x)$ tend, is clear, if it be borne in mind that the limit $y = s(x)$ is obtained by the special mode of first fixing a value of x, and then letting n increase indefinitely; thus, for example, $s(0) = \lim\limits_{n=\infty} s_n(0) = 0$; whereas, as we have seen, $\lim\limits_{n=\infty, x=0} s_n(x)$ is indeterminate between limits which have been found above. The difficulty which has been frequently felt in understanding how a series, of which the terms are continuous, such as the series here considered, can represent a function which is not continuous, will be removed if the point just explained be fully grasped*, that *the sum of the series at a point x*

* Some criticisms of Dirichlet's determination of the sum of a Fourier's series at a point of discontinuity, made by Schläfli, *Crelle's Journal*, vol. LXXII ((1870) p. 284), and by Du Bois-Reymond, *Math. Annalen*, vol. VII ((1874) p. 244), where it is maintained that the sum of the series is indeterminate, are due to a lack of appreciation of this point.

means the limit obtained by first fixing the abscissa x, and then afterwards making the number of terms increase indefinitely.

It has already been shewn, in § 82, that the points $x = 0, 2\pi, -2\pi, \ldots$, must be points of non-uniform continuity of the series; moreover, other examples have been already given, in which the peaks of the approximation curves $y = s_n(x)$ remain of finite height above the curve $y = s(x)$, however great n may be. That the portions of the limit of the graphs $y = s_n(x)$, in the present case, have a length greater than π, the measure of discontinuity of the function, was pointed out by Willard Gibbs*.

In this and all similar cases, the non-coincidence of the upper and lower double limits of $s_n(x)$, at a point ξ, with the upper and lower limits of $s(x)$ as $x \sim \xi$, is spoken of as Gibbs' phenomenon. The phenomenon, however, had been discovered earlier, in the case of the series $\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots$, by H. Wilbraham†, at the point $x = \frac{1}{2}\pi$. The phenomenon has been fully discussed by Gronwall‡, Dunham Jackson§, and Bôcher‖.

The expression $\int_0^{(n+\frac{1}{2})x} \frac{\sin z}{z} dz - \frac{1}{2}\pi + \frac{2\theta A}{2n+1}$, which has been found above, for $s_n(x) - s(x)$, provided $0 < x \leqq b < 2\pi$, may be employed to shew that the series converges uniformly in any interval (a, b), such that $0 < a < b < 2\pi$. For, by choosing n so great that $\int_0^{(n+\frac{1}{2})x} \frac{\sin z}{z} dz$, for $x \geqq a$, differs from $\frac{1}{2}\pi$ by less than a prescribed number $\frac{1}{2}\epsilon$, which is possible on account of the convergence of the integral, and further choosing n so great that $\frac{2A}{2n+1} < \frac{1}{2}\epsilon$, it is seen that n can be chosen so great that, for the chosen value of n, and for all greater values, $|s_n(x) - s(x)| < \epsilon$, for all values of x in the interval (a, b). This expresses the fact that the series converges uniformly in the interval (a, b). It is clear that the smaller a is taken, the greater must be the value of n, so that $(n + \frac{1}{2})a$ may be sufficiently large to satisfy the requirement that $\left|\int_0^{(n+\frac{1}{2})a} \frac{\sin z}{z} dz - \frac{1}{2}\pi\right| < \frac{1}{2}\epsilon$; and that this value of n increases indefinitely as a is indefinitely diminished. This is a verification of the fact that the convergence of the series is non-uniform at the point $x = 0$.

326. Let $f(x)$ be defined for the interval $(0, \pi)$, by the specifications

$$f(x) = c, \text{ for } 0 \leqq x < \tfrac{1}{2}\pi;\ f(x) = -c, \text{ for } \tfrac{1}{2}\pi \leqq x \leqq \pi.$$

* See an interesting discussion on this subject in *Nature*, vol. LVIII (1898), pp. 544, 569; vol. LIX (1899), pp. 200, 271, 319, 606; vol. LX, pp. 52, 100, in which Gibbs, Michelson, Love, Baker and Poincaré took part.

† *Camb. and Dublin Math. Journ.* new series, vol. III; old series, vol. VII (1848), pp. 198–200.

‡ *Math. Annalen*, vol. LXXII (1912), p. 228.

§ *Rend. di Palermo*, vol. XXXII (1911), p. 257.

‖ *Crelle's Journal*, vol. CXLIV (1914), p. 41. See also Fejér, *Crelle's Journal*, vol. CXLII (1913), p. 165, where methods are given for determining the saltus, and the functional limits, at a point, from a Fourier's series.

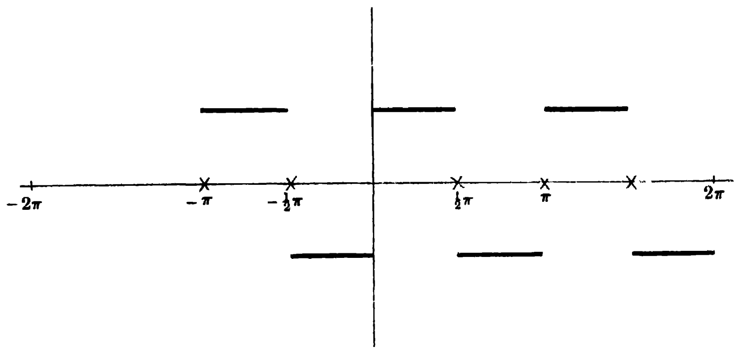
To find the sine series for this function, we have

$$\int_0^\pi f(x) \sin nx\,dx = c\int_0^{\frac{1}{2}\pi} \sin nx\,dx - c\int_{\frac{1}{2}\pi}^\pi \sin nx\,dx$$
$$= \frac{c}{n}(\cos n\pi - 2\cos \tfrac{1}{2}n\pi + 1).$$

This integral vanishes if n is odd, and also if n is a multiple of 4, but if $n = 4m + 2$, it has the value $4c/n$. The series is therefore

$$\frac{8c}{\pi}\left(\tfrac{1}{2}\sin 2x + \tfrac{1}{6}\sin 6x + \tfrac{1}{10}\sin 10x + \ldots\right).$$

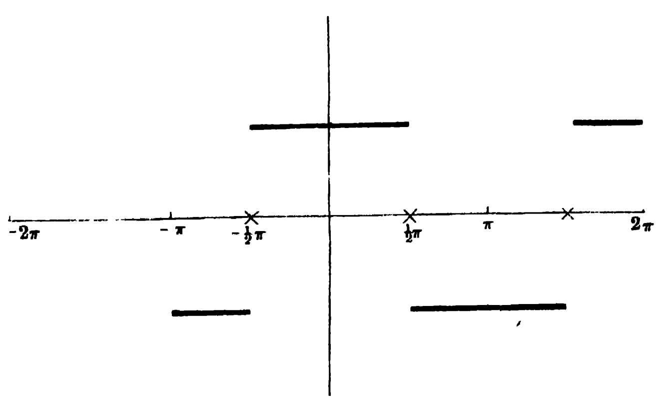
For unrestricted values of x, this series represents the ordinates of the series of straight lines in the next figure, except that it vanishes at the



points 0, $\frac{1}{2}\pi$, π, $-\frac{1}{2}\pi$, $-\pi$, ... It will be observed that, if the meaning of $f(x)$ be altered, so that it denotes the sum of the sine series for every value of x for which that sum is continuous, then at the point π, for example,

$$f(\pi + 0) = c,\ f(\pi - 0) = -c,$$

and the series represents at the point π the arithmetic mean of these two values.



In a similar manner, we find that the function defined for the interval $(0, \pi)$ as before, is represented, for the interval $(0, \pi)$, by the cosine series

$$\frac{4c}{\pi}\left(\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots\right).$$

For unrestricted values of x, the series represents the ordinates of the straight lines in the figure, except that its sum vanishes at the points

$$\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \ldots.$$

327. Let $\qquad f(x) = x, \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\pi,$

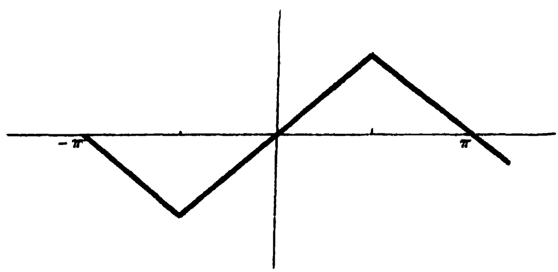
and $\qquad f(x) = \pi - x, \text{ for } \tfrac{1}{2}\pi \leqq x \leqq \pi.$

In this case we find that

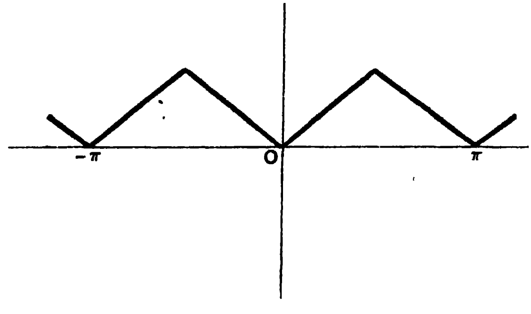
$$\int_0^\pi f(x)\sin nx\,dx = \int_0^{\frac{1}{2}\pi} x \sin nx\,dx + \int_{\frac{1}{2}\pi}^{\pi} (\pi - x)\sin nx\,dx$$
$$= \frac{2}{n^2}\sin \tfrac{1}{2}n\pi.$$

Hence the sine series is

$$\frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right).$$



For general values of x, the series represents the ordinates of the line in the figure. The broken line in the interval $(-\pi, \pi)$ is repeated indefinitely in both directions.



The cosine series, which represents the same function for the interval $(0, \pi)$, will be found to be

$$\frac{1}{4}\pi - \frac{2}{\pi}\left(\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right).$$

This series represents, for general values of x, the ordinates of the line in the second figure. As before, the broken line in the interval $(-\pi, \pi)$ is to be repeated indefinitely in both directions.

EXAMPLES

(1) Prove that the series

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \ldots$$

represents, for the interior of the interval $(-\pi, \pi)$, the function $\frac{1}{2}x$.

For any value of x which is not a multiple of π, the series represents $\frac{1}{2}(x - 2k\pi)$, where k is a positive or negative integer so chosen that $x - 2k\pi$ lies between π and $-\pi$. The sum of the series vanishes for all values of x which are multiples of π.

(2) Prove that the series

$$\cos x - \tfrac{1}{4}\cos 2x + \tfrac{1}{9}\cos 3x - \tfrac{1}{16}\cos 4x + \ldots$$

represents the function $\frac{1}{12}\pi^2 - \frac{1}{4}x^2$, for the interval $(-\pi, \pi)$.

(3) Prove that

$$\tfrac{1}{4}\pi = \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \ldots, \qquad \text{for } 0 < x < \pi;$$

$$\tfrac{1}{4}\pi = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots, \qquad \text{for } -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi.$$

(4) Prove that

$$\tfrac{1}{4}\pi x = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots, \qquad \text{for } -\tfrac{1}{2}\pi \leqq x \leqq \tfrac{1}{2}\pi.$$

(5) Prove that

$$e^{kx} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{k^2+n^2}(1 - e^{k\pi}\cos n\pi)\sin nx, \qquad \text{for } 0 < x < \pi,$$

$$e^{kx} = \frac{e^{k\pi}-1}{k\pi} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{e^{k\pi}\cos n\pi - 1}{k^2+n^2}\cos nx, \qquad \text{for } 0 \leqq x \leqq \pi.$$

(6) Prove that

$$\frac{\pi}{2}\frac{\sin kx}{\sin k\pi} = \frac{\sin x}{1^2-k^2} - \frac{2\sin 2x}{2^2-k^2} + \frac{3\sin 3x}{3^2-k^2} - \ldots, \qquad \text{where } 0 \leqq x < \pi,$$

$$\frac{\pi}{2}\frac{\cos kx}{\sin k\pi} = \frac{1}{2k} - \frac{k\cos x}{k^2-1^2} + \frac{k\cos 2x}{k^2-2^2} - \ldots, \qquad \text{where } 0 \leqq x \leqq \pi;$$

k not being integral.

(7) Prove that

$$\frac{\pi}{2}\frac{\sinh kx}{\sinh k\pi} = \frac{\sin x}{1^2+k^2} - \frac{2\sin 2x}{2^2+k^2} + \frac{3\sin 3x}{3^2+k^2} - \ldots, \qquad \text{where } 0 \leqq x < \pi;$$

$$\frac{\pi}{2k}\frac{\cosh k(\pi - x)}{\sinh k\pi} = \frac{1}{2k^2} + \frac{\cos x}{1^2+k^2} + \frac{\cos 2x}{2^2+k^2} + \frac{\cos 3x}{3^2+k^2} + \ldots, \qquad \text{where } 0 \leqq x \leqq \pi$$

DIRICHLET'S INVESTIGATION OF FOURIER'S SERIES

328. As a preliminary to the consideration of Dirichlet's integral, some properties of the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin mz}{\sin z}\, dz$$

are required.

We have

$$\int_0^{\frac{\pi}{2}} \frac{\sin mz}{\sin z}\, dz = \int_0^{\frac{\pi}{2}} [1 + 2\cos 2z + 2\cos 4z + \dots + 2\cos 2nz]\, dz = \frac{\pi}{2}.$$

If we divide the interval $\left(0, \frac{\pi}{2}\right)$ of integration into the portions

$$\left(0, \frac{\pi}{m}\right), \left(\frac{\pi}{m}, \frac{2\pi}{m}\right), \dots \left(\frac{r\pi}{m}, \frac{\overline{r+1}\,\pi}{m}\right), \dots \left(\frac{n\pi}{m}, \frac{\pi}{2}\right),$$

we see that, in these portions, the integrand $\frac{\sin mz}{\sin z}$ has alternately positive and negative signs; thus if we write

$$\rho_{r-1} = (-1)^{r-1} \int_{\frac{\overline{r-1}\,\pi}{m}}^{\frac{r\pi}{m}} \frac{\sin mz}{\sin z}\, dz,$$

$$\rho_n = (-1)^n \int_{\frac{n\pi}{m}}^{\frac{\pi}{2}} \frac{\sin mz}{\sin z}\, dz,$$

we have $\frac{\pi}{2} = \rho_0 - \rho_1 + \rho_2 + \dots + (-1)^{r-1}\rho_{r-1} + \dots + (-1)^n \rho_n$,

where all the ρ's are positive.

In ρ_{r-1}, $\sin mz$ is always of the same sign, and $\frac{1}{\sin z}$ is monotone and decreases as z increases, hence

$$\rho_{r-1} < (-1)^{r-1} \frac{1}{\sin \frac{(r-1)\pi}{m}} \int_{\frac{\overline{r-1}\,\pi}{m}}^{\frac{r\pi}{m}} \sin mz\,.dz < \frac{2}{m} \operatorname{cosec} \frac{\overline{r-1}\,\pi}{m};$$

and similarly $\rho_{r-1} > \frac{2}{m} \operatorname{cosec} \frac{r\pi}{m}$.

It follows that $\rho_{r-1} > \frac{2}{m} \operatorname{cosec} \frac{r\pi}{m} > \rho_r$.

For ρ_n, we have $\frac{1}{m} \operatorname{cosec} \frac{n\pi}{m} > \rho_n > \frac{1}{m}$,

hence $\rho_{n-1} > \frac{2}{m} \operatorname{cosec} \frac{n\pi}{m} > \rho_n$.

It follows that, if $2p < n$,

$$\frac{\pi}{2} < \rho_0 - \rho_1 + \rho_2 - \dots + \rho_{2p},$$

and

$$\frac{\pi}{2} > \rho_0 - \rho_1 + \rho_2 - \dots - \rho_{2p-1}.$$

Let us suppose that the function $F(z)$ has a finite upper boundary, for the values of z such that $0 \leqq z \leqq \frac{1}{2}\pi$, and further, that it is in the whole interval positive and monotone non-increasing; it is consequently an integrable function.

In the integral
$$\int_0^\alpha F(z) \frac{\sin mz}{\sin z}\, dz,$$
where $\alpha \leqq \frac{1}{2}\pi$, we proceed to divide the interval of integration as in the case of

$$\int_0^{\frac{\pi}{2}} \frac{\sin mz}{\sin z}\, dz$$

into alternately positive and negative portions; thus if

$$s_{r-1} = (-1)^{r-1} \int_{\frac{\overline{r-1}\pi}{m}}^{\frac{r\pi}{m}} F(z) \frac{\sin mz}{\sin z}\, dz,$$

$$s_q = (-1)^q \int_{\frac{q\pi}{m}}^{\alpha} F(z) \frac{\sin mz}{\sin z}\, dz,$$

where q is a positive integer such that

$$\frac{q\pi}{m} < \alpha \leqq \frac{\overline{q+1}\pi}{m},$$

we have

$$\int_0^\alpha F(z) \frac{\sin mz}{\sin z}\, dz = s_0 - s_1 + s_2 - \dots + (-1)^{r-1} s_{r-1} + \dots + (-1)^q s_q,$$

where $s_0, s_1, s_2, \dots s_q$ are all positive. On account of the supposition made as regards $F(z)$, we have

$$\rho_{r-1} F\left(\frac{\overline{r-1}\pi}{m}\right) \geqq s_{r-1} \geqq \rho_{r-1} F\left(\frac{r\pi}{m}\right), \text{ and } s_q \leqq \rho_q F\left(\frac{q\pi}{m}\right).$$

From these inequalities it follows that

$$s_{r-1} \geqq \rho_{r-1} F\left(\frac{r\pi}{m}\right) > \rho_r F\left(\frac{r\pi}{m}\right) > s_r;$$

and this holds for all values of r from 1 to q.

We have consequently the result, that

$$U \equiv \int_0^\alpha F(z) \frac{\sin mz}{\sin z}\, dz$$

is less than $s_0 - s_1 + s_2 - \dots - s_{2p-1} + s_{2p}$,

and greater than $s_0 - s_1 + s_2 - \dots - s_{2p-1}$, where $2p \leqq q$.

From these inequalities, with the help of those obtained above, we have

$$U > (\rho_0 - \rho_1) F\left(\frac{\pi}{m}\right) + (\rho_2 - \rho_3) F\left(\frac{3\pi}{m}\right) + \dots + (\rho_{2p-2} - \rho_{2p-1}) F\left(\frac{\overline{2p-1}\pi}{m}\right)$$

$$> F\left(\frac{2p\pi}{m}\right)(\rho_0 - \rho_1 + \rho_2 - \rho_3 + \dots + \rho_{2p-2} - \rho_{2p-1});$$

also $$U < \rho_0 F(+0) - F\left(\frac{2p\pi}{m}\right)(\rho_1 - \rho_2 + \rho_3 - \dots - \rho_{2p}).$$

On using the theorems which have been proved relating to the ρ's, we obtain

$$U > F\left(\frac{2p\pi}{m}\right)\left(\frac{\pi}{2} - \rho_{2p}\right)$$

and $$U < \rho_0 \left\{F(+0) - F\left(\frac{2p\pi}{m}\right)\right\} + \left(\frac{\pi}{2} + \rho_{2p}\right) F\left(\frac{2p\pi}{m}\right);$$

where, in accordance with the supposition made, p is any integer such that

$$2p \leqq q < \frac{ma}{\pi}.$$

Now let m and p both increase indefinitely, but in such a way that $\frac{2p}{m}$ has the limit zero. Since

$$\rho_{2p} < \frac{2}{m \sin \frac{2p\pi}{m}} < \frac{1}{p\pi} \frac{\frac{2p\pi}{m}}{\sin \frac{2p\pi}{m}},$$

we see that ρ_{2p} has zero for its limit; and hence

$$F\left(\frac{2p\pi}{m}\right)\left(\frac{\pi}{2} - \rho_{2p}\right)$$

has $\frac{\pi}{2} F(+0)$ for its limit. Again

$$\rho_0 < \frac{\pi}{2} + \rho_1 < \frac{\pi}{2} + \frac{2}{\pi} \frac{\frac{\pi}{m}}{\sin \frac{\pi}{m}};$$

and hence ρ_0 has a limiting value not greater than $\frac{\pi}{2} + \frac{2}{\pi}$. It follows that

$$\rho_0 \left\{F(+0) - F\left(\frac{2p\pi}{m}\right)\right\} + \left(\frac{\pi}{2} + \rho_{2p}\right) F\left(\frac{2p\pi}{m}\right)$$

has for its limit the value $\frac{\pi}{2} F(+0)$.

It has been proved that U lies between two numbers, each of which has $\frac{\pi}{2} F(+0)$ for limit, when m and p are indefinitely increased in such a way that $\frac{2p}{m}$ has the limit zero; hence the limit of

$$U \equiv \int_0^\alpha F(z) \frac{\sin mz}{\sin z} dz$$

is

$$\frac{\pi}{2} F(+0),$$

where α is such that $0 < \alpha \leqq \frac{1}{2}\pi$.

It follows, as a corollary from this theorem, that

$$\int_\beta^\alpha F(z) \frac{\sin mz}{\sin z} dz$$

has the limit zero, when m is indefinitely increased; where α, β are two fixed numbers, such that $0 < \beta < \alpha \leqq \frac{1}{2}\pi$.

329. We have now seen that, if $F(z)$ be a bounded and positive function which never increases as z increases from 0 to $\frac{1}{2}\pi$, the integral

$$\int_0^{\frac{\pi}{2}} F(z) \frac{\sin mz}{\sin z} dz$$

converges to the value $\frac{\pi}{2} F(+0)$, as m is increased indefinitely. The function $F(z)$ may be freed from the condition that it must be positive in the whole interval. For if $F\left(\frac{\pi}{2}\right)$ is negative, we may apply the theorem to the function $C + F(z)$, where the constant C is chosen so that

$$C + F\left(\frac{\pi}{2}\right)$$

is positive; thus

$$\int_0^{\frac{\pi}{2}} \{C + F(z)\} \frac{\sin mz}{\sin z} dz$$

converges to the limit

$$\frac{\pi}{2}\{C + F(+0)\}.$$

Now

$$C \int_0^{\frac{\pi}{2}} \frac{\sin mz}{\sin z} dz$$

converges to the limit $\frac{\pi}{2} C$; hence $\int_0^{\frac{\pi}{2}} F(z) \frac{\sin mz}{\sin z} dz$ converges to $\frac{\pi}{2} F(+0)$, where $F(z)$ is not restricted to be positive.

Again, the theorem holds for a function $F(z)$ which is monotone and never diminishes; for we can apply the theorem to the monotone function $-F(z)$ which never increases.

The theorem has now been established, that *if $F(z)$ be any bounded, monotone function, defined for the interval $(0, \frac{1}{2}\pi)$, then*

$$\int_0^{\frac{\pi}{2}} F(z) \frac{\sin mz}{\sin z}\, dz$$

converges, as the odd integer m is increased indefinitely, to the value $\frac{\pi}{2} F(+0)$.

The theorem also holds if the upper limit of the integral be any fixed number a, such that $0 < a \leqq \frac{1}{2}\pi$.

It has been shewn, in I, § 244, that any function with bounded variation is expressible as the difference of two monotone functions. Hence the results which have been established can be immediately extended to functions of this class. We have, therefore, the theorem that, *if $F(z)$ be a function defined for the interval $(0, \frac{1}{2}\pi)$, and with bounded variation, then the integrals*

$$\int_0^{\alpha} F(z) \frac{\sin mz}{\sin z}\, dz, \quad \int_{\alpha}^{\beta} F(z) \frac{\sin mz}{\sin z}\, dz,$$

where $\quad 0 < \alpha \leqq \frac{1}{2}\pi, \quad 0 < \alpha < \beta \leqq \frac{1}{2}\pi,$

converge, as the odd integer m is increased indefinitely, to the values $\frac{\pi}{2} F(+0)$, 0 respectively.

If we apply this result to the two integrals contained in the expression for $s_m(x)$, the sum of the first $2n+1$ terms in Fourier's series, we obtain the theorem that, *if $f(x)$ be a function with bounded variation, defined for the interval $(-\pi, \pi)$, the sum of $2n+1$ terms of the series*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\, dx' + \sum_{n=1} \left\{\frac{1}{\pi}\cos nx \int_{-\pi}^{\pi} f(x') \cos nx' dx' + \frac{1}{\pi}\sin nx \int_{-\pi}^{\pi} f(x') \sin nx' dx'\right\}$$

converges, as n is indefinitely increased, to the value

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\}.$$

It will be remembered that a function with bounded variation is integrable, in accordance with Riemann's definition; and that it can have discontinuities of the first kind only, so that at every point the functional limits $f(x+0)$, $f(x-0)$ exist.

In the case $x = \pm\pi$, the limit to which the sum of the series converges is

$$\tfrac{1}{2}\{f(\pi - 0) + f(-\pi + 0)\}.$$

At a point x of continuity of the function $f(x)$, the limiting sum of the series is $f(x)$; at a point of discontinuity of $f(x)$, the limiting sum of the series agrees with the value of the function at the point only if

$$f(x) = \tfrac{1}{2}\{f(x+0) + f(x-0)\}.$$

At the points π, $-\pi$, the limiting sum of the series agrees with the value of the function only if $f(\pi)$, or $f(-\pi)$, is equal to

$$\tfrac{1}{2}\{f(\pi - 0) + f(-\pi + 0)\}.$$

330. It is now clear in what sense the given function $f(x)$ is represented by the corresponding Fourier's series. The representation is necessarily complete for all points at which the function is continuous, with the possible exception of the end-points $\pm\pi$, which cannot both be points of continuity of the extended function, unless $f(\pi) = f(-\pi)$. At a point of discontinuity, or at an end-point $\pm\pi$, the series represents the function only if the functional value is properly chosen in relation to the functional limits at the point; in the case of the end-points these functional limits are those of the periodic function obtained by extension of the given function beyond the domain for which it was at first defined, this extension being such that $f(x) = f(x + 2\pi)$, as explained in § 323.

The functions with bounded variation include, as a particular case, functions which satisfy the following conditions:

(1) The function is continuous in its domain at every point, with the exception of a finite number of points at which it may have ordinary discontinuities, (2) the domain may be divided into a finite number of parts, such that in any one of them the function is monotone; or in accordance with the more usual expression, the function has only a finite number of maxima and minima in its domain.

These conditions are known as *Dirichlet's conditions*, and his proof, in its original form, applied to the case only of functions which satisfy these conditions.

331. Dirichlet extended his results to the case in which there are a finite number of points in the domain $(-\pi, \pi)$ in the neighbourhood of which $|f(x)|$ has no upper boundary. In this case the Fourier's series must be so interpreted that the integrals in the coefficients are the improper integrals

$$\int_{-\pi}^{\pi} f(x)\,dx, \quad \int_{-\pi}^{\pi} \begin{matrix}\cos\\ \sin\end{matrix}\, nx\, f(x)\,dx,$$

the function being such that these improper integrals exist. From our somewhat more general point of view, we shall suppose that the function $f(x)$ is such that, when arbitrarily small neighbourhoods of these infinite singularities are excluded from the interval $(-\pi, \pi)$, in the remaining part of the interval $f(x)$ is of bounded variation; and further it will be assumed that the improper integral

$$\int_{-\pi}^{\pi} f(x)\,dx$$

exists, and is absolutely convergent. Under these conditions, it can be shewn that the theorems still hold, that the integrals

$$\int_\alpha^\beta F(z)\frac{\sin mz}{\sin z}\,dz, \text{ for } 0<\alpha<\beta\leqq \tfrac{1}{2}\pi,$$

and

$$\int_0^\alpha F(z)\frac{\sin mz}{\sin z}\,dz, \text{ for } 0<\alpha\leqq \tfrac{1}{2}\pi,$$

converge to zero, and to $\frac{\pi}{2}F(+0)$, respectively, as m is increased indefinitely.

If, between α and β, there is a point c in whose neighbourhood $|F(z)|$ has no upper boundary,

$$\int_\alpha^\beta F(z)\frac{\sin mz}{\sin z}\,dz$$

is interpreted by Dirichlet as the limit of

$$\int_\alpha^{c-\delta} F(z)\frac{\sin mz}{\sin z}\,dz + \int_{c+\epsilon}^\beta F(z)\frac{\sin mz}{\sin z}\,dz,$$

where δ, ϵ have, independently of one another, the limit zero; assuming that such limit exists.

Let $\delta'<\delta$, then

$$\left|\left[\int_\alpha^{c-\delta'} - \int_\alpha^{c-\delta}\right] F(z)\frac{\sin mz}{\sin z}\,dz\right| < \operatorname{cosec}\alpha\int_{c-\delta}^{c-\delta'} |F(z)|\,dz;$$

where the expression on the right-hand side is arbitrarily small, on account of the absolute convergence of the integral of $F(z)$, and is independent of the value of m.

Now, if $\int F(z)\,dz$ converges absolutely at the point c, we can choose δ so small that, for every $\delta'<\delta$,

$$\operatorname{cosec}\alpha\int_{c-\delta}^{c-\delta'}|F(z)|\,dz$$

is arbitrarily small; hence the integral

$$\int_\alpha^{c-\delta} F(z)\frac{\sin mz}{\sin z}\,dz,$$

for a fixed m, converges to a definite value, as δ converges to zero. Similarly it can be shewn that

$$\int_{c+\epsilon}^\beta F(z)\frac{\sin mz}{\sin z}\,dz$$

converges to a definite value, as ϵ converges to zero. It has thus been shewn that

$$\int_\alpha^\beta F(z)\frac{\sin mz}{\sin z}\,dz = \lim_{\delta=0}\int_\alpha^{c-\delta} F(z)\frac{\sin mz}{\sin z}\,dz$$
$$+\lim_{\epsilon=0}\int_{c+\epsilon}^\beta F(z)\frac{\sin mz}{\sin z}\,dz = \psi_1(m)+\psi_2(m);$$

and we have now to shew that $\psi_1(m)$, $\psi_2(m)$ converge to zero as m is increased indefinitely. It has been already seen that δ may be so chosen that, for all values of m,

$$\left|\int_a^{c-\delta} F(z)\frac{\sin mz}{\sin z}\,dz - \psi_1(m)\right| < \eta,$$

where η is a fixed arbitrarily small positive number. Now, for a fixed value of δ, m_1 may be chosen so great that, if $m \geqq m_1$,

$$\left|\int_a^{c-\delta} F(z)\frac{\sin mz}{\sin z}\,dz\right| < \zeta,$$

where ζ is arbitrarily small; hence, if $m \geqq m_1$,

$$|\psi_1(m)| < \eta + \zeta,$$

and therefore $\psi_1(m)$ converges to the limit zero; similarly $\psi_2(m)$ converges to the limit zero.

If, between α and β, there are any finite number of points such as c, we may divide the domain (α, β) into a finite number of parts, such that each part contains only one such point as c, and apply the above result to each of the integrals which are taken through one such part.

The integral $\int_0^a F(z)\frac{\sin mz}{\sin z}\,dz$ can be divided into two parts

$$\int_0^{a_1} F(z)\frac{\sin mz}{\sin z}\,dz + \int_{a_1}^a F(z)\frac{\sin mz}{\sin z}\,dz,$$

where a_1 is so chosen that all the points of infinite discontinuity of $F(z)$ are in (a_1, a); we thus see that $\int_0^a F(z)\frac{\sin mz}{\sin z}\,dz$ converges to $\frac{\pi}{2}F(+0)$, when m is indefinitely increased.

It has now been shewn that: *if $f(x)$ be such that, when the arbitrarily small neighbourhoods of a finite number of points in whose neighbourhood $|f(x)|$ has no upper boundary have been excluded, $f(x)$ becomes a function with bounded variation, then the Fourier's series*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \Sigma\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos n(x - x')\,dx'$$

converges to the value $\frac{1}{2}\{f(x+0) + f(x-0)\}$, at every point in $(-\pi, \pi)$, except at the points of infinite discontinuity of the function, provided the improper integral $\int_{-\pi}^{\pi} f(x)\,dx$ exists, and is absolutely convergent.

APPLICATION OF THE SECOND MEAN VALUE THEOREM

332. An alternative method of investigation of the limit to which the partial sum of Fourier's series, corresponding to a function of bounded variation in the interval $(-\pi, \pi)$, converges, is obtained by the employment of the second mean value theorem (I, § 422). This method was first

employed by Bonnet, who used his form of the mean value theorem*. The method was also used in his treatise by C. Neumann†, and by Jordan‡ who applied it to the case of functions of bounded variation. The method is also employed, and discussed in great detail, in Dini's treatise§.

We have need of the following lemmas:

(1) $\int_0^{\frac{1}{2}\pi} \frac{\sin mz}{\sin z} dz = \frac{1}{2}\pi$, *m being an odd integer.* This has already been proved in § 328.

(2) *If* $0 < \alpha < \beta \leqq \frac{1}{2}\pi$, $\int_\alpha^\beta \frac{\sin mz}{\sin z} dz = o(1)$, *and* $\int_\alpha^\beta \frac{\sin mz}{z} dz = o(1)$.

To prove this, we have, by the second mean value theorem,

$$\int_\alpha^\beta \frac{\sin mz}{\sin z} dz = \frac{1}{\sin\alpha}\int_\alpha^\gamma \sin mz\, dz + \frac{1}{\sin\beta}\int_\gamma^\beta \sin mz\, dz,$$

where γ is in the interval (α, β); and therefore

$$\left|\int_\alpha^\beta \frac{\sin mz}{\sin z} dz\right| < \frac{2}{m}(\operatorname{cosec}\alpha + \operatorname{cosec}\beta) < \frac{4}{m}\operatorname{cosec}\alpha,$$

from which the result follows. The second part of the theorem is proved in a similar manner.

(3) *If* $0 \leqq \alpha < \beta$, *then* $\left|\int_\alpha^\beta \frac{\sin\theta}{\theta} d\theta\right| \leqq \pi$.

By the mean value theorem, if $0 < \alpha < h$, $\left|\int_\alpha^h \frac{\sin\theta}{\theta} d\theta\right| < \frac{2}{\alpha} + \frac{2}{h}$; and therefore $\left|\int_\alpha^\infty \frac{\sin\theta}{\theta} d\theta\right| \leqq \frac{2}{\alpha}$; and if $\alpha \geqq \pi$, we have $\left|\int_\alpha^\infty \frac{\sin\theta}{\theta} d\theta\right| \leqq \frac{2}{\pi} < \frac{\pi}{2}$. It is clear that, as α increases from 0 to π, $\int_\alpha^\infty \frac{\sin\theta}{\theta} d\theta$ diminishes, since $\int_\alpha^\pi \frac{\sin\theta}{\theta} d\theta$ does so. Therefore, since $\int_0^\infty \frac{\sin\theta}{\theta} = \frac{1}{2}\pi$, we have

$$\left|\int_\alpha^\infty \frac{\sin\theta}{\theta} d\theta\right| \leqq \frac{1}{2}\pi,$$

if $\alpha < \pi$, and it has been shewn to be $< \frac{1}{2}\pi$, if $\alpha \geqq \pi$; hence

$$\left|\int_\alpha^\infty \frac{\sin\theta}{\theta} d\theta\right| \leqq \frac{1}{2}\pi,$$

for $0 \leqq \alpha$. It now follows that $\left|\int_\alpha^\beta \frac{\sin\theta}{\theta} d\theta\right| \leqq \pi$, where $0 \leqq \alpha < \beta$.

After having established these lemmas, we proceed to consider $\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z} dz$, where $F(z)$ is monotone, and non-diminishing, in the interval $(0, \frac{1}{2}\pi)$.

* *Mémoires des Savants étrangers* of the Belgian Academy, vol. XXIII.

† *Ueber die nach Kreis- Kugel- und Cylinder-functionen fortschreitenden Reihen*, Leipsic, 1881.

‡ *Cours d'Analyse*, vol. II.

§ *Sopra la Serie di Fourier*, Pisa (1880).

If μ be a fixed positive number, we have

$$\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz - F(+0)\int_0^{\frac{1}{2}\pi}\frac{\sin mz}{\sin z}\,dz$$

$$= \int_0^{\mu}\{F(z) - F(+0)\}\frac{\sin mz}{\sin z}\,dz + \int_{\mu}^{\frac{1}{2}\pi}\{F(z) - F(+0)\}\frac{\sin mz}{\sin z}\,dz.$$

On applying the second mean value theorem, we have

$$\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz - \tfrac{1}{2}\pi F(+0) = \int_0^{\mu} G(z)\frac{\sin mz}{z}\,dz$$

$$+ \{F(\mu) - F(+0)\}\int_{\mu}^{\xi_1}\frac{\sin mz}{\sin z}\,dz + \{F(\tfrac{1}{2}\pi) - F(+0)\}\int_{\xi_1}^{\frac{1}{2}\pi}\frac{\sin mz}{\sin z}\,dz,$$

where ξ_1 is some number in the interval $(\mu, \frac{1}{2}\pi)$, and $G(z)$ denotes the monotone, non-diminishing, function $\{F(z) - F(+0)\}\dfrac{z}{\sin z}$.

Again,

$$\int_0^{\mu} G(z)\frac{\sin mz}{z}\,dz = G(\mu)\int_{\xi}^{\mu}\frac{\sin mz}{z}\,dz = G(\mu)\int_{m\xi}^{m\mu}\frac{\sin z}{z}\,dz,$$

where ξ is in the interval $(0, \mu)$.

The number ξ depends on m, and on the function $G(z)$; it may happen that, as m is indefinitely increased, ξ diminishes indefinitely in such a manner that $m\xi$ has a finite limit. Whether this happens or not, we see from (3) that $\left|\int_0^{\mu} G(z)\dfrac{\sin mz}{z}\,dz\right|$ does not exceed $\pi\,|\,G(\mu)\,|$, and μ may be so chosen that this is less than the arbitrarily chosen positive number ϵ.

Since

$$\left|\int_{\mu}^{\xi_1}\frac{\sin mz}{\sin z}\,dz\right| < \frac{4}{m}\operatorname{cosec}\mu, \quad \left|\int_{\xi_1}^{\frac{1}{2}\pi}\frac{\sin mz}{\sin z}\,dz\right| < \frac{4}{m}\operatorname{cosec}\xi_1 < \frac{4}{m}\operatorname{cosec}\mu,$$

it is seen that both integrals converge to zero, as $m \sim \infty$, notwithstanding the fact that ξ_1 is dependent upon m. It now follows that μ and m_1 can be so chosen that, for $m \geqq m_1$,

$$\left|\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz - \tfrac{1}{2}\pi F(+0)\right| < 2\epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{m\sim\infty}\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz = \tfrac{1}{2}\pi F(+0).$$

Since any function that is of bounded variation in the interval $(0, \frac{1}{2}\pi)$ is expressible as the difference of two monotone non-diminishing functions, it follows that this result holds for any function $F(z)$ which is of bounded variation in the interval $(0, \frac{1}{2}\pi)$. Writing $f(x + 2z) + f(x - 2z)$ for $F(z)$, we see that

$$\lim_{n\sim\infty}\int_0^{\frac{1}{2}\pi}\{f(x+2z) + f(x-2z)\}\frac{\sin(2n+1)z}{\sin z}\,dz = \tfrac{1}{2}\pi\{f(x+0) + f(x-0)\}.$$

Thus the convergence of the Fourier's series at any point x of the interval $(-\pi, \pi)$ has been established. The following theorem has been established:

If $f(x)$ have bounded variation in the interval $(-\pi, \pi)$, the Fourier's series corresponding to $f(x)$ converges to the value $f(x)$ at any point within the interval, at which the function is continuous; it converges to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any such point at which the function is discontinuous. At the points π, $-\pi$ it converges to the value

$$\tfrac{1}{2}\{f(-\pi+0)+f(\pi-0)\}.$$

333. It is known that a convergent series of continuous functions is non-uniformly convergent in the neighbourhood of a point of discontinuity of the sum-function, but that the series is not necessarily uniformly convergent in an interval in which it is continuous. In the case of the Fourier's series corresponding to a function $f(x)$ which is of bounded variation in the interval $(-\pi, \pi)$, it can be shewn that the series converges uniformly in the whole interval $(-\pi, \pi)$, provided the function obtained by extending $f(x)$ beyond the interval, as a periodic function, is continuous in the closed interval $(-\pi, \pi)$. This requires the condition $f(\pi-0)=f(-\pi+0)$ to hold, in which case the complete continuity holds if the values of $f(\pi)$ and $f(-\pi)$ are the same as those of $f(\pi-0)$ and $f(-\pi+0)$. The function then converges uniformly to $f(x)$ in the whole interval $(-\pi, \pi)$.

It can further be shewn that, provided $f(x)$ is of bounded variation in $(-\pi, \pi)$, the series converges uniformly to $f(x)$ in any interval (a, b) in which the function is continuous, the continuity at the points a, b being on both sides.

It has been shewn in § 332 that

$$\left|\int_0^{\frac{1}{2}\pi} F(z)\frac{\sin mz}{\sin z}\,dz-\frac{\pi}{2}F(+0)\right| < \pi\,|G(\mu)| + \frac{4}{m\sin\mu}\,|F(\mu)-F(+0)|$$
$$+\frac{4}{m\sin\mu}\,|F(\tfrac{1}{2}\pi)-F(+0)|,$$

where $F(z)$ is monotone non-diminishing. Using this inequality, and the corresponding one for $-F(-z)$, and writing $f(x+2z)+f(x-2z)$ for $F(z)+F(-z)$, we have

$$|s_{2n+1}(x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}| < |G(\mu)|+|G_1(\mu)|$$
$$+\frac{4}{\pi m\sin\mu}\{|f(x+2\mu)-f(x+0)|+|f(x+\pi)-f(x+0)|$$
$$+|f(x-2\mu)-f(x-0)|+|f(x-\pi)-f(x-0)|\}$$
$$<|G(\mu)|+|G_1(\mu)|+\frac{A}{m}\operatorname{cosec}\mu;$$

where A is a fixed number dependent on the upper boundary of $f(x)$ in the whole interval $(-\pi, \pi)$, and

$$G(\mu) = \{f(x+2\mu) - f(x+0)\}\,\mu \operatorname{cosec} \mu,$$
$$G_1(\mu) = \{f(x-2\mu) - f(x-0)\}\,\mu \operatorname{cosec} \mu.$$

If now $f(x)$ is continuous and monotone in the interval (a, b), and is continuous at a and b on both sides, on account of the uniform continuity of the function, μ_1 can be so determined, that, for all points x in the interval (a, b), $|f(x+2\mu) - f(x)|$ and $|f(x-2\mu) - f(x)|$ are both less than $\frac{\epsilon}{\pi}$, provided $\mu \leqq \mu_1$. Also $\mu \operatorname{cosec} \mu < \frac{1}{2}\pi$, thus $|G(\mu)|$ and $|G_1(\mu)|$ are both less than $\frac{1}{2}\epsilon$, provided $\mu \leqq \mu_1$, for all values of x in (a, b). Therefore $|s_{2n+1}(x) - f(x)| < \epsilon + \frac{A}{m} \operatorname{cosec} \mu_1$, for all values of x. The number μ_1 having been fixed, an integer m_1 can be so determined that $\frac{A}{m} \operatorname{cosec} \mu_1 < \epsilon$, for $m \geqq m_1$, and therefore $|s_{2n+1}(x) - f(x)| < 2\epsilon$, for $m \geqq m_1$, and for all points x in (a, b). Since ϵ is arbitrary, this establishes the uniform convergence of $s_{2n+1}(x)$ to $f(x)$ in the interval (a, b). The function $f(x)$ has been taken to be monotone, but a function of bounded variation may be expressed as the difference of two monotone functions, each of which is continuous in (a, b) when $f(x)$ is so. Therefore the theorem holds for any function of bounded variation in $(-\pi, \pi)$.

It has thus been shewn that:

If $f(x)$ be of bounded variation in $(-\pi, \pi)$, the Fourier's series converges to $f(x)$, uniformly in any interval (a, b) in which $f(x)$ is continuous, the continuity at a and b being on both sides.

Returning to the general case in which the function $f(x)$, of bounded variation in $(-\pi, \pi)$, may have discontinuities in an enumerable set of points of the interval, we see that, if μ be a fixed number, $G(\mu)$ and $G_1(\mu)$ are bounded for all values of x in the interval $(-\pi, \pi)$, since $f(x)$ is a bounded function. We find that

$$|s_{2n+1}(x)| < K + \frac{A}{m} \operatorname{cosec} \mu,$$

where K depends only on the fixed number μ, and on the upper boundary of the functions $|f(x)|$ in the interval $(-\pi, \pi)$. When $f(x)$ is not monotone, the result can be, as before, immediately extended to any function of bounded variation. Since $|s_{2n+1}(x)| < K + A \operatorname{cosec} \mu$, it is seen that $|s_{2n+1}(x)|$ is bounded for all values of n.

We have accordingly established the theorem* that:

If $f(x)$ be any function of bounded variation in the interval $(-\pi, \pi)$, the Fourier's series converges boundedly to the value $\frac{1}{2}\{f(x+0) + f(x-0)\}$ throughout the interval $(-\pi, \pi)$.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX, p. 453.

THE LIMITING VALUES OF FOURIER'S COEFFICIENTS

334. The following general property of the Fourier coefficients of a summable function was first established by Lebesgue*:

If a_n, b_n denote the Fourier's coefficients corresponding to any summable function $f(x)$, then $a_n = o(1)$, $b_n = o(1)$.

This theorem is a generalization of the theorem due to Riemann† in which the function is restricted to be integrable (R). It is consequently frequently known as the Riemann-Lebesgue theorem. The case when the function is continuous‡ was treated by Stäckel.

Lebesgue's theorem can be obtained as a special case of the general convergence Theorem I, of § 279. Consider the interval (a, b), and let $\Phi(x', x, n) = \frac{\cos}{\sin} nx'$, the set G consisting in this case of a single point, so that x does not occur. The conditions (1) and (2) of the theorem are satisfied, since $\left|\frac{\sin}{\cos} nx'\right| \leqq 1$, $\lim\limits_{n\sim\infty} \int_a^\beta \frac{\sin}{\cos} nx'\, dx' = 0$; from which it follows, in accordance with the theorem, that $\lim\limits_{n\sim\infty} \int_a^b f(x') \cos nx'\, dx' = 0$, and $\lim\limits_{t\sim 0} \int_a^b f(x') \sin nx'\, dx' = 0$. It will be observed that n may diverge as any sequence of positive numbers, not necessarily integral.

It has been shewn by Lebesgue§ that this theorem cannot be made more precise; that, in fact, if $u(n)$ be any function which converges monotonely to zero, as $n \sim \infty$, a continuous function $f(x)$, such that $|f(x)| \leqq 1$, can be constructed, for which the coefficients are not of order superior to that of $u(n)$.

The following more general theorem may be given:

If $f(x)$ be summable in the finite interval (a, b), then $\int_a^\beta f(x) \frac{\sin}{\cos} nx\, dx$ converges to zero, as $n \sim \infty$, uniformly for all intervals (α, β) contained in (a, b).

To deduce the theorem from Theorem I, of § 279, let the set G be a two-dimensional set consisting of all points (α, β) such that $\alpha \leqq \beta$, $a \leqq \alpha \leqq b$, $a \leqq \beta \leqq b$. Let $\Phi(x', x, n)$, when $x \equiv (\alpha, \beta)$, be defined by $\Phi(x', x, n) = \frac{\sin}{\cos} nx'$, for all values of x' in the interval (α, β), and $\Phi(x', x, n) = 0$, when x' is not in that interval.

* *Annales sc. de l'école normale sup.* (3). vol. XX (1903), p. 471.

† *Ges. Werke*, 2nd ed. vol. I, p. 254.

‡ *Leipz. Ber.* vol. LIII (1901), p. 147; also *Nouvelles Annales* (4), vol. II (1902), p. 57.

§ *Bulletin de la soc. mat. de France*, vol. XXXVIII (1910), p. 184.

Since $|\Phi(x', x, n)| \leqq 1$, for all points x', x, and for all values of n, and since $\int_{\alpha'}^{\beta'} \Phi(x', x, n)\, dx' \leqq \frac{2}{n}$, which converges to zero, uniformly for all points x, the conditions of the theorem are satisfied. Therefore

$$\int_\alpha^\beta f(x') \frac{\sin}{\cos} nx'\, dx'$$

converges to zero uniformly for all pairs of values of α, β in the interval (a, b).

It is of interest to have a direct proof of the theorem. First let $0 < a < b$; we then prove the theorem for the particular function $f(x) = x^r$.

Since $$\left| \int_\alpha^\beta x^r \frac{\sin}{\cos} nx\, dx \right| = \beta^r \left| \int_\gamma^\beta \frac{\sin}{\cos} nx\, dx \right| \leqq \frac{2}{n} b^r,$$

where γ is in the interval (α, β), the theorem is established for this case. Next let $f(x) = P(x)$, where $P(x)$ is a finite polynomial; the integral is then the sum of a finite number of integrals, each of which converges uniformly to zero, as $n \sim \infty$. It is clear that the condition that a and b should be positive can be at once removed by changing the variable x into a new variable $\xi = x + k$; $P(x)$ then becoming a polynomial $P(\xi)$. Thus the complete theorem holds for a function which is a polynomial $P(x)$. Next, if $f(x)$ be any function, summable in (a, b), a finite polynomial $P(x)$ can be so determined that $\int_a^b |f(x) - P(x)|\, dx < \epsilon$ (see I, § 430). We have then

$$\int_\alpha^\beta f(x) \frac{\sin}{\cos} nx\, dx = \int_\alpha^\beta P(x) \frac{\sin}{\cos} nx\, dx + \int_\alpha^\beta \{f(x) - P(x)\} \frac{\sin}{\cos} nx\, dx.$$

The second integral on the right-hand side is numerically less than ϵ, whatever values α, β, n may have. The numerical value of the first integral on the right-hand side is also $< \epsilon$, provided $n > n_\epsilon$, for all values of α and β. Therefore

$$\left| \int_\alpha^\beta f(x) \frac{\sin}{\cos} nx\, dx \right| < 2\epsilon, \quad \text{if } n > n_\epsilon,$$

for all the values of α, β, in the interval (a, b). Since ϵ is arbitrary, the theorem has been established. It may be observed that the theorem holds good if n be a continuous variable which increases indefinitely.

It may be observed that the extension of Theorem I, in § 280, to the case in which the interval (a, b) is indefinitely great, when $f(x)$ is absolutely summable in the indefinite interval, furnishes a proof of the following theorem:

If $f(x)$ is absolutely summable in one of the intervals $(0, \infty)$, $(-\infty, \infty)$

$$\int_0^\infty f(x) \frac{\cos}{\sin} nx\, dx = o(1), \text{ or } \int_{-\infty}^\infty f(x) \frac{\cos}{\sin} nx\, dx = o(1).$$

335. In case the function $f(x)$ is bounded and monotone in the interval $(-\pi, \pi)$, we have,

$$\int_{-\pi}^{\pi} f(x) \frac{\sin}{\cos} nx\, dx = f(-\pi + 0) \int_{-\pi}^{a} \frac{\sin}{\cos} nx\, dx + f(\pi - 0) \int_{a}^{\pi} \frac{\sin}{\cos} nx\, dx,$$

or $\left| \int_{-\pi}^{\pi} f(x) \frac{\cos}{\sin} nx\, dx \right| < \frac{A}{n}$, where A is a fixed number depending only upon the upper boundary of $f(x)$ in $(-\pi, \pi)$. Since any function of bounded variation in $(-\pi, \pi)$ can be expressed as the difference of two monotone functions, we have clearly

$$\left| \int_{-\pi}^{\pi} f(x) \frac{\sin}{\cos} nx\, dx \right| < \frac{K}{n},$$

where K is a fixed number. It has now been proved that:

If $f(x)$ have bounded variation in the interval $(-\pi, \pi)$, then $a_n = O(n^{-1})$, $b_n = O(n^{-1})$.

It is clear that, if $f(x)$ be of bounded variation in the interval (a, b), then $\left| \int_a^b f(x) \frac{\sin}{\cos} nx\, dx \right| < \frac{K}{n}$, where K is a fixed number, independent of n. The number n may be taken to be any positive number, not necessarily integral.

EXAMPLES

(1) Let $f(x) = -x + \lim_{m \sim \infty} \int_0^x (1 + \cos x)(1 + \cos 4x) \dots (1 + \cos 4^{m-1} x)\, dx$; then $f(x)$ is a continuous periodic function, of period 2π, and it is of bounded variation in the interval $(-\pi, \pi)$.

It can be shewn that the Fourier's coefficients of this function are such that $nb_n = 1$, when n is a power of 4. This example was given by F. Riesz* to illustrate the fact that, for a continuous function of bounded variation, the condition $a_n = o\left(\frac{1}{n}\right)$, $b_n = o\left(\frac{1}{n}\right)$ are not necessarily satisfied.

(2) If $f(x)$ is of bounded variation in the infinite interval (a, ∞), and converges to zero as $x \sim \infty$, the integrals $\int_a^\infty f(x) \frac{\sin}{\cos} nx\, dx$ exist, and are $O\left(\frac{1}{n}\right)$.

We have $f(x) = P(x) - N(x)$, where $P(x)$, $N(x)$ are monotone non-increasing, and converge to zero, as $x \sim \infty$.

If $A' > A > a$, we have $\left| \int_A^{A'} P(x) \frac{\sin}{\cos} nx\, dx \right| = P(A) \left| \int_A^{A'} \frac{\sin}{\cos} nx\, dx \right| \leq \frac{2}{n} P(A)$.

Since $P(A)$ is arbitrarily small, if A be sufficiently large, the integral on the left-hand side is $<\epsilon$, for all values of A', when A is properly chosen. Therefore $\int_a^\infty P(x) \frac{\sin}{\cos} nx\, dx$ exists. Also $\int_A^\infty P(x) \frac{\sin}{\cos} nx\, dx = O\left(\frac{1}{n}\right)$; and since $\int_a^A P(x) \frac{\sin}{\cos} nx\, dx = O\left(\frac{1}{n}\right)$, we have

$$\int_a^\infty P(x) \frac{\sin}{\cos} nx\, dx = O\left(\frac{1}{n}\right).$$

A similar result holds for $N(x)$; therefore $\int_a^\infty f(x) \frac{\sin}{\cos} nx\, dx = O\left(\frac{1}{n}\right)$.

* *Math. Zeitschr.* vol. II (1918), p. 312.

(3*) If the even function $f(x)$ is continuous, and is of bounded variation, and is an integral in any interval which does not contain the point $x=0$, then $a_n = o\left(\frac{1}{n}\right)$.

Since $f(x)$ is an integral in the interval (η, π), it has a summable differential coefficient $f'(x)$, thus

$$\int_\eta^\pi f(x)\cos nx\,dx = -f(\eta)\frac{\sin n\eta}{n} - \frac{1}{n}\int_\eta^\pi f'(x)\sin nx\,dx.$$

Since $f(x)$ is continuous and of bounded variation in $(0, \eta)$, we have, expressing $f(x)$ as the difference of two monotone functions $f_1(x)$, $f_2(x)$,

$$\int_0^\eta f_1(x)\cos nx\,dx = f_1(0)\frac{\sin n\eta'}{n} + f_1(\eta)\left(\frac{\sin n\eta - \sin n\eta'}{n}\right),$$

$$\int_0^\eta f_2(x)\cos nx\,dx = f_2(0)\frac{\sin n\eta''}{n} + f_2(\eta)\left(\frac{\sin n\eta - \sin n\eta''}{n}\right),$$

where η', η'' are in the interval $(0, 1)$. We now find that

$$n\int_0^\pi f(x)\cos nx\,dx = -\int_\eta^\pi f'(x)\sin nx\,dx + [f_1(0) - f_1(\eta)]\sin n\eta' + [f_2(0) - f_2(\eta)]\sin n\eta'',$$

hence the expression on the left-hand side is less than 3ϵ, provided η be chosen sufficiently small, and n sufficiently large. It has thus been shewn that $a_n = o\left(\frac{1}{n}\right)$.

(4) If $f(x)$ is summable in every finite interval, and $g(x)$ is of bounded variation in (a, b), then $\int_a^b f(x+u)\,g(u)\,{\sin \atop \cos}\,nu\,du$ converges to zero, as $n \sim \infty$, uniformly for all values of x in any finite interval.

It is sufficient to assume that $g(u)$ is monotone, then

$$\int_a^b f(x+u)\,g(u)\,{\sin \atop \cos}\,nu\,du = g(a)\int_a^A f(x+u)\,{\sin \atop \cos}\,nu\,du + g(b)\int_A^b f(x+u)\,{\sin \atop \cos}\,nu\,du,$$

where A is in the interval (a, b), and depends on n and x. It is sufficient to shew that the result holds for each of the integrals on the right-hand side. Let $x+u=v$, then it is easily seen to be sufficient that the integrals $\int_\alpha^\beta f(v)\,{\sin \atop \cos}\,nv\,dv$ converge to zero, as $n \sim \infty$, uniformly for all values of α and β in a finite interval. This has been shewn in § 334 to be the case.

336. Let the function $f(x)$ be assumed to have, in the whole interval $(-\pi, \pi)$, a differential coefficient $f^{(r)}(x)$ of order r, which is continuous in the interval, or more generally a differential coefficient $f^{(r-1)}(x)$ which is an indefinite integral.

The integral $\int_{-\pi}^{\pi} f(x)\,{\sin \atop \cos}\,nx\,dx$ may then be expressed by r successive integrations by parts, and its value depends upon that of

$$\frac{1}{n^r}\int_{-\pi}^{\pi} f^{(r)}(x)\,{\sin \atop \cos}\,nx\,dx,$$

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. x (1911), p. 256.

which exists, since $f^{(r)}(x)$ exists almost everywhere in the interval, and is summable. As the integral converges to zero, we have the following theorem:

If $f(x)$ have a continuous differential coefficient $f^{(r)}(x)$, of order r, or more generally, if $f^{(r-1)}(x)$ exists and is an indefinite integral, then $a_n = o(n^{-r})$, $b_n = o(n^{-r})$.

337. Let us consider the function $f(x) = |x|^{-\nu}\phi(x)$, where $0 < \nu < 1$, and $\phi(x)$ is of bounded variation in $(-\pi, \pi)$.

We have

$$\int_{-\pi}^{\pi} \frac{\phi(x)}{|x|^\nu} \sin nx\, dx = \int_{-\mu}^{\mu} \frac{\phi(x)}{|x|^\nu} \sin nx\, dx + \int_{-\pi}^{\pi} \psi(x) \sin nx\, dx,$$

where $\psi(x) = \dfrac{\phi(x)}{|x|^\nu}$ in the intervals $(-\pi, -\mu)$, (μ, π) and has the value 0 in the interval $(-\mu, \mu)$. The function $\psi(x)$ is of bounded variation in the interval $(-\pi, \pi)$, and therefore $\int_{-\pi}^{\pi} \psi(x) \sin nx\, dx = O(n^{-1})$. We now consider the integral $\int_0^\mu \dfrac{\chi(x)}{x^\nu} \sin nx\, dx$, where $\chi(x)$ denotes $\phi(x) + \phi(-x)$. Dividing the interval $(0, \mu)$ into the two parts $\left(0, \dfrac{1}{\sqrt{n}}\right)$ and $\left(\dfrac{1}{\sqrt{n}}, \mu\right)$, we have

$$\int_{\frac{1}{\sqrt{n}}}^{\mu} \frac{\chi(x)}{x^\nu} \sin nx\, dx = \int_{\frac{1}{\sqrt{n}}}^{\mu} \frac{P(x)}{x^\nu} \sin nx\, dx - \int_{\frac{1}{\sqrt{n}}}^{\mu} \frac{Q(x)}{x^\nu} \sin nx\, dx,$$

where $P(x)$, $Q(x)$ are two non-increasing monotone functions. The expression on the right-hand side is, applying Bonnet's form of the second mean theorem, numerically less than a fixed multiple of $\dfrac{1}{n} n^{\frac{1}{2}\nu}$, or $n^{\frac{1}{2}\nu - 1}$. Also

$$\int_0^{\frac{1}{\sqrt{n}}} \frac{P(x)}{x^\nu} \sin nx\, dx = P(0) \int_0^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx + \left[P\left(\frac{1}{\sqrt{n}}\right) - P(0)\right] \int_\xi^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx,$$

where ξ is in the interval $\left(0, \dfrac{1}{\sqrt{n}}\right)$; also a similar result holds for $Q(x)$. We thus find that

$$\left|\int_0^{\frac{1}{\sqrt{n}}} \frac{\chi(x)}{x^\nu} \sin nx\right| < A \left|\int_0^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx\right| + B \left|\int_\xi^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx\right|,$$

where A and B are fixed numbers, independent of n; we now have

$$\int_0^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx = \frac{1}{n^{1-\nu}} \int_0^{\sqrt{n}} \frac{\sin x}{x^\nu}\, dx,$$

$$\int_\xi^{\frac{1}{\sqrt{n}}} \frac{\sin nx}{x^\nu}\, dx = \frac{1}{n^{1-\nu}} \int_{n\xi}^{\sqrt{n}} \frac{\sin x}{x^\nu}\, dx.$$

It is known that

$$\int_0^\infty \frac{\sin x}{x^\nu}\, dx = \frac{\pi}{2 \sin \frac{1}{2}\nu\pi \,.\, \Gamma(\nu)};$$

and if the indefinite interval of integration be divided into parts $(0, \pi)$, $(\pi, 2\pi)$, ..., the integral is represented by a series $I_1 - I_2 + I_3 - \dots$, when I_m denotes

$$(-1)^{m+1} \int_0^\pi \frac{\sin x}{(x + \overline{m-1}\,\pi)^\nu}\,; \text{ thus } I_1 > I_2 > I_3 \dots.$$

It now easily follows that $\int_\alpha^\beta \frac{\sin x}{x^\nu}\, dx$ is numerically less than a fixed number, independent of α and β. It is thus seen that

$$n^{1-\nu} \int_0^{\frac{1}{\sqrt{n}}} \frac{\chi(x)}{x^\nu} \sin nx\, dx$$

is less than a fixed number, independent of n, and it then follows that

$$\left| \int_0^\mu \frac{\chi(x)}{x^\nu} \sin nx\, dx \right| = O(n^{\frac{1}{2}\nu - 1}) + O(n^{\nu-1}) = O(n^{\nu-1}).$$

We have now

$$\int_{-\pi}^{\pi} \frac{\phi(x)}{|x|^\nu} \sin nx\, dx = O(n^{-1}) + O(n^{\nu-1}) = O(n^{\nu-1}).$$

A corresponding result can be obtained when $\cos nx$ takes the place of $\sin nx$. There is no loss of generality in taking any point β of the interval as the singular point instead of the point O, because the interval $(-\pi, \pi)$ can be replaced by the interval $(\beta - \pi, \beta + \pi)$, the function $f(x)$ being taken to be periodic. The following theorem has now been established:

If $f(x)$ be defined as $\frac{\phi(x)}{|x - \beta|^\nu}$, where β is any point interior to $(-\pi, \pi)$, ν is any positive number less than unity, and $\phi(x)$ is of bounded variation in the interval $(-\pi, \pi)$, the Fourier's coefficients of $f(x)$ have the property $a_n = O(n^{\nu-1})$, $b_n = O(n^{\nu-1})$.

338. The following theorem is of use in some parts of the theory:

If $f(x)$ be summable in (a, b), then

$$\lim_{n\sim\infty} \int_a^b f(x) \,|\sin nx|\, dx = \lim_{n\sim\infty} \int_a^b f(x)\,|\cos nx|\, dx = \frac{2}{\pi} \int_a^b f(x)\, dx,$$

where n is unrestricted.

Let the finite polynomial $P(x)$ be so determined that

$$\int^b |f(x) - P(x)|\, dx < \eta.$$

If
$$P(x)=\sum_{s=0}^{s=m} A_s x^s,$$
we have
$$\int_a^b P(x)\,|\sin nx|\,dx=\sum_{s=0}^{s=m} A_s\int_a^b x^s\,|\sin nx|\,dx.$$
Dividing (a, b) into intervals (α, β) in each of which $\sin nx$ is of fixed sign, we have
$$\int_a^b P(x)\,|\sin nx|\,dx=\sum_{s=0}^{s=m} A_s\left\{\sum_{(\alpha,\beta)}\int_\alpha^\beta x^s\,|\sin nx|\,dx\right\}$$
$$=\sum_{s=0}^{s=m} A_s\left\{\sum_{(\alpha,\beta)}\frac{\delta_{(\alpha,\beta)}}{n}\xi^s_{(\alpha,\beta)}\right\},$$
where $\delta_{(\alpha,\beta)}$ is equal to 2, for all the intervals (α, β) except possibly the two extreme intervals, in which it may be < 2; and $\xi_{(\alpha,\beta)}$ is some number in the interval (α, β). It now follows that
$$\lim_{n\sim\infty}\int_a^b P(x)\,|\sin nx|\,dx=\frac{2}{\pi}\sum_{s=0}^{s=m} A_s\left\{\int_a^b x^s dx\right\}=\frac{2}{\pi}\int_a^b P(x)\,dx.$$
Since $\int_a^b f(x)\,|\sin nx|\,dx$, $\int_a^b P(x)\,|\sin nx|\,dx$ differ from one another by less than η; and since $\int_a^b f(x)\,dx$, $\int_a^b P(x)\,dx$ also differ from one another by less than η, it follows, since η is arbitrary, that
$$\lim_{n\sim\infty}\int_a^b f(x)\,|\sin nx|\,dx=\frac{2}{\pi}\int_a^b f(x)\,dx.$$
The case in which $\cos nx$ takes the place of $\sin nx$ can be treated in the same manner.

339. If $f(x)$ have a D-integral in the interval (a, b), let $\phi(x)$ denote the continuous function $\int_a^x f(x)\,dx$. Employing the method of integration by parts (I, § 474) we have
$$\int_a^b f(x)\cos nx\,dx=\Big[\phi(x)\cos nx\Big]_a^b+n\int_a^b \phi(x)\sin nx\,dx.$$
Now $\phi(x)$ is summable in (a, b), and $\Big[\phi(x)\cos nx\Big]_a^b$ is bounded; we thus see that $\frac{1}{n}\int_a^b f(x)\cos nx\,dx$ converges to zero, as $n\sim\infty$. Similarly $\frac{1}{n}\int_a^b f(x)\sin nx\,dx$ can be shewn to converge to zero. We thus obtain as the analogue of Lebesgue's theorem of § 334, the proposition:

If $f(x)$ have a D-integral in $(-\pi, \pi)$, the Fourier's (D) coefficients a_n, b_n have the property $a_n=o(n)$, $b_n=o(n)$.

The statement in this theorem cannot be improved. It has in fact been shewn* by Titchmarsh that, if $\lambda(n)$ be a positive monotone decreasing function of n which converges to zero, as $n \sim \infty$, then, however slowly $1/\lambda(n)$ tends to ∞, a function which has a single point of non-summability, and which has a non-absolutely convergent integral, exists for which $a_n \neq o\{n\lambda(n)\}$, $b_n \neq o\{n\lambda(n)\}$.

CONDITIONS OF CONVERGENCE AT A POINT OR IN AN INTERVAL

340. It has been shewn in § 323 that the partial sum of the Fourier's series corresponding to a function $f(x)$, summable in the interval $(-\pi, \pi)$, is given by

$$s_{2n+1}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x') \frac{\sin(2n+1)\frac{x'-x}{2}}{\sin\frac{x'-x}{2}}\,dx'.$$

In order to consider the behaviour of the Fourier's series at any point x_1, interior to $(-\pi, \pi)$, let μ be a fixed positive number, so small that the interval $(x_1 - \mu, x_1 + \mu)$ is interior to $(-\pi, \pi)$, and let the function $f_1(x)$ be defined to be equal to $f(x)$ in the interval $(x_1 - \mu, x_1 + \mu)$, and to be zero in the rest of the interval $(-\pi, \pi)$. Let $f_2(x)$ be such that

$$f_1(x) + f_2(x) = f(x);$$

so that $f_2(x)$ has the value zero in the interval $(x_1 - \mu, x_1 + \mu)$, and has the value $f(x)$ in the rest of the interval $(-\pi, \pi)$.

In the general convergence Theorem I, of § 279, let G consist of the single point x_1, and let $\Phi(x', x_1, n)$ be defined to be zero in the interval $(x_1 - \mu, x_1 + \mu)$, and to have the value $\dfrac{\sin(2n+1)\frac{x'-x_1}{2}}{\sin\frac{x'-x_1}{2}}$ within the two intervals $(-\pi, x-\mu)$, $(x+\mu, \pi)$.

We have then $|\Phi(x', x_1, n)| \leq \operatorname{cosec}\frac{1}{2}\mu$, for all values of x' and n; and $\int_\alpha^\beta \Phi(x', x_1, n)\,dx'$ converges to zero, as $n \sim \infty$, since $\operatorname{cosec}\frac{x'-x_1}{2}$ is summable in the intervals in which $\Phi(x', x, n)$ is not zero, and thus the theorem of § 279 is applicable to the function $f_2(x)$. It follows that, since the conditions of Theorem I are satisfied,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_2(x') \frac{\sin(2n+1)\frac{x'-x_1}{2}}{\sin\frac{x'-x_1}{}}\,dx' ,$$

* *Proc. Lond. Math. Soc.* (2), vol. XXII (1924), *Records*, p. xxv.

converges to zero. Since $f(x') = f_1(x') + f_2(x')$ it follows that $\lim_{n\sim\infty} s_{2n+1}(x_1)$ depends only on $\lim_{n\sim\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x') \frac{\sin(2n+1)\frac{x'-x_1}{2}}{\sin\frac{x'-x_1}{2}} dx'$; or on

$$\lim_{n\sim\infty} \frac{1}{2\pi}\int_{x_1-\mu}^{x_1+\mu} f(x') \frac{\sin(2n+1)\frac{x'-x}{2}}{\sin\frac{x'-x}{2}} dx';$$

and this is independent of the values of $f(x')$ outside the interval

$$(x_1 - \mu, x_1 + \mu).$$

We have accordingly established the following theorem which, in the case of functions that are integrable (R), was given by Riemann*:

The behaviour of the Fourier's series corresponding to the summable function $f(x)$, as regards convergence, divergence, or oscillation, at a particular point, depends only on the values of the function $f(x)$ in an arbitrarily small neighbourhood of the point.

It will be seen that this theorem is an immediate consequence of the Riemann-Lebesgue theorem (§ 334) as applied to the function

$$f_2(x) \operatorname{cosec} \frac{x - x_1}{2}.$$

It has been assumed in the proof that x_1 is an interior point of $(-\pi, \pi)$. This does not involve any real limitation, because, when $f(x)$ is defined to have the period 2π, we may take for the interval any interval of length 2π, instead of $(-\pi, \pi)$; and such interval can be chosen so that either of the points $\pi, -\pi$ is interior to it.

If the function $f(x)$ is of bounded variation in the interval

$$(x_1 - \mu, x_1 + \mu),$$

the function $f_1(x)$ is of bounded variation in the interval $(-\pi, \pi)$. Applying the results of § 332, to $f_1(x)$, we obtain the following theorem:

If, for a summable function $f(x)$, a neighbourhood of the point x_1 can be determined so that $f(x)$ is of bounded variation in it, the Fourier's series converges at the point x_1 to the value $\frac{1}{2}\{f(x_1 + 0) + f(x_1 - 0)\}$, which is equal to $f(x_1)$ in case the function is continuous at x_1.

This sufficient condition of convergence was given by Jordan†, and is known as Jordan's condition.

341. Next, let an interval (a, b) be taken, interior to $(-\pi, \pi)$; and let μ be a positive number such that $(a - \mu, b + \mu)$ is interior to $(-\pi, \pi)$.

* See his memoir, "Ueber die Darstellbarkeit," *Math. Werke*, p. 227.

† *Cours d'Analyse*, vol. II, 2nd ed., p. 237.

Let $f_1(x) = f(x)$ in the interval $(a - \mu, b + \mu)$, and let it have the value zero in the rest of the interval $(-\pi, \pi)$. Let $f_2(x)$ be given by

$$f(x) = f_1(x) + f_2(x),$$

so that $f_2(x)$ has the value zero in the interval $(a - \mu, b + \mu)$, and the value $f(x)$ in the intervals $(-\pi, a - \mu)$, $(b + \mu, \pi)$.

If in the Theorem I of § 279, we take G to consist of all the points x, of the interval (a, b), and $\Phi(x', x, n)$ to be defined as

$$\frac{\sin(2n+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}$$

within the two intervals $(-\pi, a - \mu)$, $(b + \mu, \pi)$, and to have the value zero in the interval $(a - \mu, b + \mu)$, it can be shewn that this function satisfies the conditions of the theorem. For $|\Phi(x', x, n)| \leqq \operatorname{cosec} \frac{1}{2}\mu$, for all values of n, and for all the values of x; thus condition (1) is satisfied. Again, if (α, β) be any interval contained in $(b + \mu, \pi)$,

$$\int^{\beta} \Phi(x', x, n)\, dx' = 2\int_{\frac{1}{2}(\alpha - x)}^{\frac{1}{2}(\beta - x)} \frac{\sin(2n+1)z}{\sin z}\, dz,$$

and by the second theorem of § 334, since $\operatorname{cosec} z$ is summable in the interval $(\frac{1}{2}\mu, \frac{1}{2}\pi - \frac{1}{2}a)$ which contains the interval $\{\frac{1}{2}(\alpha - x), \frac{1}{2}(\beta - x)\}$, it follows that this integral converges to zero, as $n \sim \infty$, uniformly for all values of x in (a, b). The corresponding result can be shewn to hold if (α, β) is contained in $(-\pi, a - \mu)$. Taking the summable function $f_2(x)$, it follows from the result of Theorem I, that

$$\int_{-\pi}^{\pi} f_2(x') \frac{\sin(2n+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}\, dx'$$

converges to zero, uniformly in the interval (a, b), of x.

Therefore, in the interval (a, b), $\lim\limits_{n\sim\infty} s_{2n+1}(x)$ depends only on

$$\frac{1}{2\pi}\lim_{n\sim\infty}\int_{-\pi}^{\pi} f_1(x') \frac{\sin(2n+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}\, dx'$$

or on

$$\frac{1}{2\pi}\lim_{n\sim\infty}\int_{a-\mu}^{b+\mu} f(x') \frac{\sin(2n+1)\dfrac{x'-x}{2}}{\sin\dfrac{x'-x}{2}}\, dx'.$$

We thus have obtained the following theorem*:

If $f(x)$ be summable in $(-\pi, \pi)$, and (a, b) be any finite interval contained

* See Hobson, *Proc. Lond. Math. Soc.* (2), vol. v (1907), p. 282.

within $(-\pi, \pi)$, ***the behaviour of the Fourier's series corresponding to*** $f(x)$ ***in the interval*** (a, b) ***as regards convergence, divergence, and oscillation, depends only on the values of the function*** $f(x)$ ***in the interval*** $(a-\mu, b+\mu)$, ***where*** μ ***is an arbitrarily small positive number.***

In case $f(x)$ is of bounded variation in the interval $(a-\mu, b+\mu)$, $f_1(x)$ is of bounded variation in $(-\pi, \pi)$, and consequently the results of §§ 332, 333 can be applied to the function $f_1(x)$. We obtain accordingly the following theorem:

If $f(x)$ ***be summable in*** $(-\pi, \pi)$, ***and*** (a, b) ***be an interval which is contained in another interval*** (a', b'), ***in which*** $f(x)$ ***is of bounded variation, the Fourier's series converges uniformly to the value*** $f(x)$ ***in the interval*** (a, b), ***in case*** $f(x)$ ***be continuous in*** (a, b), ***the continuity at a and b being on both sides. If*** $f(x)$ ***is not continuous in*** (a, b), ***the series converges boundedly in the interval*** (a, b) ***to the value*** $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

342. It has now been shewn that, in all cases, the question of the convergence of the Fourier's series at a point x depends upon the convergence of $\frac{1}{\pi}\int_0^\epsilon \{f(x+2z)+f(x-2z)\}\frac{\sin(2n+1)z}{\sin z}dz$, where ϵ is an arbitrarily small positive number. In fact, it has been shewn that

$$\lim_{n\sim\infty}\frac{1}{\pi}\int_\epsilon^{\frac{1}{2}\pi}\{f(x+2z)+f(x-2z)\}\frac{\sin(2n+1)z}{\sin z}dz=0.$$

Since $\int_0^{\frac{1}{2}\pi}\frac{\sin(2n+1)z}{\sin z}dz=\frac{1}{2}\pi$, for all values of n, and

$$\lim_{n\sim\infty}\int_\epsilon^{\frac{1}{2}\pi}\frac{\sin(2n+1)z}{\sin z}dz=0,$$

we see that

$$\lim_{n\sim\infty}\int_0^\epsilon\frac{\sin(2n+1)z}{\sin z}dz=\tfrac{1}{2}\pi.$$

Thus the condition that the Fourier's series may converge at the point x to the value $\lim\limits_{t\sim 0}\frac{1}{2}\{f(x+t)+f(x-t)\}$ is that

$$\lim_{n\sim\infty}\int_0^\epsilon\left[f(x+2z)+f(x-2z)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}\right]\frac{\sin mz}{\sin z}dz=0.$$

At a point of continuity of $f(x)$ this reduces to the condition

$$\lim_{n\sim\infty}\int_0^\epsilon\{f(x+2z)+f(x-2z)-2f(x)\}\frac{\sin mz}{\sin z}dz=0.$$

It can be shewn that, in this integral, $\frac{\sin mz}{\sin z}$ can be replaced by $\frac{\sin z}{z}$.

For the function

$$[f(x+2z)+f(x-2z)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}]\left(\frac{1}{\sin z}-\frac{1}{z}\right)$$

is summable in the interval $(0, \epsilon)$; and therefore, by Lebesgue's theorem,

$$\lim_{n\sim\infty}\int_0^\epsilon [f(x+2z)+f(x-2z)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}]\left(\frac{\sin mz}{\sin z}-\frac{\sin mz}{z}\right)dz=0.$$

Thus the condition of convergence to $\frac{1}{2}\lim_{t\sim 0}\{f(x+t)+f(x-t)\}$ is

$$\lim_{n\sim\infty}\int_0^\epsilon [f(x+2z)+f(x-2z)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}]\frac{\sin mz}{z}\,dz=0.$$

Writing for convenience $2z=t$, $f(x+t)+f(x-t)=\phi(t)$, the condition of convergence is

$$\lim_{n\sim\infty}\int_0^\epsilon \frac{\phi(t)-\phi(+0)}{t}\sin\tfrac{1}{2}mt\,dt=0.$$

This condition is certainly satisfied in case the function $\dfrac{\phi(t)-\phi(+0)}{t}$ is summable in the interval $(0, \epsilon)$, on account of Lebesgue's theorem (§ 334); and the condition of summability is satisfied in particular if $\dfrac{\phi(t)-\phi(+0)}{t}$ is bounded in $(0, \epsilon)$, or if $|\phi(t)-\phi(+0)| \leq At^{1-\alpha}$, where $\alpha < 1$, and A is a fixed number, in the interval $(0, \epsilon)$.

In case $f(x+0)$, $f(x-0)$ both exist, it will be a sufficient condition of convergence of the series to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$, that both $\dfrac{f(x+t)-f(x+0)}{t}$ and $\dfrac{f(x-t)-f(x-0)}{t}$ should be summable in the interval $(0, \epsilon)$.

We thus obtain the following sufficient conditions of convergence of the Fourier's series at the point x.

(*a*) *If ϵ can be so chosen that $\dfrac{\phi(t)-\phi(+0)}{t}$ is summable in the interval $(0, \epsilon)$, where $\phi(t)$ denotes $f(x+t)+f(x-t)$, then the Fourier's series is convergent at the point x. This condition is satisfied when $f(x+0)$, $f(x-0)$ both have definite values, and $\dfrac{f(x+t)-f(x+0)}{t}$, $\dfrac{f(x-t)-f(x-0)}{t}$ are both summable in $(0, \epsilon)$; or else when $f(x+0)$, $f(x-0)$ are not definite but $\phi(+0)$ is so, and $\dfrac{\phi(t)-\phi(+0)}{t}$ is summable in $(0, \epsilon)$. In either case the series converges to $\frac{1}{2}\lim_{t\sim 0}\{f(x+t)+f(x-t)\}$.*

(*b*) *If x be a point of continuity of $f(x)$, the series converges at the point x to the value $f(x)$ if $\dfrac{f(x+t)+f(x-t)-2f(x)}{t}$ is summable in the interval*

$(0, \epsilon)$; and in particular if $\dfrac{f(x+t)-f(x)}{t}$, $\dfrac{f(x-t)-f(x)}{t}$ *are both summable in that interval.*

This condition, which is known as Dini's condition, is satisfied, in particular, in case the four derivatives of $f(x)$ at the point x are all finite, and in particular if $f(x)$ have a finite differential coefficient. We thus have:

(c) *At a point of continuity of the function $f(x)$ the Fourier's series converges at the point x to the value $f(x)$ if $f(x)$ have a finite differential coefficient at the point, or if all the four derivatives $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, $D_-f(x)$ are finite.*

Further we have the following condition:

(d) *The Fourier's series converges at a point to $\phi(+0)$, if, for all values of t not greater than some fixed positive number ϵ, $|\phi(t)-\phi(+0)| \leqq At^k$, where A and k are fixed positive numbers.*

At a point of continuity of $f(x)$, the series converges if

$$|f(x+t)-f(x)| \leqq At^k,$$

where k, A are positive numbers, provided t is numerically less than some fixed positive number ϵ. At a point of ordinary discontinuity it is sufficient that both $|f(x+t)-f(x+0)|$ and $|f(x-t)-f(x-0)|$ should satisfy this condition.

This condition was given by Lipschitz*, and was also given by Dini.

A more general sufficient condition of summability of $\dfrac{\phi(t)-\phi(+0)}{t}$, in the neighbourhood of $t=0$, is that, in a sufficiently small interval $(0, \epsilon)$,

$$|\phi(t)-\phi(+0)| \leqq \frac{A}{\log\frac{1}{t}\log\log\frac{1}{t}\cdots\left[\log\log\cdots\frac{1}{t}\right]^{1+\alpha}},$$

where A and α are positive numbers; we therefore obtain the following sufficient condition of convergence:

(e) *The Fourier's series converges, at a point x, to the value*

$$\tfrac{1}{2}\lim_{t\sim 0}\{f(x+t)+f(x-t)\},$$

if, for all positive values of t not exceeding some fixed number ϵ, the condition

$$|\phi(t)-\phi(+0)| \leqq \frac{A}{\log\frac{1}{t}\log\log\frac{1}{t}\cdots\left[\log\log\cdots\frac{1}{t}\right]^{1+\alpha}}$$

be satisfied; where A and α are fixed positive numbers. In particular it is sufficient that both $|f(x+t)-f(x+0)|$ and $|f(x-t)-f(x-0)|$ *should satisfy this condition.*

* *Crelle's Journal*, vol. LXIII (1864), p. 296.

It may be observed that, of the two tests of convergence of Dirichlet's integral, at a point, that of Jordan and that of Dini, neither includes the other.

For, considering the function $f(x) = \left(\log \frac{1}{|x|}\right)^{-1}$, this does not satisfy Dini's condition that $\left|\frac{f(x)}{x}\right|$ is summable in the neighbourhood of the point $x = 0$; but it satisfies Jordan's condition that it is of bounded variation.

Again the function $f(x) = |x|^p \sin \frac{1}{|x|}$, where $0 < p \leqq 1$, satisfies Dini's condition, but not Jordan's.

343. The condition $\lim\limits_{m\sim\infty} \int_0^\epsilon \frac{\phi(t) - \phi(+0)}{t} \sin \frac{1}{2}mt\, dt = 0$ may be transformed so as to yield a sufficient condition of convergence of a very general character.

We have
$$\int_0^\epsilon \frac{\phi(t) - \phi(+0)}{t} \sin \tfrac{1}{2}mt\, dt = \int_0^{\frac{2\pi}{m}} \chi(t) \sin \tfrac{1}{2}mt\, dt$$
$$+ \sum_{p=1}^{p=q} \int_{\frac{2(2p-1)\pi}{m}}^{\frac{4p\pi}{m}} \left\{\chi(t) - \chi\left(t + \frac{2\pi}{m}\right)\right\} \sin \tfrac{1}{2}mt\, dt$$
$$+ \int_{\frac{2(2q+1)\pi}{m}}^{\epsilon} \chi(t) \sin \tfrac{1}{2}mt\, dt,$$
where $\chi(t)$ denotes $\frac{\phi(t) - \phi(+0)}{t}$, and $4(q+1)\frac{\pi}{m} > \epsilon \geqq \frac{2(2q+1)\pi}{m}$. The last integral is numerically less than the integral of $|\chi(t)|$ over the interval $\left(\epsilon - \frac{2\pi}{m}, \epsilon\right)$, and therefore it converges to zero, as $m \sim \infty$. The first integral is numerically less than π times the upper boundary of $|\phi(t) - \phi(+0)|$ in the interval $\left(0, \frac{2\pi}{m}\right)$ of t, and this also converges to zero, as $m \sim \infty$.

The remaining expression is numerically less than
$$\int_{\frac{2\pi}{m}}^{\epsilon} \left|\chi(t) - \chi\left(t + \frac{2\pi}{m}\right)\right| dt;$$
and thus the series converges if
$$\lim_{\delta\sim 0} \int_\delta^\epsilon |\chi(t) - \chi(t+\delta)|\, dt = 0.$$

We have thus obtained the sufficient condition of convergence in the following form:

At any point x at which $\lim\limits_{t\sim 0}\{f(x+t)+f(x-t)\}$ exists, the Fourier's series converges to the value $\frac{1}{2}\lim\limits_{t\sim 0}\{f(x+t)+f(x-t)\}$, if

$$\lim_{\delta\sim 0}\int_\delta^\epsilon |\chi(t)-\chi(t+\delta)|\,dt=0,$$

where $\chi(t)$ denotes

$$\frac{1}{t}[f(x+t)+f(x-t)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}].$$

In particular, at a point of continuity of $f(x)$ the series will converge to $f(x)$ if $\lim\limits_{\delta\sim 0}\int_\delta^\epsilon |\chi_1(t)-\chi_1(t+\delta)|\,dt=0$, and $\lim\limits_{\delta\sim 0}\int_\delta^\epsilon |\chi_2(t)-\chi_2(t+\delta)|\,dt=0$, where $\chi_1(t)=\dfrac{f(x+t)-f(x)}{t}$, $\chi_2(t)=\dfrac{f(x-t)-f(x)}{t}$.

This condition, which contains the preceding conditions, was given by Lebesgue*.

The condition may be stated in the equivalent form that

$$\lim_{\delta\sim 0}\int_\delta^\epsilon \frac{|F(t)-F(t+\delta)|}{t}\,dt=0;$$

where $F(t)$ denotes $\phi(t)-\phi(+0)$, or

$$f(x+t)+f(x-t)-\lim_{t\sim 0}\{f(x+t)+f(x-t)\}.$$

For, if δ_1 denotes a number such that $\delta<\delta_1<\epsilon$, the difference between the two integrals does not exceed

$$\int_{\delta_1}^\epsilon \frac{|F(t)-F(t+\delta)|}{t}\,dt+\int_{\delta_1}^\epsilon |\chi(t)-\chi(t+\delta)|\,dt$$
$$+\int_\delta^{\delta_1}\left(\frac{1}{t}-\frac{1}{t+\delta}\right)|F(t+\delta)|\,dt;$$

the first integral is less than $\dfrac{1}{\delta_1}\displaystyle\int_{\delta_1}^\epsilon |F(t)-F(t+\delta)|\,dt$, therefore the two integrals converge to zero, as $\delta\sim 0$, since $F(t)$, $\chi(t)$ are summable in the interval (δ_1,ϵ) (see I, § 431). The number δ_1 can be fixed so that

$$|F(t+\delta)|<\epsilon,$$

in the interval (δ,δ_1); thus the third integral is less than $\epsilon\log 2$. Thus, since the difference between the two integrals is less than an arbitrarily chosen number, when δ is taken sufficiently small, the equivalence has been established.

At a point at which $\phi(+0)$ does not exist, the preceding investigation

* *Math. Annalen*, vol. LXI (1905), p. 251. In this memoir there is contained a detailed account and comparison of the various criteria.

can be applied to the function $f(x+t)+f(x-t)-2f(x)$. In this case, we have $\chi(t) = \dfrac{f(x+t)+f(x-t)-2f(x)}{t}$, and the integral

$$\int_0^{\frac{2\pi}{m}} \chi(t)\sin\tfrac{1}{2}mt\,dt$$

is numerically less than $\frac{1}{2}m\displaystyle\int_0^{\frac{2\pi}{m}} |f(x+t)+f(x-t)-2f(x)|\,dt$. This will have the limit zero if the condition is satisfied that

$$\int_0^t |f(x+t)+f(x-t)-2f(x)|\,dt$$

has as differential coefficient, at the point $t=0$, the number zero. We have thus the following theorem, which includes the preceding theorem:

At any point x the Fourier's series converges to $f(x)$ if

$$\int_0^t |f(x+t)+f(x-t)-2f(x)|\,dt$$

has, at the point $t=0$, a differential coefficient of which the value is zero, and if also $\lim\limits_{\delta\sim 0}\displaystyle\int_\delta^\epsilon |\chi(t)-\chi(t+\delta)|\,dt = 0$, *where $\chi(t)$ denotes*

$$\frac{f(x+t)+f(x-t)-2f(x)}{t}.$$

344. Returning to the expression

$$\sum_{p=1}^{p=q}\int_{\frac{2(2p-1)\pi}{m}}^{\frac{4p\pi}{m}} \left\{\chi(t)-\chi\left(t+\frac{2\pi}{m}\right)\right\}\sin\tfrac{1}{2}mt\,dt;$$

this expression is equivalent to

$$\int_0^{\frac{2\pi}{m}}\sum_{p=1}^{p=q}\left\{\chi\left(t+\frac{\overline{4p-2}\,\pi}{m}\right)-\chi\left(t+\frac{4p\pi}{m}\right)\right\}\sin\tfrac{1}{2}mt\,dt$$

or to
$$\int_0^{2\pi}\sum_{p=1}^{p=q}\left\{\frac{F\left(\dfrac{t+\overline{4p-2}\,\pi}{m}\right)}{t+(4p-2)\pi}-\frac{F\left(\dfrac{t+4p\pi}{m}\right)}{t+4p\pi}\right\}\sin\tfrac{1}{2}t\,dt,$$

where $F(t)$ denotes $\phi(t)-\phi(+0)$. This is less numerically than

$$\int_0^{2\pi}\sum_{p=1}^{p=q}\left\{\left|\frac{F\left(\dfrac{t+\overline{4p-2}\,\pi}{m}\right)-F\left(\dfrac{t+4p\pi}{m}\right)}{t+\overline{4p-2}\,\pi}\right|dt\right.$$
$$\left.+\,2\int_0^{2\pi}\sum_{p=1}^{p=q}\left|\frac{F\left(\dfrac{t+4p\pi}{m}\right)}{(4p-2)\,4p\pi}\right|\right\}dt.$$

The second integral is less than the maximum of $\left|F\left(\frac{t+4p\pi}{m}\right)\right|$ for all the values 1, 2, ... q, of p, or of the maximum of $|F(t)|$ in the interval $(0, \epsilon)$. The first integral is less than $\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{2q}\right)$ multiplied by the upper boundary of $\left|F\left(\frac{t+\overline{4p-2}\pi}{m}\right)-F\left(\frac{t+4p\pi}{m}\right)\right|$ for all values of t in the interval $(0, 2\pi)$ and all the numbers 1, 2, 3, ... q. Now

$$\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{2q}$$

is equal to $(C_q+\log 2q)$, when C_q tends, as q increases indefinitely, to a fixed number, Mascheroni's constant.

Writing $\delta=\frac{2\pi}{m}$, the upper boundary of

$$\left|F\left(\frac{t+\overline{2p-2}\pi}{m}\right)-F\left(\frac{t+4p\pi}{m}\right)\right|$$

is that of $|F(t)-F(t+\delta)|$ in the interval $(0, \epsilon)$. Thus the first integral converges to zero, as $m \sim \infty$, if the maximum of $|\{F(t)-F(t+\delta)\}\log\delta|$ for all values of t such that $t+\delta$ is in the interval $(0, \epsilon)$ converges to zero, as $\delta \sim 0$.

The second integral may be taken to be arbitrarily small, by choosing ϵ sufficiently small.

We have now established the following sufficient condition of convergence at a point x, of the Fourier's series:

At a point at which $f(x+t)+f(x-t)$ has a definite limit, as $t \sim 0$, the Fourier's series converges to $\frac{1}{2}\lim\limits_{t\sim 0}\{f(x+t)+f(x-t)\}$, if an interval $(0, \epsilon)$ can be determined such that

$$|\{f(x+t)+f(x-t)-f(x+t+\delta)-f(x-t-\delta)\}\log\delta|$$

converges to zero, as $\delta \sim 0$, uniformly for all values of t in the interval $(0, \epsilon)$. This condition will be satisfied in particular if both

$$|\{f(x+t)-f(x+t+\delta)\}\log\delta| \quad \textit{and} \quad |\{f(x-t)-f(x-t+\delta)\}\log\delta|$$

converge to zero, uniformly for all values of t, in the interval $(0, \epsilon)$ of t.

This condition was given by Dini*.

In this condition a condition given† by Lipschitz is included. Thus it is sufficient for convergence at the point x that

$$|f(x+t)+f(x-t)-f(x+t+\delta)-f(x-t-\delta)|<C\delta^k,$$

in the interval $(0, \epsilon)$, of t, where C and k are fixed positive numbers.

* *Serie di Fourier*, p. 49. † *Crelle's Journal*, vol. LXIII (1864), p. 308.

345. In accordance with the theorem proved in § 297, if, in an interval $(0, \mu)$, $\frac{1}{t}\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt$, or $\chi_1(t)$, has bounded variation in $(0, \mu)$, then provided $|tF(\mu, n)|$ is bounded for all values and n and t in $(0, \mu)$, we have

$$\lim_{n\sim\infty}\int_0^\mu \{f(x+t)+f(x-t)\}\,F(t, n)\,dt = \chi(+0)\lim_{n\sim\infty}\int_0^\mu F(t, n)\,dt,$$

where $F(t, n)$ satisfies the conditions of Theorem I, of § 279, in every interval (μ', μ) of t, where $0 < \mu' < \mu$.

In the present case $F(t, n) = \frac{\sin(2n+1)t}{\sin t}$, and thus $tF(t, n)$ is bounded.

The following sufficient condition of convergence, first given by de la Vallée Poussin*, has thus been obtained:

The Fourier's series corresponding to $f(x)$ is convergent at any point x for which $\frac{1}{t}\int_0^t \{f(x+t)+f(x-t)\}\,dt$ has bounded variation in some interval $(0, \mu)$ of t.

This criterion includes the case in which $f(x)$ has bounded variation in the neighbourhood of the point x. To see this we need only consider the case of a monotone function; the functions $\frac{1}{t}\int_0^t f(x+t)\,dt$, $\frac{1}{t}\int_0^t f(x-t)\,dt$ are then also monotone, because the mean value of an increasing function increases when the function increases. Therefore

$$\frac{1}{t}\int_0^t \{f(x+t)+f(x-t)\}\,dt$$

has bounded variation when $f(x)$ has bounded variation in the interval $(x-\mu, x+\mu)$.

Again, it will be shewn that if $\int_0^t \left|\frac{\phi(t)}{t}\right| dt$ exists in a neighbourhood of $t = 0$, where $\phi(t) = f(x+t)+f(x-t)-2f(x)$, then also $\int_0^t |\chi_1'(t)|\,dt$ exists in that neighbourhood; and thus $\chi_1(t)$ is an indefinite integral, and accordingly of bounded variation in the neighbourhood.

We have

$$\chi_1'(t) = \frac{f(x+t)+f(x-t)-2f(x)}{t} - \frac{1}{t^2}\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt,$$

it being assumed that $f(x)$ is continuous at x, or else that

$$\lim_{t\sim 0}\{f(x+t)+f(x-t)\}$$

* *Rendiconti di Palermo*, vol. XXXI (1911), p. 296. Another proof of the theorem was given by W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. X (1911), p. 266.

exists and is taken to be the value of $2f(x)$. Thus

$$\chi_1'(t) = \frac{\phi(t)}{t} - \frac{1}{t^2}\int_0^t \phi(t)\,dt,$$

and thus $$\int_0^\epsilon |\chi_1'(t)|\,dt \leq \int_0^\epsilon \left|\frac{\phi(t)}{t}\right| dt + \int_0^\epsilon \frac{dt}{t^2}\int_0^t |\phi(t')|\,dt'.$$

The integrand in the repeated integral being positive, we may change the order of integration, after changing t' into ut; the second integral on the right-hand side then becomes

$$\int_0^\epsilon \frac{dt}{t}\int_0^1 |\phi(ut)|\,du, \text{ or } \int_0^1 du \int_0^\epsilon |\phi(ut)|\,\frac{1}{t}\,dt,$$

or $\int_0^1 du \int_0^{\epsilon u} |\phi(t)|\,\frac{dt}{t}$, which is less than $\int_0^\epsilon \left|\frac{\phi(t)}{t}\right| dt$. It follows that $\int_0^\epsilon |\chi_1'(t)|\,dt$ exists when $\int_0^\epsilon \left|\frac{\phi(t)}{t}\right| dt$ does so. Therefore de la Vallée Poussin's criterion includes Dini's criterion (*b*) of § 342; and it has been shewn to include that of Jordan.

346. The following test of convergence has been given* by W. H. Young:

If, at the point x, $f(x+t)+f(x-t)$, or $\phi(t)$, converges to a unique limit $\phi(+0)$, as $t \sim 0$, it is sufficient for the convergence of the Fourier's series at the point x, to the value $\frac{1}{2}\phi(+0)$ that, in some neighbourhood of the point,

$$\int_0^t |d\{t\phi(t)\}| = O(t).$$

If $\phi(x)$ be a function of bounded variation in an interval (a, b), the total variation may be denoted by $\int_a^b |d\phi(x)|$, which represents, as in the definition of the R-integral, the limit of the sum of the absolute differences of $\phi(x)$ at the ends of a mesh of a net D_n, belonging to a system of nets, as $n \sim \infty$. In order that the notation may be justified, it is necessary, in order that the total variation so defined may be independent of the particular system of nets employed (see I, § 246), that $\phi(x)$ should have no external saltus at any of its points of discontinuity; and we may assume that this is the case, since the set of points of discontinuity is enumerable, and thus a change of the values of the function at points of this enumerable set is sufficient for the removal of any external saltus which may originally exist. Thus, it is assumed in the above test that the function $t\phi(t)$ has bounded variation in some neighbourhood of the point $t = 0$.

In order to prove the validity of the test, it will be sufficient to shew that, when it is satisfied, Lebesgue's test, given in § 343, is satisfied. That

* *Comptes Rendus*, vol. CLXIII (1916), pp. 187, 975; also *Proc. Lond. Math. Soc.* (2), vol. XVII (1916), p. 206.

this is the case has been proved* by Hardy, in connection with a general discussion of the relations between the tests given by Dini, Jordan, de la Vallée Poussin, Young, and Lebesgue.

Let it then be assumed that, in some neighbourhood of the point $t = 0$, $t\phi(t)$ is of bounded variation, thus that $t\{\phi(t) - \phi(+0)\} = g_1(t) - g_2(t)$, when $g_1(t)$, $g_2(t)$ are both monotone non-diminishing functions, and that the total variation of $t\phi(t)$ in $(0, t)$, when divided by t, is bounded in a neighbourhood of the point $t = 0$.

We have, denoting $\phi(t) - \phi(+0)$ by $F(t)$,

$$\int_\delta^\epsilon \frac{|F(t) - F(t+\delta)|}{t}\,dt \leq \int_\delta^{m\delta} \frac{|F(t) - F(t+\delta)|}{t}\,dt$$

$$+\int_{m\delta}^\epsilon \left|\frac{g_1(t)}{t} - \frac{g_1(t+\delta)}{t+\delta}\right|\frac{dt}{t} + \int_{m\delta}^\epsilon \left|\frac{g_2(t)}{t} - \frac{g_2(t+\delta)}{t+\delta}\right|\frac{dt}{t},$$

where $1 < m$, and $m\delta < \epsilon$.

The first integral I_1 on the right-hand side does not exceed

$$\int_\delta^{m\delta} \frac{|F(t)|}{t}\,dt + \int_\delta^{m\delta} \frac{|F(t+\delta)|}{t}\,dt,$$

and is therefore not greater than $2\mu \log m$, where μ is the upper boundary of $F(t)$ in the interval $(0, \overline{m+1}\delta)$.

The second integral I_2 does not exceed

$$\int_{m\delta}^\epsilon \frac{g_1(t+\delta) - g_1(t)}{t(t+\delta)}\,dt + \int_{m\delta}^\epsilon |g_1(t)|\left(\frac{1}{t} - \frac{1}{t+\delta}\right)\frac{dt}{t}.$$

Of these parts, the second integral is less than $k\int_{m\delta}^\epsilon \left(\frac{1}{t} - \frac{1}{t+\delta}\right)dt$, or than $k \log \frac{m+1}{m}$, where k is the upper boundary of $\frac{g_1(t)}{t}$ in the interval $(0, \epsilon + \delta)$, and is a finite number, since $\frac{g_1(t) + g_2(t)}{t}$ is bounded in a neighbourhood of $t = 0$, and the numbers δ, ϵ can be so chosen that $\delta + \epsilon$ is in that neighbourhood. The first part of I_2 can be expressed as

$$\int_{(m+1)\delta}^{\epsilon+\delta} \frac{g_1(t)}{t(t-\delta)}\,dt - \int_{m\delta}^\epsilon \frac{g_1(t)}{t(t+\delta)}\,dt,$$

or as
$$\int_{(m+1)\delta}^{\epsilon+\delta} \frac{g_1(t)}{t(t+\delta)}\,dt + 2\delta\int_{(m+1)\delta}^{\epsilon+\delta} \frac{g_1(t)}{t(t^2-\delta^2)} - \int_{m\delta}^\epsilon \frac{g_1(t)}{t(t+\delta)}\,dt,$$

which is less than

$$\int_\epsilon^{\epsilon+\delta} \frac{g_1(t)}{t(t+\delta)}\,dt - \int_{m\delta}^{(m+1)} \frac{g_1(t)}{t(t+\delta)}\,dt + 4\delta k\int_{(m+1)\delta}^{\epsilon+\delta} \frac{dt}{t^2}.$$

* See *Messenger of Math.* vol. XLIX (1919), p. 154.

This is less numerically than $k\log\frac{\epsilon+2\delta}{\epsilon+\delta}+k\log\frac{m+2}{m+1}+\frac{4k}{m+1}$; thus I_2 is less than $k\left(\log\frac{m+1}{m}+\log\frac{m+2}{m+1}+\frac{4}{m+1}+\log\frac{\epsilon+2\delta}{\epsilon+\delta}\right)$.

First m may be so chosen that $k\left(\log\frac{m+2}{m}+\frac{4}{m+1}\right)$ is less than an arbitrarily chosen positive number η, then δ may be so chosen that $k\log\frac{\epsilon+2\delta}{\epsilon+\delta}<\eta$; thus $I_2<2\eta$. Similarly by proper choice of m and δ we have $I_3<2\eta$.

Also δ may be so chosen that $2\mu\log m<\eta$, since $F(t)$ converges to zero, as $t\sim 0$.

It has now been shewn that $\int_\delta^\epsilon\frac{|F(t)-F(t+\delta)|}{t}dt<5\eta$, provided δ is sufficiently small, and therefore Lebesgue's test is satisfied.

Thus Young's test of convergence at a point is included in Lebesgue's test.

Young's test includes that of Jordan, for assuming that Jordan's test is satisfied, we have

$$\begin{aligned}\int_0^t|d\{tF(t)\}| &\leq \int_0^t|F(t)|\,dt+\int_0^t t\,|dF(t)|\\ &\leq o(t)+t\int_0^t|dF(t)|\\ &=O(t).\end{aligned}$$

That Young's test does not include that of Dini is seen by considering the function $|x|^p\sin\frac{1}{|x|}$, where $p>0$ in the neighbourhood of $x=0$. We have

$$d(tF(t))=2\left\{(p+1)\,t^p\sin\frac{1}{t}-t^{p-1}\cos\frac{1}{t}\right\}dt,$$

and the condition $\int_0^t t^{p-1}\left|\cos\frac{1}{t}\right|dt=O(t)$ is only satisfied if $p\geqq 1$, whereas Dini's test is satisfied when $p>0$. Since de la Vallée Poussin's test includes that of Dini (§ 345) it follows that Young's test does not include that of de la Vallée Poussin. Conversely, it has been shewn by Hardy (*loc. cit.*) that de la Vallée Poussin's test does not include that of Young. He has shewn that, if $F(t)=\sin\left(\log\frac{1}{t}\right)\Big/\log\frac{1}{t}$, then Young's test is satisfied, but de la Vallée Poussin's test is not satisfied. A proof has been given by Hardy that Lebesgue's test includes that of de la Vallée Poussin. It thus appears that Lebesgue's test includes all the other four.

347. If we employ the general convergence theorem of § 279 in a modified form, since $t\dfrac{\sin(2n+1)t}{\sin t}$ is bounded, we obtain the following theorem:

If $f_1(x).f_2(x)$ is summable in the interval $(-\pi,\pi)$ and, in some neighbourhood of a point x, $f_1(x)$ is of bounded variation; and further if $f_2(x+0)$, $f_2(x-0)$ exist, and $\dfrac{f_2(x+t)-f_2(x+0)}{t}$, $\dfrac{f_2(x-t)-f_2(x-0)}{t}$ are summable in some interval $(0,\mu)$ of t, then the Fourier's series corresponding to $f_1(x).f_2(x)$ is convergent at the point x.

SUFFICIENT CONDITIONS OF UNIFORM CONVERGENCE OF FOURIER'S SERIES

348. Sufficient conditions will now be investigated that the Fourier's series, corresponding to a given summable function $f(x)$, may be uniformly convergent in an interval (a, b), contained in $(-\pi,\pi)$. It has already been seen that this will also cover the case in which (a, b) contains one of the points π, $-\pi$ as an end-point or an interior point, because any interval of length 2π may be substituted for $(-\pi,\pi)$ without essential change, the function $f(x)$ being taken to be periodic.

It is convenient to employ the following theorem* which may be deduced from the general Theorem I, of § 279.

The function $f(x)$ being summable in the interval $(-\pi,\pi)$, each of the four integrals $\int_\alpha^\beta f(x\pm 2z)\,\chi(z)\,{\sin \atop \cos}\, mz\,dz$, taken through any interval (α,β) such that $0\leq\alpha<\beta\leq\frac{1}{2}\pi$, converges to the limit zero, as the positive number m is indefinitely increased, uniformly for all values of x in the interval $(-\pi,\pi)$; the function $\chi(z)$ being any function that is bounded in the interval (α,β). The function $f(x)$ is assumed to be such that $f(x\pm 2\pi)=f(x)$.

There is no restriction on the number m.

It will be sufficient to consider $\int_\alpha^\beta f(x+2z)\,\chi(z)\sin mz\,dz$; the cases of the other three integrals can be treated in exactly the same manner.

Taking $(-2\pi, 2\pi)$ as the interval for which $f(x')$ is defined, let the set G, in § 279, consist of the points x of the interval $(-\pi,\pi)$. Let $\Phi(x', x, n)$ denote $\chi\left(\dfrac{x'-x}{2}\right)\sin m\dfrac{x'-x}{2}$ for $x+2\alpha\leq x'\leq x+2\beta$, and let

$$\Phi(x', x, n)=0$$

in the remainder of the interval $(-2\pi, 2\pi)$. Since $\left|\chi\left(\dfrac{x'-x}{2}\right)\sin m\dfrac{x'-x}{2}\right|$ is less than a fixed positive number, for all values of x', x, and n, the condition (1) is satisfied.

* See Hobson, *Proc. Lond. Math. Soc.* (2), vol. v (1907), p. 277. The restriction there made, and also in the first edition of this work, § 458, that $\chi(z)$ is of bounded variation, is unnecessary.

Again

$$\int_A^B \chi\left(\frac{x'-x}{2}\right)\sin m\frac{x'-x}{2}\,dx' = \int_{\frac{1}{2}(A-x)}^{\frac{1}{2}(B-x)} \chi(z)\sin mz\,dr,$$

and since $(\frac{1}{2}(A-x), \frac{1}{2}(B-x))$, is contained in the interval

$$(\tfrac{1}{2}(A-\pi), \tfrac{1}{2}(B+\pi)),$$

whatever value x may have, it follows from the theorem in § 334, that the integral converges to zero, as $m \sim \infty$, uniformly for all values of x in the interval $(-\pi, \pi)$; thus condition (2) is satisfied. Therefore

$$\int_{x+2\alpha}^{x+2\beta} f(x')\chi\left(\frac{x'-x}{2}\right)\sin m\frac{x'-x}{2}\,dx',$$

or

$$\int_\alpha^\beta f(x+2z)\chi(z)\sin mz\,dz$$

converges to zero, as $m \sim \infty$, uniformly for all values of x in $(-\pi, \pi)$.

349. Let it be supposed that, in an interval (a, b), the function $f(x)$ is continuous, the continuity at the points a and b being on both sides; and let $f(x+2z)+f(x-2z)-2f(x)$ be denoted by $F(z)$. In accordance with the theorem of § 348, $\int_\mu^{\frac{1}{2}\pi}\frac{F(z)}{z}\sin mz\,dz$ converges to zero, as $m \sim \infty$, uniformly for all values of x in $(-\pi, \pi)$, since $\frac{1}{z}$ is bounded in the interval $(\mu, \frac{1}{2}\pi)$. In order that the series may converge uniformly, for all values of $f(x)$ in the interval (a, b), it is necessary that $\int_0^\mu \frac{F(z)}{z}\sin mz\,dz$ should converge uniformly to zero in the interval (a, b), of x. Since

$$\left|\int_0^\mu \frac{F(z)}{z}\sin mz\,dz\right| < \int_0^\mu \left|\frac{F(z)}{z}\right| dz,$$

it will be sufficient that, for all values of x in (a, b), $\int_0^\mu \left|\frac{F(z)}{z}\right| dz$ should exist and be less than a number ϵ_μ, independent of x, which is such that $\lim\limits_{\mu\sim 0}\epsilon_\mu = 0$. For, in that case, μ can be chosen so small that $\epsilon_\mu < \eta$; and thus $\left|\int_0^{\frac{1}{2}\pi}\frac{F(z)}{z}\sin mz\,dz\right| < 2\eta$, for all values of x in (a, b), provided m is not less than some fixed number m_η. As η is arbitrary, the condition of uniform convergence is then satisfied. We have thus obtained the following theorem:

It is a sufficient condition for the uniform convergence of the Fourier's series in an interval (a, b) in which $f(x)$ is continuous, the continuity at the points a, b being on both sides, that

$$\int_0^\mu \left|\frac{f(x+2z)+f(x-2z)-2f(x)}{z}\right| dz$$

should exist for all values of x in (a, b), and should converge to zero, as $\mu \sim 0$, uniformly for all values of x in (a, b). The condition is satisfied in particular, if both the integrals

$$\int_0^\mu \left| \frac{f(x+2z)-f(x)}{z} \right| dz, \quad \int_0^\mu \left| \frac{f(x-2z)-f(x)}{z} \right| dz$$

exist, and converge to zero, as $\mu = 0$, uniformly for all values of x in (a, b).

From this theorem we obtain at once the following sufficient conditions as special cases:

If, in the interval (a, b), in which $f(x)$ is continuous, being continuous at a and b on both sides, one of the four derivatives (and therefore each of the other three) of $f(x)$ is bounded, the series converges uniformly to $f(x)$, in the interval (a, b).

If, in the interval (a, b), in which $f(x)$ is continuous, being continuous at a and b on both sides, the condition is satisfied that $|f(x+t) - f(x)| \leqq A\,|t|^k$, for all values of x in (a, b), and for all values of t not numerically greater than some fixed positive number, where A, k are positive numbers independent of x, then the series converges uniformly in (a, b) to the value $f(x)$.

The condition may be replaced by

$$|f(x+t) - f(x)| < \frac{A}{\frac{1}{|t|}\cdot\log\frac{1}{|t|}\cdot\log\log\frac{1}{t}\cdots\left\{\log\log\ldots\frac{1}{|t|}\right\}^{1+k}},$$

where A, k are positive numbers independent of x.

Corresponding to the theorem given in § 343, relating to the convergence of the series at a single point, the following theorem may be obtained:

In the interval (a,b), in which $f(x)$ is continuous, the continuity at a, b being on both sides, the series will converge uniformly to $f(x)$ in (a, b), if $\int_\delta^\epsilon |\chi(t) - \chi(t+\delta)|\,dt$ converges to zero, as $\delta \sim 0$, uniformly for all values of x in (a, b), where $\chi(t)$ denotes $\dfrac{f(x+t)+f(x-t)-2f(x)}{t}$.

A slight modification of the proof of the theorem in § 343 is sufficient to prove this result. That the first and third integrals converge as $m \sim \infty$, uniformly for all values of x in (a, b), follows from the fact that, ϵ being sufficiently small, $|f(x+t)+f(x-t)-2f(x)|$ is bounded in the interval $(0, \epsilon)$ of t, for all values of x in (a, b).

The proof of the theorem in § 344 suffices to establish the following test:

If $f(x)$ be continuous in (a, b), the continuity at a and b being on both sides, it is sufficient in order that the series may converge uniformly in (a, b) to the value $f(x)$, that an interval $(a-\epsilon, b+\epsilon)$ enclosing (a, b) can be determined such that

$$|f(x+t)+f(x-t)-f(x+t+\delta)-f(x-t-\delta)|\log 1/\delta$$

converges to zero, as $\delta \sim 0$, uniformly for all pairs of points $(x+t, x+t+\delta)$ or $(x-t, x-t-\delta)$ in the interval $(a-\epsilon, b+\epsilon)$. This condition is satisfied if $|f(x)-f(x+\delta)|$, $|f(x)-f(x-\delta)|$ both converge to zero, uniformly for points in the interval $(a-\epsilon, b+\epsilon)$, as $\delta \sim 0$.

350. Let it be assumed that the function $f(x)$ everywhere satisfies the Lipschitz condition $|f(x+t)-f(x)| \leqq A|t|^k$, where A is a positive constant, and $0 < k < 1$. The Fourier's series then converges uniformly to the continuous function $f(x)$. In order to determine the order of the coefficients a_n, b_n, we have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f\left(x - \frac{\pi}{2n}\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\sum_{p=0}^{p=n-1}\int_{\frac{2p\pi}{n}}^{(2p+1)\frac{\pi}{n}}\left[f\left(x-\frac{\pi}{2n}\right)-f\left(x+\frac{\pi}{2n}\right)\right]\sin nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\sum_{p=0}^{p=n-1}\int_{\frac{2p\pi}{n}}^{(2p+1)\frac{\pi}{n}}\left[f(x)-f\left(x+\frac{\pi}{n}\right)\right]\sin nx\,dx;$$

by means of the Lipschitz condition, we have then

$$|a_n| < \frac{nA}{\pi}\left(\frac{\pi}{n}\right)^k\int_0^{\frac{\pi}{n}}\sin nx\,dx,$$

$$|b_n| < \frac{nA}{\pi}\left(\frac{\pi}{n}\right)^k\int_0^{\frac{\pi}{n}}\sin nx\,dx.$$

From this it follows that $a_n = O\left(\frac{1}{n^k}\right)$, $b_n = O\left(\frac{1}{n^k}\right)$; thus the following theorem has been* established:

If $f(x)$ satisfies the Lipschitz condition $|f(x+t)-f(x)| \leqq A|t|^k$, then

$$a_n = O\left(\frac{1}{n^k}\right), \quad b_n = O\left(\frac{1}{n^k}\right).$$

In case $k=1$, $|f(x+t)-f(x)| \leqq A|t|$, it can be shewn that $f(x)$ is an indefinite integral of a summable function. For, if we consider, in the interval $(-\pi, \pi)$, a set of intervals, finite or infinite, of which the measure is $< \epsilon$, we see that the sum of the absolute variations of $f(x)$ over the intervals of the set is $\leqq A\epsilon$, and therefore the function $f(x)$ is an indefinite L-integral. It follows that the Fourier's series converges uniformly in $(-\pi, \pi)$ to $C + \int_{-\pi}^{x} f'(x)\,dx$, where C is a constant. It will be shewn in § 360 that the summable function $f'(x)$ has, for its Fourier's series, the series

$$\tfrac{1}{2}a_0' + \sum_{n=1}(-na_n\sin nx + nb_n\cos nx),$$

* See Lebesgue, *Bulletin de la soc. mat. de France*, vol. XXXVIII (1910), p. 190.

where a_0' is a constant. Applying the Riemann-Lebesgue theorem, it follows that $na_n = o(1)$, $nb_n = o(1)$. Thus we have the theorem* that:

If $f(x)$ satisfies the condition $|f(x+t) - f(x)| \leq A\,|t|$, where A is a positive number, then $a_n = o\left(\frac{1}{n}\right)$, $b_n = o\left(\frac{1}{n}\right)$.

It has been shewn by Lebesgue (*loc. cit.*) that the difference between $f(x)$ and the sum of the first $2n+1$ terms of the Fourier's series is $< \frac{3A\pi \log n}{n}$. Other investigations relating to Dirichlet's integral have been made by Kronecker†, Hölder, and Brodén‡. At the present time these have only historical interest.

POINTS OF NON-CONVERGENCE OF FOURIER'S SERIES FOR A CONTINUOUS FUNCTION

351. The continuity of a summable function at a particular point is neither necessary nor sufficient for the convergence, at that point, of the corresponding Fourier's series. The first example of a continuous function, for which the Fourier's series fails to converge at a particular point, was given by Du Bois-Reymond§, who also constructed a continuous function for which the Fourier's series fails to converge at the points of an everywhere-dense set. It is the most important outstanding question in the Theory of Fourier's series whether a continuous function can exist for which the Fourier's series fails to converge at all points of a set of positive measure, or at the points of a set of measure equal to that of the whole interval, or at every point of the interval. A function has been constructed‖ by Kolmogoroff for which the Fourier's series fails to converge at the points of a set of measure 2π; but this function is not continuous, and neither is its square summable in the interval.

It was suggested¶ by Fatou that trigonometrical series of the form $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$ might exist, such that $a_n = o(1)$, $b_n = o(1)$, which converge only at points belonging to a set of measure zero. Such a series was actually constructed** by Lusin. A simple example†† of such series was given by Hardy and Littlewood, who proved that the series $\sum_{n=1} n^{-\alpha} \cos n^2\pi x$, $\sum_{n=1} n^{-\alpha} \sin n^2\pi x$, where $0 < \alpha \leq \frac{1}{2}$, are convergent when x is a rational number of one of the forms $\frac{2p+1}{2q+1}$, $\frac{2p}{4q+3}$, in the case of

* See Fatou, *Acta Math.* vol XXX (1906), p. 398.

† *Berliner Sitzungsber.* 1885, "Ueber das Dirichlet'sche Integral," by Kronecker; and in the same volume, "Ueber eine neue hinreichende Bedingung..." by Hölder.

‡ *Math. Annalen*, vol. LII (1899), p. 177.

§ *Abhandlungen der bayer. Akad.* vol. XII, Abthg 2 (1876).

‖ *Fundamenta Math.* vol. IV (1923), p. 324.

¶ *Acta Math.* vol. XXX (1906), p. 398. ** *Rendiconti di Palermo*, vol. XXXII (1911), p. 386.

†† *Acta Math.* vol. XXXVII (1914), p. 232.

the cosine series, and of one of the forms $\frac{2p+1}{2q+1}$, $\frac{2p}{4q+1}$ in the case of the sine series; but that the series do not converge for any irrational value of x. It was shewn that, at a point at which the series is not convergent, it is not summable by any Cesàro mean. Since the series are non-summable at points of a set of measure greater than zero, it follows from § 368 that they are not Fourier's series. It can be shewn that, when $\alpha > \frac{1}{2}$, the series are Fourier's series, and that they converge almost everywhere. A series for which $a_n = o(1)$, $b_n = o(1)$ has been constructed* by Steinhaus which is nowhere convergent.

The general condition that, at a particular point of continuity of the function, the Fourier's series should fail to converge, has been investigated by Lebesgue†, and by Haar‡. The former of these also investigated the condition that, although the series converges at the point, the convergence should be non-uniform in every neighbourhood of the point.

A method of constructing continuous functions for which the Fourier's series exhibit these singularities at a point, or in an everywhere-dense set of points, was given by Fejér§. This method is of great simplicity as compared with that of Du Bois-Reymond, although the latter was simplified by Schwarz.

352. In accordance with § 299, in order that a continuous function $f(x)$ may exist, for which the Fourier's series will not converge at a particular point, or that it may be convergent at the point, but may not converge uniformly in any neighbourhood of the point, it is sufficient to shew that the integral

$$\int_0^{\frac{\pi}{2}} \left| \frac{\sin(2n+1)t}{\sin t} \right| dt$$

increases indefinitely with n.

Taking the portions of the integral, in which $(2n+1)t$ lies in the intervals

$$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right), \ldots \left(r\pi + \frac{\pi}{4}, r\pi + \frac{3\pi}{4}\right) \ldots$$

in all of which intervals $|\sin(2n+1)t| > \frac{1}{\sqrt{2}}$, we see that the integral exceeds in value

$$\sum_{r=0}^{n-1} \frac{1}{\sqrt{2}} \int_{\frac{r\pi+\frac{1}{4}\pi}{2n+1}}^{\frac{r\pi+\frac{3}{4}\pi}{2n+1}} \frac{dt}{t}, \text{ or } \frac{1}{\sqrt{2}} \sum_{r=0}^{n-1} \log\left(1 + \frac{\frac{1}{2}}{r+\frac{1}{4}}\right),$$

* *Comptes Rendus soc. sc. de Varsovie* (1912), p. 223.

† *Annales de Toulouse* (3), vol. I (1909), p. 76; *Comptes Rendus*, vol. CXLI (1905), p. 875.

‡ *Math. Annalen*, vol. LXIX (1910), p. 336.

§ *Crelle's Journal*, vol. CXXXVII (1909), p. 1, vol. CXXXVIII (1909), p. 22; *Rend. di Palermo*, vol. XXVIII (1909), p. 402.

and this exceeds $\frac{1}{\sqrt{2}} \log \prod_{r=0}^{n-1} \left(1 + \frac{1}{2r + \frac{1}{2}}\right)$, which diverges, as n is indefinitely increased.

The numbers $\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left| \frac{\sin (2n + 1) t}{\sin t} \right| dt$, for $n = 1, 2, 3, \ldots$ have been termed by Fejér the Lebesgue constants for Fourier's series. He shewed* that these numbers $\rho_1, \rho_2, \ldots$ are given by the asymptotic expression

$$\rho_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left| \frac{\sin (2n + 1) t}{\sin t} \right| dt = \frac{4}{\pi^2} \log n + c_0 + \frac{c_1}{n} + O\left(\frac{1}{n^2}\right),$$

where c_0, c_1 are determinate numbers. A complete investigation of the asymptotic expression for ρ_n has been given† by Gronwall, who shewed that the divergence of the sequence $\{\rho_n\}$ is monotone.

353. The method given by Fejér for the construction of Fourier's series for continuous functions which exhibit these phenomena depends upon the following Lemma:

The series

$$\frac{\cos (r + 1) x}{n} + \frac{\cos (r + 2) x}{n - 1} + \ldots + \frac{\cos (r + n) x}{1}$$

$$- \frac{\cos (r + n + 1) x}{1} - \frac{\cos (r + n + 2) x}{2} - \ldots - \frac{\cos (r + 2n) x}{n}$$

is less in absolute value than a fixed positive number λ, independent of x and of the integers n and r.

The expression is equal to

$$2 \sin \left(r + n + \frac{1}{2}\right) x \left\{\sin \frac{x}{2} + \frac{1}{2} \sin \frac{3x}{2} + \ldots + \frac{1}{n} \sin \frac{(2n - 1) x}{2}\right\}.$$

Let $s_n (x)$ denote

$$\sin x + \frac{1}{2} \sin 3x + \ldots + \frac{1}{n} \sin (2n - 1) x;$$

and let $\bar{s}_n (x)$ denote

$$\sin x + \frac{1}{3} \sin 3x + \ldots + \frac{1}{2n - 1} \sin (2n - 1) x;$$

then

$$\bar{s}_n (x) - \frac{1}{2} s_n (x) = \frac{1}{1.2} \sin x + \frac{1}{3.4} \sin 3x + \ldots + \frac{1}{(2n - 1) 2n} \sin (2n - 1) x,$$

hence $\left| \bar{s}_n (x) - \frac{1}{2} s_n (x) \right| < \frac{1}{1.2} + \frac{1}{3.4} + \ldots < 1$, for all values of n and x. Since $\bar{s}_n (x)$ is the partial sum of a Fourier's series (see Ex. 3, § 327) of bounded variation, $| \bar{s}_n (x) |$ is bounded with respect to (n, x) (see § 333).

* *Crelle's Journal*, vol. CXXXVIII (1910), pp. 22, 30.

† *Math. Annalen*, vol. LXXII (1912), p. 244.

It now follows that $|s_n(x)|$ is bounded with respect to (n, x), and thus the Lemma has been proved.

In order to define a Fourier's series, representing a function which is continuous, but such that the series does not converge at the point $x = 0$, let $g_m = 2 . 2^{m^2}$; and consider the series $\sum\limits_{n=1} A_n \cos nx$, of which the terms are grouped as follows:

$$\sum_{n=1}^{n=g_1} A_n \cos nx + \sum_{n=g_1+1}^{g_1+g_2} A_n \cos nx + \ldots + \sum_{n=g_1+g_2+\ldots+g_{m-1}+1}^{g_1+g_2+\ldots+g_m} A_n \cos nx + \ldots.$$

In the mth group, let A_n be so chosen that the group is

$$\frac{1}{m^2}\left\{\frac{\cos(g_1+g_2+\ldots+g_{m-1}+1)x}{\frac{1}{2}g_m} + \ldots + \frac{\cos(g_1+g_2+\ldots+g_{m-1}+\frac{1}{2}g_m)}{1}\right.$$
$$\left. - \frac{\cos(g_1+g_2+\ldots+g_{m-1}+\frac{1}{2}g_m+1)}{1} - \ldots - \frac{\cos(g_1+g_2+\ldots+g_m)}{\frac{1}{2}g_m}\right\}.$$

This is less, in absolute magnitude, than $\frac{\lambda}{m^2}$, and therefore the series, so grouped, is uniformly convergent, and the original ungrouped series is consequently (§ 321) the Fourier's series for a continuous function.

At the point $x = 0$, the value of the $(g_1 + g_2 + \ldots + g_{m-1} + \frac{1}{2}g_m)$th partial sum is $\frac{1}{m^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{2^{m^2}}\right)$, which is $> \frac{1}{m^2}\log 2^{m^2} > m \log 2$; and this increases indefinitely with m. Therefore the sum does not converge. The $(g_1 + g_2 + \ldots + g_m)$th partial sum is zero.

The series $\sum\limits_{n=1} A_n \sin nx$ converges at $x = 0$, the coefficients being taken to be those defined above; it will be shewn that both the series $\Sigma A_n \cos nx$, $\Sigma A_n \sin nx$ converge uniformly in the interval $(\epsilon, 2\pi - \epsilon)$, but that the series $\sum\limits_{n=1} A_n \sin nx$ does not converge uniformly in the interval $(0, \epsilon)$. Consider a partial remainder $R_{p,q}(x) = \sum\limits_{n=p}^{n=q} A_n \begin{matrix}\cos\\ \sin\end{matrix} nx$. We have in the interval $(\epsilon, 2\pi - \epsilon)$

$$\left|\frac{1}{2} + \sum_1^n \cos rx\right| = \left|\frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}}\right| < \frac{1}{2}\operatorname{cosec}\frac{\epsilon}{2} < \frac{\pi}{2\epsilon},$$

hence $\left|\sum\limits_\alpha^\beta \cos rx\right| < \frac{\pi}{\epsilon}$; and similarly $\left|\sum\limits_\alpha^\beta \sin rx\right| < \frac{\pi}{\epsilon}$.

Therefore
$$\left|\frac{\cos(r+1)x}{n} + \ldots + \frac{\cos(r+n)x}{1}\right|$$
$$= \left|1 . s_1 + \tfrac{1}{2}(s_2 - s_1) + \ldots + \frac{1}{n}(s_n - s_{n-1})\right|$$
$$= \left|(1 - \tfrac{1}{2})s_1 + (\tfrac{1}{2} - \tfrac{1}{3})s_2 + \ldots + \frac{1}{n}s_n\right|,$$

where $s_1 = \cos(r+n)x$, $s_2 = \cos(r+n)x + \cos(r+n-1)x, \ldots$; hence the expression on the left-hand side is less than $\left(1-\frac{1}{n}\right)\frac{\pi}{\epsilon}+\frac{\pi}{n\epsilon}$, which is less than $\frac{\pi}{\epsilon}$. If we split each group of terms in $R_{p,q}(x)$ into its terms with positive and with negative coefficients, the absolute value of the sum of the terms in the group is $< \frac{1}{m^2}\cdot\frac{2\pi}{\epsilon}$; and this holds good if only a part of the group is contained in $R_{p,q}(x)$. It now follows that

$$|R_{p,q}(x)| < \frac{2\pi}{\epsilon}\left(\frac{1}{m_p^2}+\frac{1}{(m_p+1)^2}+\ldots+\frac{1}{m_q^2}\right) < \frac{2\pi}{\epsilon}\sum_p^\infty \frac{1}{m_p^2} < \frac{2\pi}{\epsilon}\cdot\frac{1}{m_p-1},$$

for all values of q. It now follows that the series are both uniformly convergent in the interval $(\epsilon, 2\pi-\epsilon)$.

To shew that $\sum_{n=1} A_n \sin nx$ does not converge uniformly in the interval $(0, \epsilon)$, observing that the part of the mth group which has positive coefficients is

$$\frac{1}{m^2}\left\{\frac{\sin(g_1+g_2+\ldots+g_{m-1}+1)x}{2^{m^2}}+\ldots+\frac{\sin(g_1+g_2+\ldots+\frac{1}{2}g_m)x}{1}\right\},$$

consider the point $x = \frac{\pi}{2(g_1+g_2+\ldots+\frac{1}{2}g_m)}$; this is in the interval $(0, \epsilon)$ if m is sufficiently large, and all the terms in the bracket are positive. Denoting $g_1+g_2+\ldots+g_{m-1}$ by μ, the expression becomes

$$\frac{1}{m^2}\left\{\frac{\sin(\mu+1)x}{2^{m^2}}+\ldots+\frac{\sin(\mu+\frac{1}{2}g_m)x}{1}\right\},$$

and observing that $\mu+\frac{1}{4}g_m > \frac{1}{2}(\mu+\frac{1}{2}g_m)$, the expression is greater than

$$\frac{1}{m^2\sqrt{2}}\left\{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{\frac{1}{2}2^{m^2}+1}\right\},$$

and this exceeds $\frac{1}{m^2\sqrt{2}}\log\frac{1}{2}2^{m^2}$, or $\frac{m\log 2}{\sqrt{2}} - \frac{1}{m^2\sqrt{2}}\log 2$, which increases independently with m. Since these are partial remainders which increase indefinitely with m, for some point in $(0, \epsilon)$, the series cannot converge uniformly in $(0, \epsilon)$.

354. In order to construct a series which fails to converge at an everywhere-dense set of points, taking the series $\sum_{n=1} A_n \cos nx$, defined in § 353, in the first group of terms write $1!\,x$ for x, in the second group write $2!\,x$ for x, and generally in the mth group write $m!\,x$ for x; we have then a series $\sum_{n=1} A_n \cos\lambda_n x$, where $\lambda_n = 1!\,n$ when $1 \leq n \leq g_1$; $\lambda_n = 2!\,n$, when

$$g_1+1 \leq n \leq g_1+g_2, \ldots.$$

The series then fails to converge at every point $x = \pm\frac{m\pi}{n}$, where m and n

are integers; but the series converges uniformly when it is grouped, and thus it corresponds to a continuous function, and therefore (see § 321) the series is a Fourier's series. When $x = \frac{m\pi}{n}$, the part of the νth group which has positive coefficients, if ν be sufficiently large, has the value

$$\frac{1}{\nu^2}\left(1 + \tfrac{1}{2} + \dots + \frac{1}{2^{\nu^3}}\right),$$

and this increases indefinitely with ν. The series cannot therefore be convergent at the point.

355. Let $\phi_n(x)$ have the value $+1$, or -1, according as $\frac{\sin \frac{1}{2}(2n+1)x}{\sin \frac{1}{2}x}$ is positive or negative, and when it is zero, let $\phi_n(x) = 0$. The nth partial sum of the Fourier's series corresponding to $\phi_n(x)$ is at the point $x = 0$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(2n+1)\frac{x}{2}}{\sin\frac{x}{2}}\right| dx$, and this diverges, as $n \sim \infty$ (see § 352).

Let $\psi_n(x) = \sin\frac{2n+1}{2}x$, for $-\pi \leq x \leq \pi$, then the nth partial sum of the Fourier's series corresponding to $\psi_n(x)$ has, at $x = 0$, the value $\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^2(2n+1)t}{\sin t}dt$ which diverges with n. Let $\psi(\mu, x)$ denote the even function, defined in the interval $(-\pi, \pi)$, which has the value $\sin \mu x$ in the interval $(0, \pi)$; and let $\frac{1}{2}a_0 + \Sigma a_n \cos nx$ denote the Fourier's series corresponding to $\psi(\mu, x)$; we find then

$$a_n = \frac{1 + (-1)^{n+1}\cos\mu\pi}{\pi}\cdot\frac{2\mu}{\mu^2 - n^2}, \quad a_0 = \frac{1 - \cos\mu\pi}{\pi}\cdot\frac{2}{\mu};$$

hence a_n is positive if $n < \mu$, and negative if $n > \mu$. It is easily seen that $\psi(2, x)$, $\psi(4, x)$, ..., have the same properties, as regards the partial remainders at the point $x = 0$, as $\psi_1(x)$, $\psi_2(x)$, ... If we define a function by means of the expression $\sum_{m=1}^{\infty}\frac{\sin 2^{m^3}x}{m^2}$ in $(0 < x < \pi)$, it is seen from the properties of $\psi(2, x)$, $\psi(4, x)$, ..., that the cosine series for this function does not converge at the point $x = 0$.

Again, if we take $f_1(x) = \sum_{m=1}^{\infty}\frac{\sin(2^m+1)\frac{x}{2}}{m^2}$, then $f_1(x)$ is continuous in the interval $(-\pi, \pi)$, but as is seen from the properties of $\psi(\frac{1}{2}, x)$, $\psi(\frac{3}{2}, x)$, the cosine series for $f_1(x)$ does not converge at the point $x = 0$.

356. An example, due to Schwarz*, will be given here of a function which is everywhere continuous, but for which the Fourier's series fails to converge at a certain point. It will here be shewn† that the series is, at that point, oscillatory, with an infinite upper sum.

Let the product $1.3.5 \dots (2\lambda + 1)$ be denoted by $[2\lambda + 1]$, and let the function $\phi(z)$ be defined for the interval $(0, \alpha)$, where $0 < \alpha \leqq \frac{1}{2}\pi$, in the following manner: In the interval $(\pi/[\lambda], \pi/[\lambda - 1])$, let $\phi(z) = c_\lambda \sin [\lambda] z$, where c_λ is a constant, depending upon the value of λ; let λ have all values $\lambda_1, \lambda_1 + 1, \lambda_1 + 2, \dots$, where λ_1 is a fixed integer, and we may suppose α so chosen that $\alpha = \pi/[\lambda_1 - 1]$; also let $\phi(0) = 0$. If the sequence $c_{\lambda_1}, c_{\lambda_1+1}, c_{\lambda_1+2}, \dots$, be so chosen that it converges to the limit zero, the function $\phi(z)$ is continuous at the point $z = 0$, but it has an indefinitely great number of oscillations in an arbitrarily small neighbourhood of that point. If the constants c_λ satisfy the further condition, that $c_\lambda \log (2\lambda + 1)$ becomes indefinitely great, as λ is indefinitely increased, it will be shewn that the integral

$$\int_0^\alpha \phi(z) \frac{\sin (2n+1) z}{z} dz$$

will increase indefinitely, as n has successively the values of integers in a certain sequence. Thus the Fourier's series, corresponding to the continuous function defined by $f(x) = 0$, for $-\pi \leqq x \leqq 0$, and $f(x) = \phi(\frac{1}{2}x)$, for $0 \leqq x \leqq 2\alpha$, and $f(x) = 0$, for $2\alpha \leqq x \leqq \pi$, does not converge at the point $x = 0$.

Let $2n + 1 = 1.3.5 \dots (2\mu + 1) = [\mu]$; then

$$\int_0^\alpha \phi(z) \frac{\sin [\mu] z}{z} dz$$

may be written in the form

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2 [\mu] z}{z} dz + \sum_{r=\lambda_1}^{\mu-1} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin [r] z \sin [\mu] z}{z} dz$$
$$+ \sum_{r=\mu+1}^{\infty} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin [r] z \sin [\mu] z}{z} dz.$$

The first integral may be written in the form

$$\tfrac{1}{2} c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{1 - \cos 2 [\mu] z}{z} dz,$$

which is equivalent to

$$\tfrac{1}{2} c_\mu \log (2\mu + 1) - \tfrac{1}{2} c_\mu \frac{[\mu]}{\pi} \int_{\pi/[\mu]}^{\beta} \cos 2 [\mu] z \, dz,$$

where β is some number between $\pi/[\mu]$ and $\pi/[\mu - 1]$.

* See the history of the theory of Fourier's series, by Sachs, *Schlömilch's Zeitschr.* Supplement, vol. XXV (1880), p. 231.

† See Hobson, *Proc. Lond. Math. Soc.* (2), vol. III (1904), p. 55.

Now let $c_\mu \log(2\mu+1)$ increase indefinitely with μ. This is consistent with c_μ having the limit zero; for we have only to take

$$c_\mu = \{\log(2\mu+1)\}^{-s},$$

where s is some fixed positive number, less than unity.

Since
$$c_\mu \frac{[\mu]}{\pi}\int_{\pi/[\mu]}^{\beta} \cos 2[\mu] z\, dz$$
is numerically not greater than c_μ/π, we see that, with the supposition made as to c_μ, the expression

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \frac{\sin^2[\mu] z}{z}\, dz$$

becomes indefinitely great, as μ is increased indefinitely.

To evaluate
$$\sum_{r=\lambda_1}^{\mu-1} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin[r] z \sin[\mu] z}{z}\, dz,$$
we see, by writing $\sin[r] z . \sin[\mu] z$ as half the difference of two cosines, and applying the second mean value theorem to each integral, that the absolute value of the expression is less than

$$\sum_{r=\lambda_1}^{\mu-1} c_r \frac{[r]}{\pi}\left\{\frac{1}{[\mu]-[r]} + \frac{1}{[\mu]+[r]}\right\},$$

or than

$$\sum_{r=\lambda_1}^{\mu-1} \frac{c_r}{\pi}\frac{[r]}{[\mu-1]}\left\{\frac{1}{2\mu+1-[r]/[\mu-1]} + \frac{1}{2\mu+1+[r]/[\mu-1]}\right\},$$

which is less than
$$\frac{c_{\lambda_1}}{\pi}\sum \frac{[r]}{[\mu-1]}.\frac{1}{\mu};$$
and this is less than

$$\frac{c_{\lambda_1}}{\pi\mu}\left\{1 + \frac{1}{2\mu-1} + \frac{1}{(2\mu-1)(2\mu-3)} + \dots\right\}.$$

Therefore the absolute value of the integral is less than $2c_\lambda/\pi\mu$; and this becomes indefinitely small, as μ is indefinitely increased; and therefore the limiting value of the expression is zero.

Lastly, we have to consider the expression

$$\sum_{r=\mu+1}^{\infty} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin[r] z \sin[\mu] z}{z}\, dz.$$

Since
$$\left|\frac{\sin[\mu] z}{z}\right| < [\mu], \text{ and } |\sin[r] z| \leqq 1,$$
the absolute value of the expression is less than $\pi c_{\mu+1}$; and this has the limit zero, when μ is indefinitely increased.

It has now been shewn that

$$\int_0^{a} \frac{\phi(z)\sin[\mu] z}{z}\, dz,$$

increases indefinitely with μ, where $[\mu] = 1.3.5 \dots (2\mu+1)$, provided c_λ has the value $\{\log(2\lambda+1)\}^{-s}$, where $0 < s < 1$.

357. We proceed to consider the case in which $2n + 1 = (2p + 1)[\mu - 1]$, where p is an integer which varies with μ in such a manner that it always lies between 0 and μ.

In this case, as before, we divide the integral

$$\int_0^a \phi(z) \frac{\sin(2n+1)z}{z} dz$$

into three parts,

$$c_\mu \int_{\pi/[\mu]}^{\pi/[\mu-1]} \sin[\mu] z \frac{\sin(2p+1)[\mu-1]z}{z} dz$$
$$+ \sum_{r=\lambda_1}^{\mu-1} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin[r]z \sin(2p+1)[\mu-1]z}{z} dz$$
$$+ \sum_{r=\mu+1}^{\infty} c_r \int_{\pi/[r]}^{\pi/[r-1]} \frac{\sin[r]z \sin(2p+1)[\mu-1]z}{z} dz.$$

The first part is equal to

$$\frac{c_\mu[\mu]}{2\pi} \int_{\pi/[\mu]}^{\beta} [\cos\{[\mu-1](2\mu-2p)z\} - \cos\{[\mu-1](2\mu+2p+2)z\}] dz,$$

where β is a number between $\pi/[\mu]$ and $\pi/[\mu-1]$; and this expression is less, in absolute value, than

$$\frac{c_\mu[\mu]}{\pi}\left\{\frac{1}{[\mu-1](2\mu-2p)} + \frac{1}{[\mu-1](2\mu+2p+2)}\right\},$$

or than

$$\frac{c_\mu}{\pi}\left\{\frac{1+1/2\mu}{1-p/\mu} + \frac{1+1/2\mu}{1+1/\mu+p/\mu}\right\}.$$

If, now, p increases with μ in such a manner that p/μ is always less than some fixed number which is less than unity, then this expression diminishes indefinitely, as μ is indefinitely increased. It would also be sufficient that

$$p/\mu = 1 - \kappa\{\log(2\mu+1)\}^{-s'},$$

where $s' < s$, and $c_\mu = \{\log(2\mu+1)\}^{-s}$; the positive number κ being fixed.

The second part of the above integral is less, in absolute value, than

$$\sum_{r=\lambda_1}^{\mu-1} \frac{c_r[r]}{\pi}\left\{\frac{1}{(2p+1)[\mu-1]-[r]} + \frac{1}{(2p+1)[\mu-1]+[r]}\right\},$$

or than $\dfrac{c_{\lambda_1}}{\pi} \sum \dfrac{[r]}{[\mu-1]}\left\{\dfrac{1}{2p+1-[r]/[\mu-1]} + \dfrac{1}{2p+1+[r]/[\mu-1]}\right\}$,

and this is less than

$$\frac{c_{\lambda_1}}{p\pi}\left\{1 + \frac{1}{2\mu-1} + \frac{1}{(2\mu-1)(2\mu-3)} + \dots\right\},$$

or than $2c_{\lambda_1}/p\pi$. Therefore the expression diminishes indefinitely, as p is indefinitely increased.

That the third part of the above integral has the limit zero is seen from the fact that its absolute value is less than $c_{\mu+1}(2p+1)[\mu-1]\pi/[\mu]$, or than $\pi c_{\mu+1}(2p+1)/(2\mu+1)$.

It has now been proved that

$$\int_0^a \phi(z)\frac{\sin(2n+1)z}{z}\,dz$$

has the limit zero, if $2n+1$ increases indefinitely through a sequence of the form

$$[\mu_1-1](2p_1+1),\quad [\mu_2-1](2p_2+1),\quad [\mu_3-1](2p_3+1),\ \ldots$$

where $\mu_1, \mu_2, \mu_3, \ldots$ is an increasing sequence of integers, and $p_1, p_2, p_3, \ldots$ are such that $p/\mu \leqq 1-\kappa\{\log(2\mu+1)\}^{-s'}$, where $s'<s$.

It has now been shewn that the sum of the Fourier's series oscillates; the limit being infinite, or zero, according as one or other of two particular sequences of values of n is chosen.

THE ABSOLUTE CONVERGENCE OF TRIGONOMETRICAL SERIES

358. Let $\frac{1}{2}a_0+\sum\limits_{n=1}(a_n\cos nx+b_n\sin nx)$ be a trigonometrical series which converges absolutely at the point ξ. We have then

$$\left[\tfrac{1}{2}a_0+\sum_{n=1}^{n}\{a_n\cos n(\xi+h)+b_n\sin n(\xi+h)\}\right]$$
$$+\left[\tfrac{1}{2}a_0+\sum_{n=1}^{n}\{a_n\cos n(\xi-h)+b_n\sin n(\xi-h)\}\right]$$
$$=2\left[\tfrac{1}{2}a_0+\sum_{n=1}^{n}(a_n\cos n\xi+b_n\sin n\xi)\right]-4\sum_{n=1}^{n}\sin^2\tfrac{1}{2}nh\,(a_n\cos n\xi+b_n\sin n\xi);$$

and it follows that the expression on the left-hand side converges as $n\sim\infty$. If either of the expressions in the two brackets is convergent as $n\sim\infty$, then the other is so, and if either is non-convergent the other is so also. Also, if either converges to a sum-function which is continuous with respect to h, for $h=h_1$, the other has the same property. It has thus been shewn* that:

The points of continuity of the sum-function of a trigonometrical series, and the points of convergence of the series, are symmetrical with respect to a point of absolute convergence of the series.

Since

$$|\,a_n\cos n(\xi-h)+b_n\sin n(\xi-h)\,|$$
$$\leqq|\,a_n\cos n(\xi+h)+b_n\sin n(\xi+h)\,|+|\,a_n\cos n\xi+b_n\sin n\xi\,|,$$

it follows that, if the trigonometrical series is absolutely convergent at the two points ξ, $\xi+h$, it is also absolutely convergent at $\xi-h$. By continued application of this result it then appears that the trigonometrical series must be absolutely convergent at all the points $\xi\pm\iota h$, where ι is any integer. In case h/π is an irrational number, and (α,β) is any interval

* See Fatou, *Acta Math.* vol. XXX (1906), p. 398.

contained in $(-\pi, \pi)$, ι can be so determined that $\xi \pm \iota h$ differs by some multiple of 2π from a number in the interval (α, β). It follows that:

If a trigonometrical series is absolutely convergent at two points x_1, x_2, such that $x_1 - x_2$ is incommensurable with π, then it converges absolutely at the points of an everywhere-dense set.

359. The following theorem was given* by Lusin:

If a trigonometrical series $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ is absolutely convergent at all points of a set of which the measure is positive, then $\sum\limits_{n=1} |a_n|$ and $\sum\limits_{n=1} |b_n|$ are convergent, and the trigonometrical series is absolutely convergent everywhere.

If $\frac{1}{2}|a_0| + \Sigma\rho_n |\cos(nx - \alpha_n)|$, where $\rho_n = (a_n^2 + b_n^2)^{\frac{1}{2}}$, converges in a set E, such that $m(E) > 0$, there exists a perfect set P, contained in E, of measure $p\ (> 0)$, in which the series converges uniformly (see § 99). If $s(x)$ denote the sum of the series, we have

$$\int_{(P)} s(x)\,dx = p \cdot \tfrac{1}{2}|a_0| + \sum_{n=1}^{\infty} \rho_n \int_{(P)} |\cos n(x - \alpha_n)|\,dx.$$

It will be shewn that $\int_{(P)} |\cos n(x - \alpha_n)|\,dx$ is not less than a fixed positive number K_p, independent of n.

If θ be between 0 and $\frac{1}{2}\pi$, we have $|\cos n(x - \alpha_n)| \geqq \cos\theta$, when $x - \alpha_n$ is in an interval $\left(\frac{r\pi - \theta}{n}, \frac{r\pi + \theta}{n}\right)$, where r is a positive or negative integer, including zero. The condition is satisfied in each interval of length $\frac{2\theta}{n}$ belonging to a set, consecutive intervals of the set being separated by an interval of length $\frac{\pi - 2\theta}{n}$. It follows that, in the interval $(-\pi, \pi)$, the condition $|\cos n(x - \alpha_n)| \geqq \cos\theta$ is satisfied at all points belonging to a set of which the measure is 4θ. If θ be taken to be $> \frac{2\pi - p}{4}$, there is a set of points of positive measure q, contained in P, at which

$$|\cos n(x - \alpha_n)| \geqq \cos\theta.$$

It then follows that $\int_{(P)} |\cos n(x - \alpha_n)|\,dx$ exceeds a fixed number K_p, independent of n.

We have then $\sum\limits_{n=1}^{\infty} \rho_n \leqq \frac{1}{K_p}\int_{(P)} s(x)\,dx$; and therefore $\sum\limits_{n=1} \rho_n$ is convergent. The result in the theorem then follows at once.

It is clear that the series converges uniformly in $(-\pi, \pi)$; hence it is the Fourier's series of a continuous function.

* *Comptes Rendus*, vol. CLV (1912), p. 580.

It follows that:

Unless a trigonometrical series is the Fourier's series of a continuous function, it can only converge absolutely at the points of a set of measure zero.

The following Lemma will be applied:

If E be a measurable set of points contained in $(-\pi, \pi)$, and there are in $(-\pi, \pi)$ an infinite number of points with respect to which E is symmetrical (when E is repeated periodically beyond the interval $(-\pi, \pi)$), then E has either measure zero or measure 2π.

Let $P_1, P_2, \ldots P_r, \ldots$ be an enumerably infinite set of points of symmetry; and let us consider any pair P_r, P_s of these points. Let E_{rs} be the component of E in the interval P_r, P_s, of length δ_{rs}; then if $m(E_{rs}) = 0$, it is clear from the double symmetry of E that $m(E) = 0$. If $m(E_{rs}) = l_{rs} > 0$, we see that $m(E) \geqq nl_{rs}$, where n is the integer such that

$$n\delta_{rs} \leqq 2\pi < (n+1)\,\delta_{rs};$$

and thus $m(E) > (2\pi - \delta_{rs})\dfrac{l_{rs}}{\delta_{rs}}$. Since the set $\{P_n\}$ is not finite, it contains an infinite number of pairs of points P_r, P_s for which δ_{rs} is less than an arbitrarily chosen number η, hence $m(E) > 2\pi\dfrac{l_{rs}}{\delta_{rs}} - \eta$. If possible, let $\dfrac{l_{rs}}{\delta_{rs}} < h < 1$, for all pairs of values of r and s for which $\delta_{rs} < \eta$; thus every point of E in δ_{rs} has a neighbourhood $\delta_{rs}\,(<\eta)$, for which the component of E in that neighbourhood has measure $< h\delta_{rs}$. Now any fixed point P, of E, corresponds, on account of the symmetry of the set E with respect to the points P_r, P_s, to a point of δ_{rs}; hence P has a neighbourhood of length δ_{rs} in which the component of E has measure $< h\delta_{rs}$, where $h < 1$; and this for every pair of values of r and s for which $\delta_{rs} < \eta$. Since δ_{rs} has indefinitely small values, this is contrary to the fact that P may be so chosen that E has metric density 1 in its neighbourhood (see I, § 140). It is thus impossible that $h < 1$; and r, s can be so chosen that $\dfrac{l_{rs}}{\delta_{rs}}$ is arbitrarily near unity. Hence $m(E) > 2\pi(1-\zeta) - \eta$; where η and ζ are arbitrarily chosen positive numbers. It thus follows that $m(E) = 2\pi$. From the Lemma, combined with the results in § 358, we obtain the following theorems given by Lusin:

A trigonometrical series having in the interval $(-\pi, \pi)$ an enumerable set of points of absolute convergence is either almost everywhere convergent, or almost everywhere non-convergent.

For the set of points of convergence must, in accordance with the Lemma, have either measure 2π, or measure zero.

A trigonometrical series having two points of absolute convergence, of which the distance is incommensurable with π, is either almost everywhere convergent, or almost everywhere non-convergent.

The theorem has been given* by S. Bernstein that:

If the function $f(x)$ satisfies the Lipschitz condition that, for any pair of points x_1, x_2 in the interval $(-\pi, \pi)$, $|f(x_1) - f(x_2)| \leqq \lambda |x_1 - x_2|^\alpha$, where $0 < \alpha < 1$, and λ is a positive constant, then, provided $\alpha > \frac{1}{2}$, $\Sigma |a_n|$, $\Sigma |b_n|$ are convergent, so that the Fourier's series converges uniformly and absolutely to $f(x)$. There are Fourier's series of functions which satisfy the condition for a value of α $(< \frac{1}{2})$ which do not converge absolutely at all points.

The more general theorem has been given† by Szász that:

If the function $f(x)$ satisfies the Lipschitz condition, $\sum\limits_{n=1} (|a_n|^k + |b_n|^k)$ is convergent if $k > \dfrac{2}{2\alpha + 1}$, but may diverge if $k < \dfrac{2}{2\alpha + 1}$.

THE INTEGRATION OF FOURIER'S SERIES

360. Let $\frac{1}{2}a_0 + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be the Fourier's series corresponding to a function $f(x)$, summable in the interval $(-\pi, \pi)$, and of period 2π. No assumption is made as regards the convergence of the series.

The function $g(x) \equiv \int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x$ is continuous and of bounded variation in any finite interval; also it is periodic, and of period 2π, in the variable x. It can consequently be represented everywhere by a Fourier's series $\frac{1}{2}a_0' + \Sigma (a_n' \cos nx + b_n' \sin nx)$ which converges uniformly to $g(x)$.

We have $a_n' = \dfrac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos nx\,dx$, $b_n' = \dfrac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin nx\,dx$. Since $g(x)$, $\sin nx$ are both indefinite integrals, the formula of integration by parts can be applied to the expression for a_n'. Thus

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos nx\,dx = -\frac{1}{n\pi}\int_{-\pi}^{\pi} \sin nx \,.\, Dg(x)\,dx,$$

where $Dg(x)$ is any one of the four derivatives of $g(x)$ (see I, § 420). Now, at almost every point of $(-\pi, \pi)$, the four derivatives of $g(x)$ are all equal to $f(x) - \frac{1}{2}a_0$ (see I, § 405). Accordingly, we have

$$a_n' = -\frac{1}{n\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx = -\frac{1}{n} b_n;$$

in a similar manner it can be shewn that $b_n' = \dfrac{1}{n} b_n$. It has now been shewn that the series

$$\tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$$

* *Comptes Rendus*, vol. CLVIII (1914), p. 1661.

† *Münch. Sitzungsber.* (1922), p. 135.

converges uniformly in any interval to the function

$$\int_{-\pi}^{x} f(x)\,dx - \tfrac{1}{2}a_0 x,$$

where $$a_0' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,dx.$$

The following theorem has been established:

If $f(x)$ be any summable function, periodic, and of period 2π, then, if (x_1, x_2) be any finite interval, $\int_{x_1}^{x_2} f(x)\,dx$ is represented by

$$\left[\tfrac{1}{2}a_0 x_2 + \sum_{n=1}^{\infty} \frac{a_n \sin nx_2 - b_n \cos nx_2}{n}\right] - \left[\tfrac{1}{2}a_0 x_1 + \sum_{n=1}^{\infty} \frac{a_n \sin nx_1 - b_n \cos nx_1}{n}\right]$$

which is obtained by integrating the series $\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ term by term.

This theorem, which in this general form was given* by Fatou, was a generalization of an earlier form of the theorem due to Lebesgue. It is remarkable in view of the fact that the Fourier's series which is integrated is not necessarily known to converge.

It follows, by letting $x = 0$, that the series $\sum_{n=1} \frac{b_n}{n}$ is always convergent. The necessary and sufficient conditions that $\sum_{n=1} \frac{a_n}{n}$ should be convergent have been given† by Hardy and Littlewood.

To obtain the converse of the above theorem, let it be assumed that the series $\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is such that the series

$$\sum_{n=1} \frac{a_n \sin nx - b_n \cos nx}{n}$$

converges in $(-\pi, \pi)$ everywhere to a function $F(x)$ which is an indefinite L-integral of a summable function. The function $F(x)$ being continuous and of bounded variation, is representable by a Fourier's series which converges everywhere to the value of the function. As there cannot be two distinct trigonometrical series which have this property (see § 320), $\sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$ is the Fourier's series corresponding to $F(x)$. Therefore

$$\frac{a_n}{n} = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx\,dx, \quad \frac{b_n}{n} = -\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx, \quad 0 = \int_{-\pi}^{\pi} F(x)\,dx.$$

* *Acta Math.* vol. XXX (1906), p. 384.

† *Math. Zeitschr.* vol. XIX (1924), p. 95.

Let $DF(x)$, one of the four derivatives of $F(x)$, be denoted by $f(x)$; we have then, on integration by parts as before,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Therefore $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ is the Fourier's series corresponding to the function $f(x) + \frac{1}{2}a_0$ or $DF(x) + \frac{1}{2}a_0$.

Combining this result with the previous one, the following theorem has been established*:

The necessary and sufficient condition that the series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

should be a Fourier's series of some function $f(x)$, summable in $(-\pi, \pi)$, is that the integrated series

$$\tfrac{1}{2}a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$$

should converge throughout the interval $(-\pi, \pi)$ to a function which is the indefinite L-integral of a summable function. The function $f(x)$ differs by a null-function from any one of the four derivatives of the L-integral.

361. The theorem of § 360 can be applied to shew that two non-equivalent functions, summable in the interval $(-\pi, \pi)$, cannot have one and the same Fourier's series. For if $f(x)$, $\phi(x)$ be two summable functions which have the Fourier's series $\frac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$, it follows from the first theorem of § 360 that $\int_\Delta f(x)\,dx = \int_\Delta \phi(x)\,dx$, where Δ is any finite interval. Since the integral of $f(x) - \phi(x)$, taken over every interval, is zero, it follows (I, § 394) that the two functions $f(x)$, $\phi(x)$ are equivalent.

The following theorem has accordingly been established:

There cannot exist two non-equivalent functions $f(x)$ such that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n,\quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n,$$

for $n = 0, 1, 2, 3, \ldots$, where $a_0, a_1, b_1, a_2, b_2, \ldots$ are given numbers.

362. If $f_n(x)$ denote the finite sum

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \ldots + (a_n \cos nx + b_n \sin nx)$$

of the first $n + 1$ terms of the Fourier's series corresponding to the function $f(x)$, the theorem of § 360 may be expressed in the form

$$\lim_{n\sim\infty}\int_{x_1}^{x_2}\{f(x) - f_n(x)\}\,dx = 0,$$

where (x_1, x_2) is any finite interval.

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 423.

Let it now be assumed that $\{f(x)\}^2$ is summable in the interval $(-\pi, \pi)$; then

$$\int_{-\pi}^{\pi} \{f(x) - f_n(x)\}^2 dx = \int_{-\pi}^{\pi} \{f(x)\}^2 dx - \pi \left[\tfrac{1}{2}a_0^2 + \sum_{r=1}^{r=n} (a_r^2 + b_r^2)\right],$$

since
$$\int_{-\pi}^{\pi} f(x) f_n(x)\, dx = \tfrac{1}{2}\pi a_0^2 + \pi \sum_{r=1}^{r=n} (a_r^2 + b_r^2),$$

$$\int_{-\pi}^{\pi} \{f_n(x)\}^2 dx = \tfrac{1}{2}\pi a_0^2 + \pi \sum_{r=1}^{r=n} (a_r^2 + b_r^2).$$

As $\int_{-\pi}^{\pi} \{f(x) - f_n(x)\}^2 dx$ is essentially positive, it follows that the series $\tfrac{1}{2}a_0^2 + \sum_{r=1} (a_r^2 + b_r^2)$ is convergent, and has for its limiting sum a number $\leq \frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 dx$. This result is known* as Bessel's inequality.

We have now $\int_{-\pi}^{\pi} \{f(x) - f_n(x)\}^2 dx \leq K^2$, for all values of n, where K is some fixed number, independent of n.

If e be any measurable set of points in $(-\pi, \pi)$, we have, by Schwarz's inequality,

$$\left| \int_{(e)} \{f(x) - f_n(x)\}\, dx \right| \leq \left\{ \int_{(e)} dx \int_{(e)} \{f(x) - f_n(x)\}^2 dx \right\}^{\frac{1}{2}} \leq K\sqrt{m(e)}.$$

Let the set e be enclosed in intervals $\delta_1, \delta_2, \ldots$ of a set Δ, of non-overlapping intervals, such that $m(\Delta) - m(e) < \epsilon$; and let F denote the set of points $\Delta - e$; thus $m(F) < \epsilon$. If Δ_r be a finite set of the intervals $\delta_1, \delta_2, \ldots$, such that $m(\Delta) - m(\Delta_r) < \epsilon$, we have

$$\int_{(e)} \{f(x) - f_n(x)\}\, dx = \left\{ \int_{(\Delta_r)} + \int_{(\Delta - \Delta_r)} - \int_{(F)} \right\} [f(x) - f_n(x)]\, dx.$$

Since Δ_r consists of a finite set of intervals,

$$\lim_{n \sim \infty} \int_{(\Delta_r)} \{f(x) - f_n(x)\}\, dx = 0.$$

Also

$$\left| \left\{ \int_{(\Delta - \Delta_r)} - \int_{(F)} \right\} [f(x) - f_n(x)]\, dx \right| \leq K\sqrt{m(\Delta - \Delta_r)} + K\sqrt{m(F)} < 2K\epsilon^{\frac{1}{2}},$$

for all values of n. It follows that

$$\overline{\lim_{n \sim \infty}} \left| \int_{(e)} \{f(x) - f_n(x)\}\, dx \right| \leq 2K\epsilon^{\frac{1}{2}}.$$

Since ϵ is arbitrary, we obtain the result

$$\lim_{n \sim \infty} \int_{(e)} \{f(x) - f_n(x)\}\, dx = 0,$$

and thus the integral of $f(x)$, over any measurable set e, is obtained by

* *Astron. Nachrichten*, vol. VI (1828), pp. 333–348.

term by term integration, over the set e, of the Fourier's series corresponding to $f(x)$.

In accordance with the definition (§ 201) of complete term by term integration of a series, the following theorem has been established:

If $f(x)$ be a function of which the square is summable in $(-\pi, \pi)$, the Fourier's series corresponding to $f(x)$ is completely integrable term by term, giving as its sum, when taken over a measurable set of points, the integral of $f(x)$ over that set.

363. The method of § 360 can be applied to prove the following theorem relating to the integration of a Fourier's D-series corresponding to a function for which the set of points of non-summability has measure zero.

If $f(x)$ be a function which has a Denjoy integral, and for which the set of points of non-summability has measure zero, in the interval $(-\pi, \pi)$, the Fourier's series corresponding to $\int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x$ is

$$\tfrac{1}{2}a_0' + \Sigma \frac{a_n \sin nx - b_n \cos nx}{n}.$$

The class of functions considered here includes those which have an HL-integral in $(-\pi, \pi)$.

In this case the function $g(x) \equiv \int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x$ is continuous, although in general not of bounded variation, in $(-\pi, \pi)$. The Fourier's series corresponding to $g(x)$ has for coefficients a_n', b_n', given by

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos nx\,dx, \quad b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin nx\,dx.$$

Since $g(x)$ is of bounded variation in any closed interval which contains no points of H, the non-dense closed set of points of non-summability of $f(x)$, the Fourier's series converges uniformly to $g(x)$ in any interval interior to an interval contiguous to H, but it cannot be assumed that the series converges to $g(x)$ at points of H. In any case the Cesàro sum of the Fourier's series is everywhere $g(x)$ (see § 365). At almost all points of an interval contiguous to H, $g'(x)$ exists and has the value $f(x) - \frac{1}{2}a_0$. In the case here considered, in which H has content zero, this also holds for the whole interval $(-\pi, \pi)$. The method of integration by parts being applicable when one of the functions is a D-integral, and the other is of bounded variation (I, § 474), it follows, as in § 360, that $a_n' = -\frac{1}{n} b_n$, $b_n' = \frac{1}{n} a_n$; and the theorem has thus been established.

364. Taking the case in which the closed set H is enumerable, let it now be assumed that the series $\frac{1}{2}a_0 x + \Sigma \frac{a_n \sin nx - b_n \cos nx}{n}$ converges through

the interval $(-\pi, \pi)$ to a function $F(x)$ which is the HL-integral of a function which has H for its sole points of non-summability; except that the convergence is not assumed to hold at the points of H. In case π and $-\pi$ belong to H, $F(x)$ will be defined at these points so as to satisfy the condition $F(\pi) - F(-\pi) = \pi a_0$; otherwise this condition follows from the periodicity of the series. If (a, b) be an interval interior to an interval contiguous to H, the series converges everywhere in (a, b) to the continuous function $F(x)$ which has bounded variation in (a, b). The Fourier's series of $F(x) - \frac{1}{2}a_0 x$ converges to $F(x) - \frac{1}{2}a_0 x$ at every point not belonging to H. Since there cannot be two trigonometrical series both of which converge, at every point not belonging to an enumerable set, to the same function (see § 442) it follows that $\Sigma \dfrac{a_n \sin nx - b_n \cos nx}{n}$ is the Fourier's series of $F(x) - \frac{1}{2}a_0 x$, and therefore

$$\frac{a_n}{n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \{F(x) - \tfrac{1}{2}a_0 x\} \sin nx\,dx, \quad -\frac{b_n}{n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \{F(x) - \tfrac{1}{2}a_0 x\} \cos nx\,dx.$$

Since $F(x) - \frac{1}{2}a_0 x$ has almost everywhere a differential coefficient $f(x) - \frac{1}{2}a_0$, where $F(x) = \int_{-\pi}^{x} f(x)\,dx$, we find, by integration by parts,

$$\frac{b_n}{n} = -\frac{1}{\pi}\left[\{F(x) - \tfrac{1}{2}a_0 x\}\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x)dx \sin nx,$$

or
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx;$$

also
$$\frac{a_n}{n} = \frac{1}{\pi}\left[\{F(x) - \tfrac{1}{2}a_0 x\}\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx,$$

or $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$. Hence the series $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ is the Fourier's HL-series corresponding to $f(x)$.

The following theorem has now been established*:

The necessary and sufficient condition that a trigonometrical series $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ *should be the Fourier's HL-series corresponding to a function with only an enumerable set of points of non-summability is that the integrated series* $\frac{1}{2}a_0 x + \sum_{n=1} \dfrac{a_n \sin nx - b_n \cos nx}{n}$ *should, except at an enumerable set of points, converge throughout* $(-\pi, \pi)$ *to a function* $F(x)$ *which is the HL-integral of a function with only an enumerable set of points of non-summability. Also, in case* $\pi, -\pi$ *are points of non-summability, the condition* $F(\pi) - F(-\pi) = \pi a_0$ *must be added.*

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 425.

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365. If the Fourier's series corresponding to a summable function $f(x)$ be summed by the method of arithmetic means (§ 27) we find, since the sum $s_n(x)$ of the first n terms of the Fourier's series is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \frac{1}{\pi}\sum_{r=1}^{r=n-1}\int_{-\pi}^{\pi} f(x')\cos r(x'-x)\,dx',$$

that the Cesàro's partial sum $S_n(x)$ is given by

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2} + \sum_{r=1}^{r=n-1}\frac{n-r}{n}\cos r(x'-x)\right\} f(x')\,dx',$$

from which it is easily found that

$$S_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 f(x')\,dx'.$$

If we apply the Theorem I, of § 279, to the case in which

$$F(x', x, n) = \frac{1}{2n\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2,$$

for all values of x for which $|x'-x| > \mu$, and $F(x', x, n) = 0$ for all values of x for which $|x'-x| \leq \mu$, where x is in the set G which consists of all the points of the interval $(-\pi+\mu, \pi-\mu)$, it can be verified that the conditions (1) and (2) of the Theorem I are satisfied.

For $\frac{1}{2n\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 < \frac{1}{2n\pi}\operatorname{cosec}^2\frac{1}{2}\mu$, for all values of n, and of x (in G) such that $|x'-x| > \mu$; and for the other values of x, $F(x', x, n) = 0$, for all values of n. Therefore the condition (1) is satisfied.

Also $\int_{-\pi}^{x-\mu}\frac{1}{2n\pi}\left\{\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 dx'$, where $x > -\pi+\mu$, does not exceed $\frac{1}{n}\operatorname{cosec}^2\frac{1}{2}\mu$, and therefore converges to zero, as $n \sim \infty$, uniformly for all the values of x in G.

Similarly
$$\int_{x+\mu}^{\pi}\frac{1}{2n\pi}\left\{\frac{\sin\frac{n}{2}(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 dx'$$

converges to zero, as $n \sim \infty$, uniformly for all values of x in the interval $(-\pi+\mu, \pi-\mu)$ of x. It follows that $\int_\alpha^\beta F(x', x, n)\,dx'$ converges uniformly to zero, for all values of x in $(-\pi+\mu, \pi-\mu)$, where (α, β) is any interval in $(-\pi, \pi)$; thus the condition (2) of Theorem I is satisfied.

Hence
$$\frac{1}{2n\pi}\left\{\int_{-\pi}^{x-\mu} + \int_{x+\mu}^{\pi}\right\}\left\{\frac{\sin\frac{n}{2}(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 f(x')\,dx',$$

converges uniformly to zero.

The behaviour of $S(x)$ at any interior point x of the interval $(-\pi, \pi)$, as regards convergence, divergence, or oscillation, accordingly depends only upon the limits of

$$\frac{1}{2n\pi}\int_{x-\mu}^{x+\mu} f(x') \left\{\frac{\sin\frac{n}{2}(x'-x)}{\sin\frac{1}{2}(x'-x)}\right\}^2 dx',$$

as $n \sim \infty$, where μ is so chosen that $x-\mu$, $x+\mu$ are interior points of $(-\pi, \pi)$; and this expression is equivalent to

$$\frac{1}{n\pi}\int_0^{\epsilon}\{f(x+2t)+f(x-2t)\}\left(\frac{\sin nt}{\sin t}\right)^2 dt,$$

where $\epsilon = \frac{1}{2}\mu$.

That the behaviour of $S(x)$ at a point x depends only upon the character of the series in an arbitrarily small neighbourhood of x is a consequence of the corresponding property of $f(x)$. For, as in § 340, $f(x)$ may be expressed as $f_1(x) + f_2(x)$; and since the series for $f_2(x)$ converges to zero at the point x, the corresponding Cesàro sum is also zero at x, and $S(x)$ depends therefore only on the function $f_1(x)$.

Taking the case in which $f(x) = 1$, in the interval $(-\pi, \pi)$, we see that

$$\lim_{n\sim\infty}\frac{1}{n\pi}\int_0^{\epsilon}\left(\frac{\sin nt}{\sin t}\right)^2 dt = \lim_{n\sim\infty}\frac{1}{n\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{\sin nt}{\sin t}\right)^2 dt,$$

and the expression on the right-hand side is

$$\lim_{n\sim\infty}\int_0^{\frac{1}{2}\pi} 2\left[\tfrac{1}{2}+\cos 2t+\cos 4t+\dots+\cos 2nt\right]dt = \tfrac{1}{2}\pi.$$

Therefore
$$\lim_{n\sim\infty}\frac{1}{n\pi}\int_0^{\epsilon}\left(\frac{\sin nt}{\sin t}\right)^2 dt = \tfrac{1}{2},$$
where $0 < \epsilon < \frac{1}{2}\pi$.

At a point at which $f(x+2t)+f(x-2t)$ has a definite limit as $t \sim 0$, the limit of

$$\frac{1}{n\pi}\int_0^{\epsilon}\{f(x+2t)+f(x-2t)\}\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

may be evaluated. For the value of the integral lies between

$$M(\epsilon)\frac{1}{n\pi}\int_0^{\epsilon}\left(\frac{\sin nt}{\sin t}\right)^2 dt, \text{ and } m(\epsilon)\frac{1}{n\pi}\int_0^{\epsilon}\left(\frac{\sin nt}{\sin t}\right)^2 dt,$$

when $M(\epsilon)$, $m(\epsilon)$ are the upper and lower boundaries of $f(x+2t)+f(x-2t)$ in the interval $(0, \epsilon)$. It follows that $\bar{S}(x)$, $\underline{S}(x)$, the upper and lower limits of $S_n(x)$, as $n \sim \infty$, lie in the interval $(\frac{1}{2}m(\epsilon), \frac{1}{2}M(\epsilon))$.

As ϵ is diminished indefinitely, $\frac{1}{2}m(\epsilon)$, $\frac{1}{2}M(\epsilon)$ have one and the same limit $\frac{1}{2}\lim\limits_{\epsilon\sim 0}\{f(x+\epsilon)+f(x-\epsilon)\}$, to which $S_n(x)$ must converge.

In case $f(x+\epsilon)+f(x-\epsilon)$ does not converge to a definite value as $\epsilon \sim 0$, it will have finite or infinite upper and lower limits

$$\overline{f(x+0)+f(x-0)}, \qquad \underline{f(x+0)+f(x-0)},$$

which certainly lie in the interval

$$(\underline{f}(x-0)+\underline{f}(x+0),\ \overline{f}(x-0)+\overline{f}(x+0)).$$

We have accordingly established the following theorem:

If $f(x)$ be a summable function, periodic and of period 2π, the Cesàro partial sum $S_n(x)$, for the Fourier's series corresponding to $f(x)$, converges at any point x at which $f(x)$ is continuous, to the value $f(x)$; at any point at which $f(x)$ has an ordinary discontinuity, to $\frac{1}{2}\{f(x+0)+f(x-0)\}$; and at any point at which $f(x+\epsilon)+f(x-\epsilon)$ has a definite limit, to

$$\tfrac{1}{2}\lim_{\epsilon\sim 0}\{f(x+\epsilon)+f(x-\epsilon)\}.$$

Moreover, at any point x, the upper and lower limits of $S_n(x)$ both lie in the interval bounded by the finite or infinite upper and lower limits of

$$\tfrac{1}{2}\{f(x+\epsilon)+f(x-\epsilon)\},\ as\ \epsilon\sim 0.$$

This theorem, so far as it applies to points of continuity, or points of ordinary discontinuity, of $f(x)$, was first established for the case in which $f(x)$ is integrable (R) in the interval $(-\pi,\pi)$, by* Fejér.

366. It has been shewn in § 27 that, if $\bar{f}(x)$, $\underline{f}(x)$ denote the upper and lower sums of the Fourier's series at the point x,

$$\underline{f}(x) \leq \underline{S}(x) \leq \bar{S}(x) \leq \bar{f}(x).$$

It follows that $s_n(x)$ cannot diverge unless $S_n(x)$ diverges, and thus that, when $S_n(x)$ converges, $f_n(x)$ must oscillate and cannot be divergent, unless it also converges. We have accordingly the following properties of Fourier's series:

At a point of continuity of $f(x)$, or at a point of ordinary discontinuity at which $f(x+0)$, $f(x-0)$ are finite, the sum of the Fourier's series is either $f(x)$ or $\frac{1}{2}\{f(x+0)+f(x-0)\}$, or else it oscillates between finite or infinite limits so that $f(x)$ or $\frac{1}{2}\{f(x+0)+f(x-0)\}$ lies in the interval bounded by these limits, but it cannot diverge. It can only diverge if the Cesàro sum $S(x)$ is $+\infty$, or $-\infty$.

Next, let (a, b) be an interval which is contained in another interval $(a-\delta, b+\delta)$ in which $f(x)$ is bounded. The limits of $S_n(x)$, as $n\sim\infty$, for points in (a, b), are given as the limits of

$$\frac{1}{n\pi}\int_0^{\epsilon}\{f(x+2t)+f(x-2t)\}\left(\frac{\sin nt}{\sin t}\right)^2 dt,$$

* *Math. Annalen*, vol. LVIII (1904), p. 51; also *Comptes Rendus*, vol. CXXXI (1900), p. 984, and vol. CXXXIV (1902), p. 762.

where ϵ may be taken to be so small that the points $a - 2\epsilon$, $b + 2\epsilon$ are in the interval $(a - \delta, b + \delta)$ for which $f(x)$ is bounded. The value of the above expression, for all values of x in (a, b), does not exceed, numerically, $\frac{2A}{n\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{\sin nt}{\sin t}\right)^2 dt$, or A, where A is the upper boundary of $|f(x)|$ in $(a - \delta, b + \delta)$.

The remaining part of $S_n(x)$ has been shewn to converge to zero, as $n \sim \infty$, uniformly for all values of x in (a, b), an interval interior to $(-\pi, \pi)$.

We have accordingly the following theorem:

In any interval (a, b), contained in an interval $(a - \delta, b + \delta)$ in which $f(x)$ is bounded, $|S_n(x)|$ is bounded for all values of n, and of x, in (a, b). If also all the points of (a, b) are points of continuity, or of ordinary discontinuity, $S_n(x)$ converges boundedly to $f(x)$ or $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$.

367. In case $f(x)$ is continuous in (a, b), an interval interior to $(-\pi, \pi)$, the continuity at a and b being on both sides, ϵ may be so determined that

$$|f(x + 2t) - f(x)|, \quad |f(x - 2t) - f(x)|$$

are both $< \eta$, for all values of x in (a, b) and for $t \leqq \epsilon$. We have then

$$S_n(x) - f(x) = \frac{1}{n\pi}\int_0^{\epsilon}\{f(x + 2t) + f(x - 2t) - 2f(x)\}\left(\frac{\sin nt}{\sin t}\right)^2 dt + \theta_n,$$

where θ_n is a number which converges uniformly to zero, for all values of x in (a, b). From this equation we deduce that

$$|S_n(x) - f(x)| < 2\eta\int_0^{\frac{1}{2}\pi}\frac{1}{n\pi}\left(\frac{\sin nt}{\sin t}\right)^2 dt + |\theta_n| < \eta + |\theta_n| < 2\eta,$$

for all values of x in (a, b), provided n is not less than some fixed value n_η. It follows that $S_n(x)$ converges uniformly to $f(x)$ in the interval (a, b).

The condition that (a, b) is interior to $(-\pi, \pi)$ may be removed by considering overlapping intervals, each of which by proper choice of the origin may be made interior to $(-\pi, \pi)$.

It has thus been established that:

In any interval (a, b), in which $f(x)$ is continuous, the continuity at a and b being on both sides, the Cesàro sum $S_n(x)$ converges uniformly to the value of the function.

This theorem illustrates the greater precision of the knowledge we have of the convergence of the Cesàro sum of the Fourier's series, as compared with the ordinary sum. For, comparing this theorem with the corresponding one for the ordinary sum, given in § 341, we observe that in the latter case the assumption that $f(x)$ is of bounded variation in an interval containing (a, b) is made, whereas it is not necessary for the validity of the above theorem.

368. A more general theorem than that of § 365 has been obtained by Lebesgue. He has in fact shewn that the Cesàro sum converges to $f(x)$ at any point x for which $\int_0^t |f(x+t)+f(x-t)-2f(x)|\,dt$ has a differential coefficient equal to zero, at the point $t=0$. This condition will be satisfied at any point x for which both the functions

$$\int_0^t |f(x+t)-f(x)|\,dt, \quad \int_0^t |f(x-t)-f(x)|\,dt$$

have differential coefficients of value zero, for $t=0$. It has been shewn in I, § 432, that these conditions are satisfied almost everywhere in the interval $(-\pi, \pi)$, of x. At a point of continuity of $f(x)$, these conditions are satisfied, and the complete condition is satisfied at any point at which $\lim\limits_{t\sim 0}[f(x+t)+f(x-t)]$ has a definite value, provided $f(x)$ is taken to have as its value half this limit. Thus Lebesgue's theorem includes the theorem of § 365.

In order to prove the theorem we have to shew that, when the condition that $\int_0^t |\phi(t)|\,dt$ has a differential coefficient of value zero, at the point $t=0$, then $\frac{1}{2n\pi}\int_0^\mu \phi(t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt = o(1)$, where $\phi(t)$ denotes $f(x+t)+f(x-t)-2f(x)$, and μ is such that $0<\mu\leqq\frac{1}{2}\pi$. Taking $\frac{1}{t}\int_0^t \phi(t)\,dt = \chi(t)$; at a point x, at which the condition is satisfied, $\chi(0)=0$, and $\chi(t)$ is continuous in the interval $(0, \mu)$.

Since

$$\frac{1}{2n\pi}\int_0^\mu \phi(t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt = \left[\frac{\mu\chi(\mu)}{2n\pi}\left(\frac{\sin\frac{1}{2}n\mu}{\sin\frac{1}{2}\mu}\right)^2\right] - \frac{1}{2n\pi}\int_0^\mu t\chi(t)\frac{d}{dt}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt,$$

it is seen that it is only necessary to shew that

$$\frac{1}{2n\pi}\int_0^\mu t\chi(t)\frac{d}{dt}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt = o(1).$$

We have $$\frac{d}{dt}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 = \frac{n\sin nt}{2\sin^2\frac{1}{2}t} - \frac{\sin^2\frac{1}{2}nt}{\sin^3\frac{1}{2}t}\cos\tfrac{1}{2}t,$$

and it will be sufficient to shew that

$$\frac{1}{n}\int_0^\mu t\chi(t)\frac{\sin^2\frac{1}{2}nt}{\sin^3\frac{1}{2}t}\cos\tfrac{1}{2}t\,dt = o(1),$$

and $$\int_0^\mu t\chi(t)\frac{\sin nt}{\sin^2\frac{1}{2}t}\,dt = o(1).$$

In order to prove the first of these results it is sufficient to shew that $\frac{t}{n}\frac{\sin^2\frac{1}{2}nt\cos\frac{1}{2}t}{\sin^3\frac{1}{2}t}$ satisfies the conditions (1) and (2) of Theorem II of § 290, and also the conditions (a) of § 292, and (b') of § 293. Since the function is numerically $<\frac{\pi}{n}\operatorname{cosec}^2\frac{1}{2}t$, in any interval interior to $(0, \pi)$, it converges

uniformly to zero, as $n \sim \infty$; therefore the conditions (1), (2) are both satisfied. In the present case conditions (b') and (a) may be taken together.

To prove that these conditions are satisfied, we have

$$\left|\int_{\lambda_1}^{\lambda_2} \frac{t}{n} \frac{\sin^2 \frac{1}{2}nt \cos \frac{1}{2}t}{\sin^3 \frac{1}{2}t} dt\right| < \frac{\pi}{n}\int_{\lambda_1}^{\lambda_2}\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt < \frac{\pi}{n}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt = \pi^2;$$

thus the conditions (a), (b') are satisfied. The first part of the result has accordingly been established.

To consider the integral $\int_0^t t\chi(t) \frac{\sin nt}{\sin^2 \frac{1}{2}t} dt$, we apply the method of § 296. The function $\frac{t \sin nt}{\sin^2 \frac{1}{2}t}$ is bounded for all values of n in any interval (α, β) which does not include the point $t = 0$, and $\int_\alpha^\beta \frac{t \sin nt}{\sin^2 \frac{1}{2}t} dt$, where $0 < \alpha < \beta \leqq \pi$, converges to zero, as $n \sim \infty$, since $t \operatorname{cosec}^2 \frac{1}{2}t$ is summable in (α, β).

Thus the conditions (1) and (2) of the theorem of § 290 are satisfied. To apply the condition (3), of § 296, we have to write for $tF_1(nt)$, the expression $t \sin nt \operatorname{cosec}^2 \frac{1}{2}t$.

The maximum $M(\alpha_n)$ of $t \sin nt \operatorname{cosec}^2 \frac{1}{2}t$ in $(0, \alpha_n)$ is less than π times the maximum of $\sin nt \operatorname{cosec} \frac{1}{2}t$, and this is less than $\pi^2 n$. If we choose $\alpha_n = \frac{\pi}{n}$, we have $\alpha_n M(\alpha_n) < \pi^3$. Again, $N(\alpha_n)$ denotes the absolute maximum of $\int t^2 \sin nt \operatorname{cosec}^2 \frac{1}{2}t\, dt$, for all intervals contained in (α_n, μ); and since $\int_\alpha^\beta t^2 \sin nt \operatorname{cosec}^2 \frac{1}{2}t\, dt = (\beta \operatorname{cosec} \frac{1}{2}\beta)^2 \int_{\alpha'}^\beta \sin nt\, dt$, where α' is in the interval (α, β), we see that $N(\alpha_n) < \frac{2\pi^2}{n}$; and thus $\frac{N(\alpha_n)}{\alpha_n} < 2\pi$. The two conditions of the theorem of § 296 being satisfied, it follows that

$$\int_0^t t\chi(t) \frac{\sin nt}{\sin^2 \frac{1}{2}t} dt = o(1).$$

It has now been proved that:

The Fourier's series corresponding to $f(x)$ is summable by Cesàro's means, at almost all points of the interval $(-\pi, \pi)$; the Cesàro sum being $f(x)$. These points include all points at which $\int_0^t |f(x+t) + f(x-t) - 2f(x)|\, dt$ has the differential coefficient zero, for $t = 0$.

369. The theory of the Cesàro sums may be applied to throw light upon the convergence of an important class of Fourier's series. If the coefficients satisfy the conditions $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$, in which case

$$a_n \cos nx + b_n \sin nx = O\left(\frac{1}{n}\right),$$

we have a class of Fourier's series which includes the series corresponding to functions of bounded variation. By Hardy's theorem (§ 54), at any point at which the series is summable $(C, 1)$ it is convergent; therefore the Fourier's series converges almost everywhere, and in particular, at every point of continuity or of ordinary discontinuity of the series. Although the convergence of the Cesàro partial sums is uniform in any interval of continuity of the function, provided the end-points are points of continuity on both sides, it does not follow that the convergence of the Fourier's series itself is uniform in the interval. It has thus been shewn that:

A Fourier's series for which $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$ is almost everywhere convergent; in particular, it converges at every point of continuity, or of ordinary discontinuity of the function.

The more general theorem has been established* by Hardy and Littlewood that:

The necessary and sufficient condition that a Fourier's series for which $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$ should converge at a point x is that

$$\frac{1}{t}\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\,dt = o(1).$$

The theorem will be proved in § 414.

370. It will be shewn that:

The Fourier's series for $f(x)$ is summable $(C, 2)$ at every point x at which $\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\,dt$ has, for $t = 0$, a differential coefficient of value zero.

This theorem is due† to Lebesgue. The set of points at which the condition is satisfied includes those for which $\int_{-\pi}^{x} f(x)\,dx$ has a differential coefficient equal to $f(x)$, and the set also contains that set of points at which $\int_0^t |f(x+t) + f(x-t) - 2f(x)|\,dt$ has a differential coefficient at the point $t = 0$, of value zero. To prove the theorem, we take for $S_n(x)$, the Cesàro partial sum, of order 1, the expression

$$S_n(x) - f(x) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} \{f(x+2t) + f(x-2t) - 2f(x)\}\left(\frac{\sin nt}{\sin t}\right)^2 dt;$$

writing $\quad u(t) = f(x+2t) + f(x-2t) - 2f(x), \quad U(t) = \int_0^t u(t)\,dt,$

* *Proc. Lond. Math. Soc.* (2), vol. XVIII (1917), p. 229.

† *Math. Annalen*, vol. LXI (1905), p. 278.

and integrating by parts, we have

$$S_n(x) - f(x) = \frac{1}{n\pi} U\left(\frac{\pi}{2}\right)\sin^2\frac{n\pi}{2} + \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} U(t)\cot t\left(\frac{\sin nt}{\sin t}\right)^2 dt$$
$$- \frac{1}{\pi}\int_0^{\frac{\pi}{2}} U(t)\operatorname{cosec} t \,.\, \frac{\sin 2nt}{\sin t}\,dt.$$

We now form the arithmetic mean $\frac{1}{n}\{S_1(x) + S_2(x) + \ldots + S_n(x)\} - f(x)$ for the expression $S_n(x) - f(x)$, and consider separately the parts of this mean corresponding to the three terms on the right-hand side. The first term converges to zero, as $n \sim \infty$, and therefore also its arithmetic mean converges to zero, as $n \sim \infty$. At a point at which $U(t)/t$ converges to zero, the second term converges to zero, as $n \sim \infty$, by the theorem of § 365, and therefore its arithmetic mean does so also. The integral in the third term can be expressed by

$$\int_0^{\frac{\pi}{2}} \chi(t)\left[\frac{\sin(2n+1)t + \sin(2n-1)t}{\sin t}\right] dt,$$

where $\chi(t) = U(t)/\sin 2t$, and for the point $t = 0$, $\chi(t)$ is continuous and $\chi(+0) = 0$. Applying the theorem of § 365, the arithmetic means of the expressions

$$\int_0^{\frac{\pi}{2}} \chi(t)\frac{\sin(2n+1)t}{\sin t}\,dt, \quad \int_0^{\frac{\pi}{2}} \chi(t)\frac{\sin(2n-1)t}{\sin t}\,dt$$

both converge to zero, as $n \sim \infty$. It has now been shewn that the sum $(H, 2)$, and consequently the sum $(C, 2)$, of the series exists, and has the value $f(x)$, at a point at which the condition stated in the theorem is satisfied.

THE PROPERTIES OF A CERTAIN CLASS OF FUNCTIONS

371. For the investigation of the Cesàro sum of order k, not equal to 1, of a Fourier's series, it is convenient to employ certain functions, of which the properties have been investigated* by W. H. Young. Only those properties of the functions which are absolutely necessary for the purpose will be given here.

The function defined for all finite values of t by

$$\frac{t^p}{\Gamma(p+1)}\left\{1 - \frac{t^2}{(p+1)(p+2)} + \frac{t^4}{(p+1)(p+2)(p+3)(p+4)} - \ldots\right\},$$

where $p \geqq 0$, will be denoted by $C_p(t)$. It will be seen that $C_0(t) = \cos t$, $C_1(t) = \sin t$, $C_2(t) = 1 - \cos t$, $\frac{dC_p(t)}{dt} = C_{p-1}(t)$. Writing tu for t, and multiplying by $(1-t)^{q-1}$, where $q > 0$, since $(1-t)^{q-1}$ is summable in the

* *Quarterly Journal*, vol. XLIII (1912), p. 161.

interval (0, 1), and the series which represents $C_0(tu)$ converges uniformly, we may apply term by term integration, and thus obtain the formula $C_{p+q}(u) = \frac{u^q}{\Gamma(q)}\int_0^1 C_p(tu)(1-t)^{q-1}dt$, $(0<q)$. Giving p the values 0, 1, we obtain the formula

$$C_q(u) = \frac{u^q}{\Gamma(q)}\int_0^1 (1-t)^{q-1}\cos tu\,dt,\ q>0,$$

$$C_q(u) = \frac{u^{q-1}}{\Gamma(q-1)}\int_0^1 (1-t)^{q-2}\sin tu\,dt,\ q>1,$$

$$C_q(u) = \frac{u^{q-2}}{\Gamma(q-2)}\int_0^1 (1-t)^{q-3}(1-\cos tu)\,dt,\ q>2.$$

It will be shewn that the function $C_q(u)$ is bounded for all values of q such that $0 \leq q \leq 2$, and for all values of $u \geqq 0$.

We have $\Gamma(q)\,C_q(u) = u^q\int_0^1 t^{q-1}\cos(1-t)\,u\,dt$, where $q>0$; thus

$$\Gamma(q)\,C_q(u) = u^q\int_0^1 t^{q-1}\cos u\cos tu\,dt + u^q\int_0^1 t^{q-1}\sin u\sin tu\,dt$$

$$= \cos u\int_0^u t^{q-1}\cos t\,dt + \sin u\int_0^u t^{q-1}\sin t\,dt$$

$$= \cos u\left\{\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^u\right\} t^{q-1}\cos t\,dt + \sin u\left\{\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^u\right\} t^{q-1}\sin t\,dt.$$

Also $\int_{\frac{1}{2}\pi}^u t^{q-1}\begin{matrix}\cos\\ \sin\end{matrix}\,t\,dt$, when $q<1$, is equal to $(\frac{1}{2}\pi)^{q-1}\int_{\frac{1}{2}\pi}^v \begin{matrix}\cos\\ \sin\end{matrix}\,t\,dt$, which is numerically less than $2(\frac{1}{2}\pi)^{q-1}$. The integrals $\int_0^{\frac{\pi}{2}} t^{q-1}\begin{matrix}\cos\\ \sin\end{matrix}\,t\,dt$ are both numerically less than $\frac{1}{q}\left(\frac{\pi}{2}\right)^q$. We have thus

$$|\Gamma(q)\,C_q(u)| < \frac{2}{q}\left(\frac{\pi}{2}\right)^q + 4\left(\frac{\pi}{2}\right)^{q-1},$$

when $0<q<1$. As q increases from 0 to 1, $\Gamma(q)$ decreases down to a minimum M which lies between 1 and 2, hence $\frac{1}{\Gamma(q)} < \frac{1}{M}$, $\frac{1}{\Gamma(q+1)} < \frac{1}{M}$, $0<q<1$, and thus $|C_q(u)| < \frac{1}{M}\left(\frac{\pi}{2}\right)^{q-1}(\pi+4) < \frac{2}{M\pi}(\pi+4)$; since $C_0(u)$ is also bounded for all values of u, it has now been shewn that $|C_q(u)|$ is bounded for $0 \leq q < 1$, $u \geqq 0$. In case $2 \geqq q \geqq 1$, by employing the formula $\Gamma(q-1)\,C_q(u) = u^{q-1}\int_0^1 t^{q-1}\cos(1-t)\,u\,dt$ in a similar manner, it can be shewn that $|C_q(u)|$ is bounded for all such values of q, and for $u \geqq 0$.

If $q > 2$, we have

$$C_q(u) = \frac{u^{q-2}}{\Gamma(q-2)} \int_0^1 (1-t)^{q-3}(1-\cos tu)\,dt$$
$$= \frac{u^{q-2}}{\Gamma(q-1)} - \frac{u^{q-2}}{\Gamma(q-2)} \int_0^1 (1-t)^{q-3} \cos tu\,dt,$$

and therefore $$\left|\frac{C_q(u)}{u^{q-2}}\right| < \frac{2}{\Gamma(q-1)} < \frac{2}{M};$$

hence $|u^{-q} C_q(u)| < \dfrac{A}{u^2}$, where A is a fixed number.

When $0 \leqq q \leqq 2$, it has been shewn that $|u^{-q} C_q(u)| < \dfrac{A}{u^q}$.

Since $$\frac{d}{du}\{u^{-q} C_q(u)\} = -qu^{-q-1} C_q(u) + u^{-q} C_{q-1}(u),$$

we have $$\left|\frac{d}{du}\{u^{-q} C_q(u)\}\right| < \frac{P}{u^{q+1}} + \frac{Q}{u^q}, \text{ for } 1 < q \leqq 2,$$

and $$\left|\frac{d}{du}\{u^{-q} C_q(u)\}\right| < \frac{P}{u^3} + \frac{Q}{u^q}, \quad \text{for } 2 < q < 3,$$

and $$\left|\frac{d}{du}\{u^{-q} C_q(u)\}\right| < \frac{P}{u^3}, \qquad \text{for } q > 3,$$

where P and Q are fixed numbers independent of u.

Therefore, when $q > 1$, $\left|\dfrac{d}{du}\{u^{-q} C_q(u)\}\right|$ is, for all values of u greater than 1, less than a fixed multiple of u^{-2}, or of u^{-q}, according as $q > 2$, or $q \leqq 2$. Since the variation of $C_q(u) \,.\, u^{-q}$ in the infinite interval $(0, \infty)$ is given by

$$\int_0^\infty \left|\frac{d}{du}\{C_q(u)\,u^{-q}\}\right| du,$$

which integral exists, as the integrand is less than a fixed multiple of u^{-2}, or of u^{-q}, it is seen that $C_q(u)\,u^{-q}$ is, for $q > 1$, of bounded variation.

If $q > 1$, we have

$$\int_0^\infty \frac{1}{t^q} C_q(t) \cos xt\,dt = \frac{1}{\Gamma(q-1)} \int_0^\infty dt \frac{\cos xt}{t} \int_0^1 \sin tu\,(1-u)^{q-2}\,du,$$

and the order of the successive integrations may, in accordance with the theorem of § 241 (2″), be changed, since $(1-u)^{q-2}$ is summable in $(0, 1)$, and $\dfrac{\sin tu \cos x}{t}$ is a bounded function of (t, u) in a rectangle $(0, 0; A, 1)$, whose integral over $(0, A)$ with respect to t converges boundedly to $\displaystyle\int_0^\infty \frac{\sin tu \cos xt}{t}\,dt$.

Therefore

$$\int_0^\infty \frac{1}{t^q} C_q(t) \cos xt\,dt = \frac{1}{\Gamma(q-1)} \int_0^1 (1-u)^{q-2}\,du \int_0^\infty \frac{\sin tu \cos xt}{t}\,dt$$
$$= \frac{1}{\Gamma(q-1)} \cdot \tfrac{1}{2}\pi \int_x^1 (1-u)^{q-2}\,du, \text{ if } x \leqq 1, \text{ and } = 0, \text{ if } x \geqq 1.$$

Therefore

$$\int_0^\infty t^{-q} C_q(t) \cos xt\, dt = \frac{\pi}{2\Gamma(q)} (1-x)^{q-1}, \text{ if } x \leq 1, \text{ and } = 0, \text{ if } x \geqq 1,$$

provided $q > 1$.

THE SUMMABILITY (C, k) OF FOURIER'S SERIES

372. It was shewn by M. Riesz*, Chapman†, and W. H. Young‡ that a Fourier's series is summable (C, k), where $k > 0$, at any point of continuity or of ordinary discontinuity of the function. The theorem was extended by Hardy§, who shewed that the summability holds good almost everywhere in the interval $(-\pi, \pi)$, of x.

Since $$f(x) \sim \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$
we have

$$\tfrac{1}{2}\{f(x+t) + f(x-t)\} \sim \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \cos mt.$$

By the theorem of § 384, since $(\omega t)^{-(1+k)} C_{1+k}(\omega t)$ is of bounded variation in the whole interval $(0, \infty)$ (see § 371), and since it is absolutely summable in $(0, \infty)$, we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty & (\omega t)^{-(1+k)} C_{1+k}(\omega t)\,[f(x+t) + f(x-t)]\,dt \\ &= \tfrac{1}{2}a_0 \int_0^\infty (\omega t)^{-(1+k)} C_{1+k}(\omega t)\,dt \\ &\quad + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \int_0^\infty (\omega t)^{-(1+k)} C_{1+k}(\omega t) \cos mt\,dt.\end{aligned}$$

Changing ωt into t, and using the theorem

$$\int_0^\infty t^{-(1+k)} C_{1+k}(t) \cos \frac{m}{\omega} t\,dt = \frac{\pi}{2\Gamma(k+1)} \left(1 - \frac{m}{\omega}\right)^k,$$

where $m < \omega$, we have

$$\begin{aligned}\tfrac{1}{2}a_0 &+ \sum_{m<\omega} \left(1 - \frac{m}{\omega}\right)^k (a_m \cos mx + b_m \sin mx) \\ &= \frac{\Gamma(k+1)}{\pi} \int_0^\infty t^{-(1+k)} C_{1+k}(t) \left[f\left(x + \frac{t}{\omega}\right) + f\left(x - \frac{t}{\omega}\right)\right] dt.\end{aligned}$$

where k is any positive number. The expression on the left-hand side is the partial Riesz's sum of the Fourier's series, of order k (see § 45).

* *Comptes Rendus*, vol. CXLIX (1909), p. 909.

† *Proc. Lond. Math. Soc.* (2), vol. IX (1911), p. 390; also *Quarterly Journal*, vol. XLIII (1911), p. 26.

‡ *Leipziger Ber.* vol. LXIII (1911), p. 377.

§ *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 365.

Let $\phi(t)$ denote $f(x+t)+f(x-t)-2f(x)$, then

$$\frac{\Gamma(1+k)}{\pi}\int_0^\infty t^{-(1+k)}C_{1+k}(t)\,\phi\left(\frac{t}{\omega}\right)dt$$

$$=\tfrac{1}{2}a_0+\sum_{m<\omega}\left(1-\frac{m}{\omega}\right)^k(a_m\cos mx+b_m\sin mx)-f(x).$$

It will be proved that

$$\lim_{\omega\sim\infty}\int_0^\infty t^{-(1+k)}C_{1+k}(t)\,\phi\left(\frac{t}{\omega}\right)dt=0$$

for any point x at which $\int_0^t|\phi(t)|\,dt$ has a differential coefficient of value zero, for $t=0$. This is the case for almost all values of x, and in particular at a point of ordinary discontinuity, provided $f(x)=\frac{1}{2}\{f(x+0)+f(x-0)\}$. Thus the theorem is established when the limit of the above integral has been shewn to be zero. It will be assumed that $0<k<1$, because the summability when $k>1$ follows from the summability $(C, 1)$ which has already been established.

The interval $(0, \infty)$ may be divided into three parts $(0, 1)$, $(1, \omega)$, and (ω, ∞). Considering these separately, we have

$$\left|\int_0^1 t^{-(1+k)}C_{1+k}(t)\,\phi\left(\frac{t}{\omega}\right)dt\right|<A\int_0^1\left|\phi\left(\frac{t}{\omega}\right)\right|dt,$$

since $t^{-(1+k)}C_{1+k}(t)$ is bounded for $t\leqq 1$. At a point at which

$$\lim_{t\sim 0}\frac{1}{t}\int_0^t|\phi(t)|\,dt=0,$$

we have $\int_0^t|\phi(t)|\,dt<\epsilon t$ provided t is sufficiently small, thus

$$\int_0^1\left|\phi\left(\frac{t}{\omega}\right)\right|dt=\omega\int_0^{\frac{1}{\omega}}|\phi(t)|\,dt<\epsilon,$$

if ω be sufficiently large, or

$$\left|\int_0^1 t^{-(1+k)}C_{1+k}(t)\,\phi\left(\frac{t}{\omega}\right)dt\right|<A\epsilon,$$

where ϵ is an arbitrarily chosen positive number, provided $\omega>\omega_\epsilon$.

Again $\int_1^\omega t^{-(1+k)}C_{1+k}(t)\,\phi\left(\frac{t}{\omega}\right)dt$, since $C_{1+k}(t)$ is bounded for $t\geqq 1$, is numerically less than

$$B\int_1^\omega t^{-(1+k)}\left|\phi\left(\frac{t}{\omega}\right)\right|dt,\text{ or than }B\omega^{-k}\int_{\omega^{-1}}^1 t^{-(1+k)}\,|\phi(t)|\,dt;$$

and this is equal to

$$B\omega^{-k}\left\{\left[t^{-(1+k)}\int_0^t|\phi(t)|\,dt\right]_{\omega^{-1}}^1+(k+1)\int_{\omega^{-1}}^1 t^{-(k+2)}\left\{\int_0^t|\phi(t)|\,dt\right\}dt\right\}$$

or to $$B\omega^{-k}\left\{\Phi(1)-\omega^{k+1}\Phi\left(\frac{1}{\omega}\right)+(k+1)\int_{\omega^{-1}}^1 t^{-(k+2)}\Phi(t)\,dt\right\},$$

where $\Phi(t) = \int_0^t |\phi(t)|\,dt$. Since $\Phi(t) < \epsilon t$, for $0 \leqq t \leqq \eta$; and dividing the last integral into parts taken over (ω^{-1}, η), $(\eta, 1)$, we see that this expression is numerically less than an arbitrarily chosen positive number ζ, provided ω is sufficiently large.

Lastly, $\int_\omega^\infty t^{-(1+k)} C_{1+k}(t)\, \phi\left(\frac{t}{\omega}\right) dt$, or $\omega^{-k} \int_1^\infty t^{-(1+k)} C_{1+k}(\omega t)\, \phi(t)\, dt$, is numerically less than a fixed multiple of $\omega^{-k} \int_1^\infty t^{-(1+k)} |\phi(t)|\, dt$, or than $\omega^{-k} \int_{-\pi}^{\pi} |\phi(t)|\, dt \cdot \left(\frac{1}{1^{1+k}} + \frac{1}{2^{1+k}} + \dots\right)$ which is less than a fixed multiple of ω^{-k}, and thus converges to 0, as $\omega \sim \infty$. The theorem has now been established that:

A Fourier's series is summable (C, k), where $k > 0$, almost everywhere in the interval $(-\pi, \pi)$, the sum (C, k) being $f(x)$. At any point of continuity of the function, the sum (C, k) is $f(x)$, and at any point of ordinary discontinuity it is $\frac{1}{2}\{f(x+0) + f(x-0)\}$; at any point at which $f(x+t) + f(x-t)$ has a definite limit, as $t \sim 0$, the sum (C, k) is $\frac{1}{2} \lim_{t \sim 0} \{f(x+t) + f(x-t)\}$.

If $F(x)$ denote an indefinite integral of $f(x)$, the theorem, when combined with that of § 370, may be stated as follows:

A Fourier's series is summable $(C, 2)$ at any point x at which

$$\lim_{t \sim 0} \frac{F(x+t) - F(x-t)}{2t}$$

has a unique value, and the set L of all such points contains a set, of measure 2π, at each point of which the series converges (C, k), where $k > 0$.

373. It is easily seen that, in any interval (a, b), in which $f(x)$ is continuous, the continuity at a and b being on both sides, the sum (C, k) is continuous. For a number δ can be so determined that $|\phi(t)| < \eta$, for all values of t such that $|t| < \delta$, and for all values of x in (a, b). The integral $\int_0^1 t^{-(1+k)} C_{1+k}(t)\, \phi\left(\frac{t}{\omega}\right) dt$ is less than $A\eta$, for $\omega^{-1} < \delta$. The integral $\int_1^\omega t^{-(1+k)} C_{1+k}(t)\, \phi\left(\frac{t}{\omega}\right) dt$ converges uniformly in (a, b) to zero, as $\omega \sim \infty$, because $\frac{1}{t} \int_0^t |\phi(t)|\, dt$ converges to zero, uniformly for all values of x in (a, b). Also $\int_\omega^\infty t^{-(1+k)} C_{1+k}(t)\, \phi\left(\frac{t}{\omega}\right) dt$ converges uniformly to 0, as $\omega \sim \infty$. Thus it has been shewn that:

In any interval (a, b) in which $f(x)$ is continuous, the continuity at a and b being on both sides, the partial sum (C, k), where $k > 0$, converges uniformly to $f(x)$ in (a, b).

374. The theorem that a Fourier's series is summable (C, k), for $k > 0$, almost everywhere, does not provide necessary and sufficient conditions that the series should be summable (C, k) at a particular point, and no such conditions, of a simple character, are known. The question has been considered by Hardy and Littlewood, of the conditions under which, at a particular point, the series is summable (C, k), for some value or another of k. They have obtained* the following theorem:

The necessary and sufficient condition that the Fourier's series corresponding to $f(x)$ should be summable (C, r), for some value or another, of r, at the point x, is that there should be an integer k such that, if

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

and
$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(t_1)\,dt_1, \quad \phi_2(t) = \frac{1}{t}\int_0^t \phi_1(t_1)\,dt, \ldots,$$

then
$$\lim_{t \sim 0} \phi_k(t) = 0.$$

The function $f(x)$ may be either summable, or may satisfy a certain more general condition of integrability.

From this theorem they have deduced that:

If $f(x)$ is bounded in a neighbourhood of the point ξ, the Fourier's series corresponding to $f(x)$ is, at the point ξ, either summable (C, k) for every positive value of k, or summable for no value of k.

375. At a point x at which the condition

$$\int_0^t |f(x+t) + f(x-t) - 2f(x)|\,dt = o(t)$$

is satisfied, the condition

$$\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\,dt = o(t)$$

is also satisfied, but the converse of this does not hold.

It has been shewn† in § 370 that, at any point at which the second condition is satisfied, the series is summable $(C, 2)$, and it was shewn‡ by W. H. Young that the series is summable (C, k), where $k > 1$.

An example has been given§ by Hahn of a function which at a particular point x satisfies the condition $\int_0^t \{f(x+t) + f(x-t) - 2f(x)\}\,dt = o(t)$, but at which the series is not summable $(C, 1)$; at this point the condition $\int_0^t |f(x+t) + f(x-t) - 2f(x)|\,dt = o(t)$ is of course not satisfied. Accordingly the first condition, although necessary, is not sufficient, for the

* *Math. Zeitschr.* vol. XIX (1924), p. 70.

† *Math. Annalen*, vol. LXI (1905), p. 274.

‡ *Proc. Lond. Math. Soc.* (2), vol. X (1912), p. 268.

§ *Deutsch. Math. Vereinig.* vol. XXV (1916), p. 359.

convergence $(C, 1)$ of the Fourier's series. The series is then however summable $(C, 2)$.

It has been shewn* by W. H. Young that the series obtained by term by term differentiation of the Fourier's series corresponding to a function of bounded variation is summable (C, k), for $k > 0$, almost everywhere, the sum (C, k) being equal to the differential coefficient of the function. For example $\sum_{n=1} \cos nx$ is a series of this kind, since $\sum_{n=1} \frac{\sin nx}{n}$ is the Fourier's series of a function of bounded variation; but $\sum_{n=1} \sin nx$ is not such a series, since $\sum_{n=1} \frac{\cos nx}{n}$ is not the Fourier's series of a function of bounded variation.

THE CESÀRO SUMMATION OF A FOURIER-DENJOY SERIES

376. Let $f(x)$ have a D-integral in the interval $(-\pi, \pi)$, then the nth Cesàro sum, of order 1, of the Fourier's D-series, corresponding to $f(x)$, is

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x')\left[\frac{\sin \frac{1}{2}n(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx'.$$

Let (a, b) be an interval interior to an interval contiguous to the set H, of points of non-summability of the function $f(x)$; and let

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x) = f(x)$ at all points of the interval (a, b), and let $f_1(x) = 0$ at all other points of $(-\pi, \pi)$. The function $f_1(x)$ is summable in the interval $(-\pi, \pi)$, and consequently (see § 368),

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f_1(x')\left[\frac{\sin \frac{1}{2}n(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx'$$

converges to $f(x)$ almost everywhere in (a, b), and in particular at any point at which $\{f(x+t) + f(x-t)\}$ converges to $2f(x)$ as $t \sim 0$. Also the convergence to $f(x)$ is uniform in any interval contained in (a, b) in which the function is continuous, the continuity at the end-points being assumed to be on both sides.

We have to examine the convergence of

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f_2(x')\left[\frac{\sin \frac{n}{2}(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx'$$

in the interval $(a+\mu, b-\mu)$, where μ is an arbitrarily chosen positive number $\left(< \frac{b-a}{2}\right)$. The function $f_2(x')$ has the value zero within the

* *Proc. Lond. Math. Soc.* (2), vol. XIII (1913), p. 23.

interval (a, b), and elsewhere it has the values of $f(x')$. The expression to be examined is equivalent to

$$\frac{1}{2n\pi}\int_{-\pi}^{a} f(x')\left[\frac{\sin \frac{1}{2}n(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx' + \frac{1}{2n\pi}\int_{b}^{\pi} f(x')\left[\frac{\sin \frac{1}{2}n(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx'.$$

Let us consider $\displaystyle\int_{-\pi}^{a} f(x')\,\Phi(x'-x, n)\,dx'$,

where $$\Phi(x'-x, n) \equiv \frac{1}{2n\pi}\left[\frac{\sin \frac{1}{2}n(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2.$$

We shall suppose x to be confined to the interval $(a+\mu, b-\mu)$; so that, in the integrand, $x - x' \geqq \mu$, $x - x' \leqq b - \mu + \pi < 2\pi - \mu$, and thus $|\sin \frac{1}{2}(x'-x)| \geqq \sin \frac{1}{2}\mu$, for all the values of x and x' concerned. Applying the theorem of § 287, we see that $\lim\limits_{n\sim\infty}\displaystyle\int_{-\pi}^{a} f(x')\,\Phi(x'-x, n)\,dx' = 0$, and the convergence is uniform for all values of x in $(a+\mu, b-\mu)$, provided (1), $\Phi(x', x, n)$ is, for each pair of values of x and n, of bounded variation in $(-\pi, a)$, and (2), the condition $\left|\dfrac{\partial\Phi(x', x, n)}{\partial x'}\right| < K$ is satisfied, where K is independent of x and n, and (3), $\Phi(x', x, n)$ converges to zero, as $n \sim \infty$, uniformly for all points x, in $(a+\mu, b-\mu)$.

It is clear that the condition (1) is satisfied; also

$$\Phi(x', x, n) < \frac{1}{2n\pi}\operatorname{cosec}^2 \tfrac{1}{2}\mu,$$

and therefore (3) is satisfied.

We have also

$$\frac{\partial\Phi(x'-x, n)}{\partial x'} = \frac{1}{2n\pi}\left[\frac{n\sin n(x'-x)}{2\sin^2 \frac{1}{2}(x'-x)} - \frac{\sin^2 \frac{1}{2}n(x'-x)\cos \frac{1}{2}(x'-x)}{2\sin^2 \frac{1}{2}(x'-x)}\right];$$

thus $$\left|\frac{\partial\Phi(x'-x, n)}{\partial x'}\right| < \frac{1}{4\pi}\operatorname{cosec}^2 \tfrac{1}{2}\mu + \frac{1}{4\pi}\operatorname{cosec}^2 \tfrac{1}{2}\mu,$$

and hence the condition (2) is satisfied.

It has now been proved that

$$\frac{1}{2n\pi}\int_{-\pi}^{a} f(x')\left[\frac{\sin \frac{n}{2}(x'-x)}{\sin \frac{1}{2}(x'-x)}\right]^2 dx'$$

converges to zero, as $n \sim \infty$, uniformly for all points x in the interval $(a+\mu, b-\mu)$.

In a similar manner the corresponding result can be proved for the integral whose limits are b and π.

It has now been shewn that, for any interval (a, b) interior to an interval contiguous to H, the partial sum $(C, 1)$ of the series converges to $f(x)$ almost everywhere in the interval (a, b), converging to

$$\tfrac{1}{2}\lim_{t\sim 0}\{f(x+t) + f(x-t)\}$$

at a point at which the limit exists. Moreover the convergence is uniform in an interval interior to (a, b), in which $f(x)$ is continuous, provided it is continuous on both sides at the end-points.

For that class of Denjoy integrals for which $m(H) = 0$, including the HL-integrals as a sub-class, the series must converge almost everywhere in $(-\pi, \pi)$.

The following theorems have accordingly been established*:

If $f(x)$ have a D-integral in $(-\pi, \pi)$, then, almost everywhere in an interval contiguous to the set H, of points of non-summability of $f(x)$, the Cesàro sum $(C, 1)$ of the corresponding Fourier's D-series exists, and is equal to $f(x)$.

In case the set H have measure zero, and in particular for all functions which have an HL-integral in $(-\pi, \pi)$, the series is summable $(C, 1)$ almost everywhere in $(-\pi, \pi)$.

In any interval of continuity of $f(x)$, provided the continuity at the end-points is on both sides, the Cesàro partial sums converge uniformly to $f(x)$.

The first theorem cannot be improved by taking the Cesàro sum (C, δ), where $\delta < 1$, instead of $(C, 1)$. For if the series were summable (C, δ) at a particular point α, which we may, without loss of generality, take to be the point $x = 0$, we should have $a_n = o(n^\delta)$ (see § 52). It has been shewn by Titchmarsh (see § 339) that a series of the type in question can be constructed for which this condition is not satisfied.

A proof has been published† by Priwaloff that every Fourier's (D) series is summable $(C, 1 + \delta)$, $(\delta > 0)$, almost everywhere, but there is a part of this proof which appears to need elucidation.

PROPERTIES OF THE FOURIER'S CONSTANTS

377. Let $f(x)$, $g(x)$ be functions such that $\{f(x)\}^2$, $\{g(x)\}^2$ are both summable in the interval $(-\pi, \pi)$. It will be shewn that

$$\int_{-\pi}^{\pi} g(x)\{f(x) - f_n(x)\}\,dx$$

converges to zero, as $n \sim \infty$; $f_n(x)$ denoting the sum of the first $2n + 1$ terms of the Fourier's series $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$, corresponding to $f(x)$.

First, let $g(x)$ be a bounded function, then a function $g_\epsilon(x)$ can be defined (I, § 385) which takes only a finite set of values in the interval, and is such that $|g(x) - g_\epsilon(x)| < \epsilon$. We then have

$$\int_{-\pi}^{\pi} g_\epsilon(x)\{f(x) - f_n(x)\}\,dx = \sum_{s=1}^{s=r} c_s \int_{(e_s)} \{f(x) - f_n'(x)\}\,dx,$$

* See Hobson, *Proc. Lond. Math. Soc.* (2), vol. XXII (1924), pp. 420–424.

† *Rendiconti d. Palermo*, vol. XLI (1916), p. 203.

where $c_1, c_2, \dots c_r$ are the values of $g_\epsilon(x)$, and where e_s is the set of points at which $g_\epsilon(x) = c_s$.

In virtue of the theorem of § 362, the expression on the right-hand side converges to zero, as $n \sim \infty$.

Thus $$\lim_{n \sim \infty} \int_{-\pi}^{\pi} g_\epsilon(x) \{f(x) - f_n(x)\}\, dx = 0.$$

Also

$$\int_{-\pi}^{\pi} g(x) \{f(x) - f_n(x)\}\, dx = \int_{-\pi}^{\pi} \{g(x) - g_\epsilon(x)\} \{f(x) - f_n(x)\}\, dx + \int_{-\pi}^{\pi} g_\epsilon(x) \{f(x) - f_n(x)\}\, dx.$$

The first integral on the right-hand side is numerically less than

$$\epsilon \int_{-\pi}^{\pi} |f(x) - f_n(x)|\, dx,$$

or than $$\epsilon \left\{2\pi \int_{-\pi}^{\pi} \{f(x) - f_n(x)\}^2\, dx\right\}^{\frac{1}{2}}$$

which is less than a fixed multiple of ϵ. The second integral converges to zero, as $n \sim \infty$. It follows that

$$\overline{\lim_{n \sim \infty}} \left| \int_{-\pi}^{\pi} g(x) \{f(x) - f_n(x)\}\, dx \right|$$

cannot exceed a fixed multiple of ϵ; and since ϵ is arbitrary, it follows that

$$\lim_{n \sim \infty} \int_{-\pi}^{\pi} g(x) \{f(x) - f_n(x)\}\, dx = 0.$$

Next, let $g(x)$ be unbounded, but such that $\{g(x)\}^2$ is summable, and consequently such that $f(x)\, g(x)$ is summable.

Let $g(x) = g_1(x) + g_2(x)$, where $g_1(x) = 0$ when $|g(x)| > N$, and $g_1(x) = g(x)$ when $|g(x)| \leq N$; where N is an arbitrarily chosen positive number.

We have

$$\int_{-\pi}^{\pi} g(x) \{f(x) - f_n(x)\}\, dx = \int_{-\pi}^{\pi} g_1(x) \{f(x) - f_n(x)\}\, dx + \int_{-\pi}^{\pi} g_2(x) \{f(x) - f_n(x)\}\, dx;$$

the first integral on the right-hand side converges to zero, as $n \sim \infty$, since $g_1(x)$ is bounded; and the second is numerically less than a fixed multiple (independent of n) of $\left\{\int_{(E)} \{g(x)\}^2\, dx\right\}^{\frac{1}{2}}$; where E denotes the set of points in which $|g(x)| > N$. For $\int_{(E)} \{f(x) - f_n(x)\}^2\, dx$ is less than a fixed number.

Since $\int_{(E)} \{g(x)\}^2 dx$ converges to zero, as $m(E)$ converges to zero, which happens when N is indefinitely increased, it follows that

$$\int_{-\pi}^{\pi} g(x)\{f(x) - f_n(x)\}\, dx$$

converges to zero, as $n \sim \infty$.

The following theorem has now been established:

If $f(x)$ be a function whose square is summable in the interval $(-\pi, \pi)$, the Fourier's series corresponding to $f(x)$ is integrable term by term over $(-\pi, \pi)$, when multiplied by any function $g(x)$ whose square is summable, and the integrated series converges to $\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$.

It may be observed that the theorem holds when the integration is taken over any measurable set e in $(-\pi, \pi)$; for we have only to replace $g(x)$ by a function which is equal to $g(x)$ in the set e, and to zero in the complementary set.

378. If the Fourier's series corresponding to $g(x)$ be denoted by

$$\tfrac{1}{2}a_0' + \Sigma\,(a_n' \cos nx + b_n' \sin nx),$$

we obtain the following theorem:

If $f(x)$, $g(x)$ be two functions such that the square of each of them is summable in $(-\pi, \pi)$, and their Fourier's constants be denoted respectively by a_0, a_1, b_1, a_2, b_2; $a_0', a_1', b_1', a_2', b_2', \ldots$ the series $\tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty} (a_n a_n' + b_n b_n')$ converges to $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$.

In particular, let $g(x) = f(x)$; we have then the theorem that:

If $f(x)$ be a function such that its square is summable in $(-\pi, \pi)$, and $a_0, a_1, b_1, \ldots$ are its Fourier's constants, the series $\tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges to $\frac{1}{\pi} \{f(x)\}^2 dx$.

It will be observed that these remarkable theorems express properties of the Fourier's constants for functions whose squares are summable, and do not involve any knowledge as to the convergence or non-convergence of the corresponding Fourier's series. They have been obtained as the result of a whole series of investigations in which the theorems were proved for the cases of functions of special classes involving greater restrictions than the sole condition that the squares of the functions should be summable. The first theorem is known as Parseval's theorem, in virtue of the fact that it was first stated* by Parseval, whose proof was valid only

* *Sav. étr.* vol. I (1806).

subject to very stringent assumptions. For the case in which the function is integrable (R), the theorems were proved independently of one another by Hurwitz*, Liapounoff†, and de la Vallée Poussin‡. The theorems were extended by Lebesgue§ to the case of bounded summable functions, by a method involving Fejér's theorem relating to arithmetic means. The theorem, as given above in its complete generality, was obtained by Fatou‖. It will be shewn in § 399 that the theorem may be extended so as to apply to any two functions $f(x)$, $g(x)$ such that $|f(x)|^p$, $|g(x)|^q$ are summable in the interval $(-\pi, \pi)$, where p and q are two positive numbers such that $\frac{1}{p}+\frac{1}{q}=1$.

379. Let $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ denote any trigonometrical series such that $\sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2)$ is convergent. Denoting by $f_p(x)$ the partial sum $\frac{1}{2}a_0 + \sum\limits_{1}^{p} (a_n \cos nx + b_n \sin nx)$, it is seen that

$$\int_{-\pi}^{\pi} \{f_q(x) - f_p(x)\}^2 \, dx = \pi \sum_{n=p+1}^{n=q} (a_n^2 + b_n^2),$$

where $q > p$, and it follows that

$$\lim_{p\sim\infty,\, q\sim\infty} \int_{-\pi} \{f_q(x) - f_p(x)\}^2 \, dx = 0;$$

and thus that the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is convergent on the average (see § 170). Consequently, a sequence

$$f_{n_1}(x), f_{n_2}(x), f_{n_3}(x), \ldots$$

can be so determined as to converge almost everywhere to a function $f(x)$ whose square is summable, and which is such that $\int_{-\pi}^{\pi} \{f(x) - f_n(x)\}^2 \, dx$ converges to 0, as $n \sim \infty$; and thus the sequence $\{f_n(x)\}$ converges on the average to $f(x)$.

It follows, as in § 172, that $\int_{-\pi}^{\pi} \{f(x)\}^2 \, dx = \lim\limits_{n\sim\infty} \int_{-\pi}^{\pi} \{f_n(x)\}^2 \, dx$, and therefore $\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 \, dx$ is the sum of the convergent series

$$\tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

* *Math. Annalen*, vol. LVII (1903), p. 425, and *Comptes Rendus*, vol. CXXXII (1901), p. 1473: see also *Annales de l'école normale sup* (3), vol. XIX (1902), p. 357.

† Stekloff states in the *Comptes Rendus* for Nov. 10, 1903, that the first theorem was communicated by Liapounoff to the Kharkow Mathematical Society in 1896.

‡ *Annales de la soc. sci. de Bruxelles*, vol. XVI (1893).

§ *Leçons sur les séries trigonométriques*, pp. 100–101.

‖ *Acta Math.* vol. XXX (1906), p. 352.

Since $\int_{-\pi}^{\pi}\{f(x)-f_n(x)\}\cos mx\,dx$ cannot numerically exceed

$$\left\{\pi\int_{-\pi}^{\pi}\{f(x)-f_n(x)\}^2\,dx\right\}^{\frac{1}{2}},$$

it follows that $\left|\int_{-\pi}^{\pi} f(x)\cos mx\,dx - \pi a_m\right|$ is less than the arbitrarily chosen number ϵ, as is seen by taking n sufficiently large.

It follows that $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$; and similarly it can be proved that $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$.

Consequently the trigonometrical series is the Fourier's series corresponding to $f(x)$. It has now been established that:

Any trigonometrical series such that the sum of the squares of its coefficients is convergent is the Fourier's series of a function $f(x)$, of which the square is summable in $(-\pi, \pi)$.

This is the converse of the theorem of § 378.

That the function $f(x)$ is unique, in the sense that every function which satisfies the condition that its Fourier's series is the given series differs from $f(x)$ only at points of a set of measure zero, has been established in § 361. It can also be proved as follows: Let $f_1(x)$, $f_2(x)$ be two summable functions to which correspond one and the same Fourier's series. By the theorem of § 368, the Cesàro sum of the series is almost everywhere equal both to $f_1(x)$ and to $f_2(x)$. Therefore the values of $f_1(x)$ and $f_2(x)$ coincide almost everywhere.

This may be stated in the form that:

If $a_0, a_1, b_1, a_2, b_2, \ldots$ be a given sequence of numbers such that the sum of their squares forms a convergent series, there exists a function $f(x)$, and it is unique except for equivalent functions, for which $a_0, a_1, b_1, a_2, b_2, \ldots$ are the Fourier's constants. Moreover the square of this function is summable.

This theorem is known as the Riesz-Fischer theorem*, for trigonometrical series. An account of various proofs of the theorem has been given† by W. H. and G. C. Young.

A simple proof of the Riesz-Fischer theorem has been obtained, but not yet published, by Pollard, who proves that, when $\{f(x)\}^2$ is not

* See F. Riesz, *Comptes Rendus*, vol. CXLIV (1907), pp. 615–619 and 734–736; also *Göttinger Nachr.* (1907), p. 116 and *Comptes Rendus*, vol. CXLVIII (1909), pp. 1303–1305. See also Fischer, *Comptes Rendus*, vol. CXLIV (1907), pp. 1022–1024.

† *Quarterly Journal*, vol. XLIV (1912), p. 49.

summable, this implies that $\frac{1}{2}a_0^2 + \sum_{n=1} (a_n^2 + b_n^2)$ is divergent, and who proves simply, by means of Cesàro summability, that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 dx \leqq \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

which is the reverse of the inequality of Bessel.

The above theorem may be expressed in the form that, if $a_0, a_1, b_1, a_2, b_2, \ldots$ be numbers such that $a_1^2 + b_1^2 + a_2^2 + b_2^2 + \ldots$ is convergent, the set of equations

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

where $n = 1, 2, 3, \ldots$ is satisfied only by a single summable function, apart from equivalent functions, and this unique solution is such that its square is summable. If the sum of the squares of the numbers is not convergent, it has not been established that there exists any solution of the equations.

It is thus seen that no summable function $F(x)$ exists which differs from zero at a set of points of positive measure, such that

$$\int_{-\pi}^{\pi} F(x)\,dx = 0, \quad \int_{-\pi}^{\pi} F(x) \frac{\cos}{\sin} nx\,dx, \text{ for } n = 1, 2, 3, \ldots.$$

This may be expressed by the statement that no summable function exists which is orthogonal to all the orthogonal functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$

the orthogonality having reference to the interval $(-\pi, \pi)$.

In other words these functions form a *complete* system of orthogonal functions for the interval $(-\pi, \pi)$ (see § 489).

A sequence of numbers $x_1, x_2, \ldots x_n, \ldots$ such that $\sum_{n=1}^{\infty} x_n^2$ is convergent is said to define a point in Hilbertian space. The above theorem shews that there is a unique correlation of the points of Hilbertian space with functions whose square is summable in a given interval, provided equivalent functions are regarded as identical.

380. Parseval's theorem, given in § 378, may be extended to the case in which one of the functions $f(x)$ is summable, but not necessarily either $\{f(x)\}^2$ or $|f(x)|^{1+q}$, for any positive value of q, and the other function $g(x)$ is measurable and bounded; provided that the series

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1} (a_n a_n' + b_n b_n')$$

is convergent.

It has been shewn in § 366 that the partial Cesàro sum

$$\tfrac{1}{2}a_0' + (a_1'\cos x + b_1'\sin x)\left(1 - \frac{1}{n}\right) + \ldots$$
$$+ (a'_{n-1}\cos \overline{n-1}\,x + b'_{n-1}\sin \overline{n-1}\,x)\left(1 - \frac{n-1}{n}\right)$$

converges boundedly, provided $g(x)$ is bounded in $(-\pi, \pi)$, and that it converges almost everywhere to $g(x)$. Denoting this partial Cesàro sum by $G_n(x)$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx = \tfrac{1}{2}a_0 a_0' + (a_1 a_1' + b_1 b_1')\left(1 - \frac{1}{n}\right) + \dots$$
$$+ (a_{n-1}a'_{n-1} + b_{n-1}b'_{n-1})\left(1 - \frac{n-1}{n}\right);$$

the sum on the right-hand side being the nth partial Cesàro sum of the series $\tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n')$.

In accordance with the theorem of § 305, since $G_n(x)$ is bounded for all values of n and x, $\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx$ converges to $\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$, and consequently the sum on the right-hand side is convergent. Therefore in any case $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$ is the Cesàro sum of the series of products of the Fourier's constants of the functions $f(x)$, $g(x)$. In case this latter series is convergent, its sum is equal to the Cesàro sum.

The following theorem has accordingly been established:

If $f(x)$ be summable, and $g(x)$ be measurable and bounded in $(-\pi, \pi)$, and if the series $\tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n')$ is convergent, its sum is $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$. In any case the Cesàro sum of the series exists, and has this value.

In the particular case in which $g(x)$ is of bounded variation in the interval $(-\pi, \pi)$, it has been shewn in § 333, that $g_n(x)$ converges boundedly to $g(x)$; and it therefore follows that

$$\lim_{n\sim\infty}\int_{-\pi}^{\pi} f(x)\, g_n(x)\, dx = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx,$$

from which Parseval's theorem follows. It has accordingly been proved that:

Parseval's theorem

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n')$$

holds for any pair of functions, one of which is summable, and the other of which is of bounded variation, in $(-\pi, \pi)$.

381. The theorem can be extended to the case in which the function $f(x)$ possesses only an HL-integral in $(-\pi, \pi)$, the other function $g(x)$ being of bounded variation.

The functions $f(x)$, $g(x)$ will be taken to be periodic, of period 2π. We have $g(x) = \lim_{n\sim\infty} G_n(x) = \lim_{n\sim\infty} \frac{1}{2n\pi}\int_{-\pi}^{\pi} g(x+z)\left(\frac{\sin \frac{1}{2}nz}{\sin \frac{1}{2}z}\right)^2 dz$, provided $g(x) = \frac{1}{2}\{g(x+0)+g(x-0)\}$, at any point of discontinuity of $g(x)$. It is known (see § 366) that $G_n(x)$ is bounded with respect to (n, x), and it will be shewn that the total variation of $G_n(x)$ in $(-\pi, \pi)$ is bounded with respect to n.

Since $g(x)$ is of bounded variation in $(-2\pi, 2\pi)$, $g(x)$ can be expressed in the form $P(x) - N(x)$, where $P(x)$ and $N(x)$ are bounded and monotone non-diminishing in the interval $(-2\pi, 2\pi)$; and thus

$$g(x+z) = P(x+z) - N(x+z),$$

where $P(x+z)$, $N(x+z)$ are monotone non-diminishing, for each value of x, in the interval $(-\pi, \pi)$ of z, provided x is in the interval $(-\pi, \pi)$.

The function $\frac{1}{2n\pi}\int_{-\pi}^{\pi} P(x+z)\left(\frac{\sin \frac{1}{2}nz}{\sin \frac{1}{2}z}\right)^2 dz$ is a monotone non-diminishing function of x, in the interval $(-\pi, \pi)$, and thus its total variation is

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi}\{P(\pi+z) - P(-\pi+z)\}\left(\frac{\sin \frac{1}{2}nz}{\sin \frac{1}{2}z}\right)^2 dz,$$

which does not exceed a fixed multiple of $\frac{1}{2n\pi}\int_{-\pi}^{\pi}\left(\frac{\sin \frac{1}{2}nz}{\sin \frac{1}{2}z}\right)^2 dz$, which $= 1$, for all values of n; therefore the total variation is bounded with respect to n. The same argument applies to the case in which $N(x+z)$ takes the place of $P(n+z)$, and therefore the total variation of $G_n(x)$ is bounded with respect to n.

In accordance with the theorem of § 310, since $|g(x) - G_n(x)|$ is bounded with respect to (n, x), and $V_{-\pi}^{\pi} G_n(x)$ is bounded with respect to n, and $G_n(x)$ converges to $g(x)$, it follows that

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \lim_{n\sim\infty}\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx.$$

Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \lim_{n\sim\infty}\left\{\tfrac{1}{2}a_0 a_0' + \left(1 - \frac{1}{n}\right)(a_1 a_1' + b_1 b_1') + \dots \right.$$
$$\left. + \left(1 - \frac{n-1}{n}\right)(a_n a_n' + b_n b_n')\right\}.$$

The following theorem* has therefore been established:

If $f(x)$ have an HL-integral in $(-\pi, \pi)$, and $g(x)$ be of bounded variation in the same interval, then $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$ is the sum of the series

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n'),$$

* See W. H. Young, *Proc. Lond. Math. Soc.* vol. IX (1911), p. 458.

provided this series converges, and it is in any case equal to the Cesàro sum $(C, 1)$ *of this series.*

The above theorem may be extended to the case of integration over any finite interval (α, β), in which $f(x)$ has an HL-integral. If (α, β) be contained in $(-\pi, \pi)$, we may take the function which has the value $f(x)$ in the interval (α, β), and the value zero in the remainder of $(-\pi, \pi)$. It thus follows that

$$\int_\alpha^\beta f(x)\, g(x)\, dx = \lim_{n\sim\infty} \int_\alpha^\beta f(x)\, G_n(x)\, dx.$$

In the general case the same result follows by dividing (α, β) into a finite number of parts, each of which is in an interval $(r\pi, \overline{r+2}\pi)$, where r is integral. We have therefore

$$\int_\alpha^\beta f(x)\, g(x)\, dx = \lim_{n\sim\infty} \left[\tfrac{1}{2} a_0' \int_\alpha^\beta f(x)\, dx + \sum_{r=1}^{r=n-1} \left(1 - \frac{r}{n}\right) \left\{ a_r' \int_\alpha^\beta f(x) \cos rx\, dx + b_r' \int_\alpha^\beta f(x) \sin rx\, dx \right\} \right].$$

It may be proved, in a similar manner that, if $f(x)$ have the period 2π, and have an HL-integral in $(-\pi, \pi)$, and $g(x)$ be any function of bounded variation in (α, β), then

$$\int_\alpha^\beta f(x)\, g(x)\, dx = \lim_{n\sim\infty} \left[\tfrac{1}{2} a_0 \int_\alpha^\beta g(x)\, dx + \sum_{r=1}^{r=n-1} \left(1 - \frac{r}{n}\right) \left\{ a_r \int_\alpha^\beta g(x) \cos rx\, dx + b_r \int_\alpha^\beta g(x) \sin rx\, dx \right\} \right].$$

These results have been extended*, by W. H. Young, to the case in which β is infinite, provided, in the second case, that $a_0 = 0$, and that $g(x) \sim 0$, as $x \sim \infty$.

THE SUBSTITUTION OF A FOURIER'S SERIES IN AN INTEGRAL

382. If $f(x)$ be a periodic function, of period 2π, and the corresponding Fourier's series be denoted by $\tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx)$, no assumption being made as to the convergence of the series, it is frequently of importance to be in possession of sufficient conditions for the validity of the process of substituting for $f(x)$ in an integral $\int_\alpha^\beta f(x)\, g(x)\, dx$, the terms of the series, and of asserting that

$$\tfrac{1}{2}a_0 \int_\alpha^\beta g(x)\, dx + \sum_{n=1}^{\infty} \left\{ a_n \int_\alpha^\beta g(x) \cos nx\, dx + b_n \int_\alpha^\beta g(x) \sin nx\, dx \right\}$$

converges to $\int_\alpha^\beta f(x)\, g(x)\, dx$, where (α, β) is any finite interval. The function $g(x)$ need not be supposed to be periodic, but is defined for the finite interval (α, β). In case (α, β) is contained in the interval $(-\pi, \pi)$, we may replace $g(x)$ by a function $g_1(x)$ which has the same values as $g(x)$ at all points of (α, β), and is zero in the intervals $(-\pi, \alpha)$, (β, π). If $f(x)$, $g_1(x)$ satisfy the conditions of the theorem of § 377, by applying that

* *Loc. cit.* p. 459.

theorem, we obtain the required justification of the term by term integration indicated above. In case (α, β) is not contained in the interval $(-\pi, \pi)$, we may suppose it to be contained in an interval $(r\pi, s\pi)$, where r and s are odd integers, positive or negative. The function $g_1(x)$ may be taken to be zero in the intervals $(r\pi, \alpha)$ and $(s\pi, \beta)$, and to be equal to $g(x)$ in the interval (α, β). If $\{f(x)\}^2$ is summable in $(-\pi, \pi)$, and $\{g(x)\}^2$ is summable in (α, β), $\{g_1(x)\}^2$ is summable in each of the intervals $(r\pi, \overline{r+1}\pi)$, $(\overline{r+1}\pi, \overline{r+2}\pi)$, ... $(\overline{s-1}\pi, s\pi)$. The theorem may then be applied to the functions $f(x)$, $g_1(x)$ in each of these intervals; then, by addition, the result is obtained. We have therefore established the following theorem:

If $\{f(x)\}^2$ be of period 2π, and summable in the interval $(-\pi, \pi)$, and $\{g(x)\}^2$ be summable in the finite interval (α, β), the integral $\int^{\beta} f(x)\, g(x)\, dx$ may be obtained by substituting for $f(x)$ its Fourier's series, and applying term by term integration. No assumption is made as regards the convergence of the Fourier's series corresponding to $f(x)$.

383. In case $f(x)$ is summable in $(-\pi, \pi)$, and $g(x)$ is of bounded variation in the interval (α, β), or in case $f(x)$ is periodic and of bounded variation in $(-\pi, \pi)$, and $g(x)$ is summable in (α, β), precisely similar reasoning, assuming the result of § 380, establishes the following result:

If the periodic function $f(x)$ be summable in $(-\pi, \pi)$, and $g(x)$ be of bounded variation in the finite interval (α, β), then $\int_{\alpha}^{\beta} f(x)\, g(x)\, dx$ may be evaluated by substituting for $f(x)$ its Fourier's series, and applying term by term integration. The same holds in case the periodic functions $f(x)$ is of bounded variation in $(-\pi, \pi)$ and $g(x)$ is summable in (α, β).

By applying the theorem obtained in § 380, we have the result that:

If the periodic function $f(x)$ be summable in $(-\pi, \pi)$, and $g(x)$ be bounded in the finite interval (α, β), then the integral $\int_{\alpha}^{\beta} f(x)\, g(x)\, dx$ may be evaluated by substituting for $f(x)$ its Fourier's series and applying term by term integration, provided the resulting series is convergent; in any case the series is summable $(C, 1)$. The same holds if $f(x)$ is bounded in $(-\pi, \pi)$ and $g(x)$ is summable in (α, β).

From the theorem of § 381, we find that:

If the periodic function $f(x)$ have an HL-integral in $(-\pi, \pi)$, and $g(x)$ is of bounded variation in the finite interval (α, β), then $\int_{\alpha}^{\beta} f(x)\, g(x)\, dx$ may be evaluated by substituting for $f(x)$ its Fourier's HL-series, and integrating term by term, provided the resulting series is convergent; in any case the series is summable $(C, 1)$. The same holds if $f(x)$ is of bounded variation in $(-\pi, \pi)$, and $g(x)$ has an HL-integral in (α, β).

384. For the case of integration over an infinite interval $(0, \infty)$, the following theorem, which is of use in the evaluation of integrals over an infinite interval, will be established:

If $f(x)$ have the period 2π, and be summable over $(0, 2\pi)$, and $g(x)$ satisfies the conditions (a), that it is of bounded variation over the interval $(0, \infty)$, and (b), that $|g(x)|$ is summable in $(0, \infty)$, then $\int_0^\infty f(x)\,g(x)\,dx$ may be calculated by substituting for $f(x)$ its Fourier's series, and integrating term by term. The conditions (a), (b) are satisfied, in particular, if (a)′, $g(x)$ is positive and monotone decreasing, and (b)′, $g(x)$ is summable over the interval $(0, \infty)$.

If $a_0 = 0$, the condition (b), or (b)′, may be replaced by the condition that $g(x)$ converges to zero, as $x \sim \infty$.

This theorem was given* by Hardy, who states that it can be obtained by the collation of results due to W. H. Young.

It will first be shewn that, if $g(x)$ satisfies the conditions (a), (b), the series $g(x) + g(x + 2\pi) + g(x + 4\pi) + \ldots$ converges, for every positive value of x, to a sum $G(x)$ which is summable and of bounded variation in the interval $(0, 2\pi)$.

Denoting $\int_{x+2n\pi}^{x+(2n+2)\pi} g(t)\,dt$ by $v_n(x)$, where $0 \leq x \leq 2\pi$, we have

$$\sum_{n=m+1}^{n=m+r} |v_n(x)| \leq \sum_{n=m+1}^{n=m+r} \int_{x+2n\pi}^{x+(2n+2)\pi} |g(t)|\,dt \leq \int_{(2m+2)\pi}^{\infty} |g(t)|\,dt;$$

thus $\sum_{n=m+1}^{n=m+r} |v_n(x)| < \epsilon$, for a sufficiently large value of m, and for all values of r. It follows that the series $\Sigma v_n(x)$ is absolutely and uniformly convergent.

We have also

$$2\pi g(x + 2n\pi) - v_n(x) = \int_{x+2n\pi}^{x+(2n+2)\pi} \{g(x + 2n\pi) - g(t)\}\,dt;$$

and thus $|2\pi g(x + 2n\pi) - v_n(x)|$ is not greater than $2\pi V_n$, where V_n is the total variation of $g(t)$ in the interval $(x + 2n\pi, x + \overline{2n+1}\pi)$. It follows that $\Sigma\{2\pi g(x + 2n\pi) - v_n(x)\}$ converges absolutely and uniformly; and consequently $\sum_{n=0} g(x + 2n\pi)$ is absolutely and uniformly convergent in $(0, 2\pi)$. Denoting its sum, in that interval, by $G(x)$, we have

$$|G(x)| \leq \sum_{n=0}^{\infty} |g(x + 2n\pi)|;$$

thus $|G(x)|$ is summable in $(0, 2\pi)$, and $\int_0^{2\pi} |G(x)|\,dx \leq \int_0^{\infty} |g(x)|\,dx$. Further, if x_1, x_2 be any two points in the interval $(0, 2\pi)$, we have

$$|G(x_1) - G(x_2)| \leq \sum_{n=0}^{\infty} |g(x_1 + 2n\pi) - g(x_2 + 2n\pi)|.$$

* *Messenger of Math.* vol. LI (1922), p. 186.

It follows that the total variation of $G(x)$ in $(0, 2\pi)$ cannot exceed the total variation of $g(x)$ in $(0, \infty)$.

If we multiply the partial sum $f_n(x)$ of the Fourier's series corresponding to $f(x)$ by $g(x)$, and integrate over the interval $(0, \infty)$, we have for $\int_0^\infty f_n(x)\, g(x)\, dx$ the expression

$$\frac{1}{2\pi}\int_0^\infty g(x)\, dx \int_0^{2\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} f(t)\, dt,$$

and since the integrand $g(x) f(t) \dfrac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}$ is absolutely summable over the domain $(0, 0; \infty, 2\pi)$, the order of integration may be reversed, and becomes

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt \int_0^\infty g(x) \frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\, dx$$

or

$$\int_0^{2\pi} f(t)\, G_n(t)\, dt;$$

where $G_n(t)$ is the sum of the first $2n+1$ terms of the Fourier's series corresponding to the function $G(t)$, defined in the interval $(0, 2\pi)$.

Since $G(t)$ is of bounded variation, $G_n(t)$ converges boundedly to $G(t)$; hence, applying the theorem of § 380, we have

$$\lim_{n\sim\infty}\int_0^{2\pi} f(t)\, G_n(t)\, dt = \int_0^{2\pi} f(t)\, G(t)\, dt.$$

It follows that

$$\lim_{n\sim\infty}\int_0^\infty f_n(x)\, g(x)\, dx = \int_0^{2\pi} f(t)\, G(t)\, dt = \int_0^\infty f(t)\, g(t)\, dt;$$

and thus the first part of the theorem has been established.

Let it next be assumed that $a_0 = 0$, and that the condition that $g(\infty) = 0$ takes the place of the condition (b).

Let $\gamma(x) = g(2m\pi)$, where $2m\pi \leq x < (2m+1)\pi$; and let

$$\bar{g}(x) = \gamma(x) - g(x).$$

It is clear that $\gamma(x)$, and consequently $\bar{g}(x)$, is of bounded variation in $(0, \infty)$. Also, we have

$$\int_{2m\pi}^{(2m+2)\pi} |\bar{g}(x)|\, dx = \int_{2m\pi}^{(2m+2)\pi} \{g(2m\pi) - g(x)\}\, dx \leqq 2\pi V_m;$$

therefore the integral $\int_0^\infty |\bar{g}(x)|\, dx$ exists. Since $\bar{g}(x)$ satisfies the conditions which $g(x)$ satisfied in the first part of the theorem, we can apply to the Fourier's series for $f(x)$, term by term integration over $(0, \infty)$, after multiplication by $\bar{g}(x)$, and the result converges to $\int_0^\infty f(x)\, \bar{g}(x)\, dx$.

Since

$$\int_0^\infty f(x)\gamma(x)\,dx = \sum_{n=0}\int_{2n\pi}^{(2n+2)\pi} f(x)\gamma(x)\,dx = \sum_{n=0}^{\infty} g(2n\pi)\int_0^{2\pi} f(x)$$

and similarly $\int_0^\infty \gamma(x)\cos nx\,dx = 0, \quad \int_0^\infty \gamma(x)\sin nx\,dx = 0,$

it is seen that the second part of the theorem is satisfied.

THE FORMAL MULTIPLICATION OF TRIGONOMETRICAL SERI

385. If two trigonometrical series

$$\tfrac{1}{2}a_0 + (a_1\cos x + b_1\sin x) + \ldots + (a_n\cos nx + b_n\sin nx) +$$

$$\tfrac{1}{2}\alpha_0 + (\alpha_1\cos x + \beta_1\sin x) + \ldots + (\alpha_n\cos nx + \beta_n\sin nx) +$$

be multiplied together, as if the series were finite, and the arranged as a trigonometrical series, we obtain the series

$$\tfrac{1}{2}A_0 + (A_1\cos x + B_1\sin x) + \ldots + (A_n\cos nx + B_n\sin nx) +$$

where

$$\tfrac{1}{2}A_0 = \tfrac{1}{4}a_0\alpha_0 + \tfrac{1}{2}\sum_{n=1}(a_n\alpha_n + b_n\beta_n),$$

$$A_n = \tfrac{1}{2}a_0\alpha_n + \tfrac{1}{2}\sum_{p=1}[a_p(\alpha_{p+n} + \alpha_{p-n}) + b_p(\beta_{p+n} + \beta_{p-n})],$$

$$B_n = \tfrac{1}{2}a_0\beta_n + \tfrac{1}{2}\sum_{p=1}[a_p(\beta_{p+n} - \beta_{p-n}) - b_p(\alpha_{p+n} - \alpha_{p-n})],$$

where it is assumed that $\alpha_{-k} = \alpha_k$, $\beta_{-k} = -\beta_k$.

In this expression, the numbers a_0, a_n, b_n and $\alpha_0, \alpha_n, \beta_n$ may be interchanged.

The series (3) is said to be the formal product of the series (1) and (2).

In case the series $\sum_{n=1}|a_n|$, $\sum_{n=1}|b_n|$, $\sum_{n=1}|\alpha_n|$, $\sum_{n=1}|\beta_n|$ are all absolutely convergent, the series (1), (2) converge absolutely and uniformly to continuous sum-functions $f_1(x), f_2(x)$. In that case the Cauchy-multiplication of the series (1) and (2) yields an absolutely and uniformly convergent series of which the sum-function is the product $f_1(x)f_2(x)$. The series may then be arranged in the form (3) without altering the character of its convergence; and therefore the series (3) converges to $f_1(x)f_2(x)$.

In general, since the process of obtaining (3) from (1) and (2) is purely formal, it is a subject for investigation what relation there may be between the sum-functions of the three series, in case they exist, or between any conventional sums of those series that may exist at particular points or in an interval.

Let the series (1) be the Fourier's series corresponding to a function $f(x)$, summable in the interval $(-\pi, \pi)$. It will be shewn that the formal product of the series (1) into a finite trigonometrical series is the Fourier's series corresponding to the function which is the product of $f(x)$ and the

function which is represented by the finite series. It is clearly sufficient to consider the cases in which the finite series consists of a single term $\cos kx$, or $\sin kx$, where k is a positive integer. Let $a_n = 0$, except when $n = k$, when $a_k = 1$, and let $\beta_n = 0$, for all values of n; the system of equations (K) then becomes

$$A_0 = a_k, \quad A_n = \tfrac{1}{2}(a_{k+n} + a_{k-n}), \quad B_n = \tfrac{1}{2}(b_{k+n} - b_{k-n}).$$

We have now

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\cos nx\,dx,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\sin nx\,dx;$$

it follows that the formal product is the Fourier's series corresponding to $f(x)\cos kx$. Similarly, it is seen that, when we take $\sin kx$, the formal product is the Fourier's series corresponding to $f(x)\sin kx$.

Next, let (1) and (2) be the Fourier's series which correspond to two summable functions $f(x)$, $g(x)$ which are such that either (1), $\{f(x)\}^2$, $\{g(x)\}^2$ are both summable in the interval $(-\pi, \pi)$, or (2), one of the functions $g(x)$ is of bounded variation in $(-\pi, \pi)$. In either case, Parseval's theorem is applicable to the two functions $f(x)\frac{\cos kx}{\sin kx}$, $g(x)$, where k denotes a positive integer. We have accordingly

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\cos kx\,dx = \tfrac{1}{2}\bar{A}_0\alpha_0 + \Sigma(\bar{A}_n\alpha_n + \bar{B}_n\beta_n),$$

where $\bar{A}_0$, $\bar{A}_n$, $\bar{B}_n$ are the Fourier's constants corresponding to the function $f(x)\cos kx$. The expression on the right-hand side is

$$\tfrac{1}{2}\alpha_0 a_k + \tfrac{1}{2}\Sigma\{\alpha_n(a_{k+n} + a_{k-n}) + \beta_n(b_{k+n} - b_{k-n})\}, \text{ or } A_k,$$

where A_k is the coefficient of $\cos kx$ in the formal product of the series (1) and (2). It thus appears that A_k is the coefficient of $\cos kx$ in the Fourier's series corresponding to the function $f(x)\,g(x)$. Similarly, it may be shewn that B_k is the coefficient of $\sin kx$ in the Fourier's series corresponding to $f(x)\,g(x)$. The following theorem has now been established:

If either (1), $\{f(x)\}^2$, $\{g(x)\}^2$ *are both summable in the interval* $(-\pi, \pi)$, *or* (2), *one of the functions* $f(x)$, $g(x)$ *is summable, and the other of bounded variation, in the interval* $(-\pi, \pi)$, *the formal product, of which the coefficients are given by* (K), *is the Fourier's series corresponding to the product* $f(x)\,g(x)$.

This theorem was given* by Hurwitz for the case in which the two functions are both integrable (R) in the interval $(-\pi, \pi)$; and by† Lebesgue in the case in which they are both summable and bounded. The theorem

* *Math. Annalen*, vol. LVII (1903), p. 45, and vol. LIX (1904), p. 553.

† *Leçons sur les séries trigonométriques*, p. 101.

may be extended to the case in which $| f(x) |^k$, $| g(x) |^{k'}$ are summable in $(-\pi, \pi)$, where k and k' are positive numbers such that $\frac{1}{k} + \frac{1}{k'} = 1$ (see § 399).

386. The theory of formal multiplication has been applied by Rajchman* and by Zygmund† to more general classes of trigonometrical series.

It has been proved by Rajchman that:

If $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$, $\frac{1}{2}\alpha_0 + \Sigma (\alpha_n \cos nx + \beta_n \sin nx)$ *be two trigonometrical series such that* $a_n = o(1)$, $b_n = o(1)$, $n^3\alpha_n = o(1)$, $n^3\beta_n = o(1)$, *then the convergence to zero, of the second series, at the point* x_0, *involves the convergence to zero, at* x_0, *of the formal product of the two series.*

If, at x_0, *the second series converges to a value different from zero, the convergence, or the summability* (C, r), *or the summability by Poisson's method* (§ 411), *or by Riemann's method* (§ 420), *of the formal product of the two series is the necessary and sufficient condition for the convergence, or for the summability by the same procedure, of the first series, at the point* x_0.

This theorem has been extended by Zygmund, who proved that:

If the two trigonometrical series are such that, for some value of $\gamma\, (\geqq 0)$, $n^{-\gamma}a_n = o(1)$, $n^{-\gamma}b_n = o(1)$, $n^{2\gamma+3}\alpha_n = o(1)$, $n^{2\gamma+3}\beta_n = o(1)$, *and further* (C) *the sum-function of the second series, and its first* k *differential coefficients, where* $k < \gamma + 1$, *all vanish at the point* x_0, *then* (1), *the formal product of the two series has its sum* (C, γ) *equal to zero;* (2), *the series conjugate to* (§ 400) *the formal product series is summable* (C, γ) *with the sum* (C, γ) *in general different from zero;* (3), *the series obtained by differentiating the formal product* p *times is, at* x_0, *summable* $(C, \gamma + p)$, *with its sum* $(C, \gamma + p)$ *equal to zero, provided the sum-function of the series possesses a sufficient number of differential coefficients which vanish at* x_0, *with an analogous result for the conjugate series;* (4), *if the conditions* (C) *are fulfilled in a closed set of points* E, *the summability is uniform in* E.

Further, analogous results hold when γ *is negative, but not integral, the second of the conditions* (C) *then disappearing.*

AN EXTENSION OF THE THEOREM OF ARITHMETIC MEANS

387. It has been shewn in § 368 that, almost everywhere in $(-\pi, \pi)$, the arithmetic mean of the partial sums of a Fourier's series converges to the value of the function. This may be stated in the form

$$\{f_0(x) - f(x)\} + \{f_1(x) - f(x)\} + \ldots + \{f_n(x) - f(x)\} = o(n)$$

at each point x at which $\int_0^t | f(x+t) + f(x-t) - 2f(x) |\, dx = o(t)$.

* *Comptes Rendus*, vol. CLXXVII (1923), p. 491.

† *Ibid.* vol. CLXXVII (1923), pp. 521, 576, 804. Zygmund has developed the method in *Math. Zeitschr.* vol. XXIV (1925), p. 47.

For the case in which $\{f(x)\}^2$ is summable, the more precise theorem has been given* by Hardy and Littlewood that:

At almost every point in the interval $(-\pi, \pi)$, *and in particular at every point at which* $f(x+t)+f(x-t)$ *is convergent as* $t \sim 0$, *and* $f(x)$ *is half the limit, the relations*

$$|f_0(x)-f(x)|+|f_1(x)-f(x)|+\ldots+|f_n(x)-f(x)|=o(n)$$
$$\{f_0(x)-f(x)\}^2+\{f_1(x)-f(x)\}^2+\ldots+\{f_n(x)-f(x)\}^2=o(n)$$

are satisfied, provided $\{f(x)\}^2$ *is summable in the interval* $(-\pi, \pi)$.

It is clear that, if the second of these relations holds good, then the first holds also; this follows from the known inequality

$$\left(\frac{|c_1|+|c_2|+\ldots+|c_n|}{n}\right)^2 \leq \frac{c_1^2+c_2^2+\ldots+c_n^2}{n}.$$

It is therefore sufficient to prove the second relation. The first relation shews that, in the case of a function whose square is summable, the average of the numbers $f_r(x)-f(x)$ tends to zero because the number of terms in $\sum\limits_0^n \{f_r(x)-f(x)\}$ which are not themselves small is small compared with n, and not merely on account of the cancelling of positive and negative terms.

Denoting $f(x+t)+f(x-t)-2f(x)$ by $\phi(t)$, we have

$$\begin{aligned} f_m(x)-f(x) &= \frac{1}{\pi}\int_0^\pi \phi(t)\sin(m+\tfrac{1}{2})t \operatorname{cosec}\tfrac{1}{2}t\,dt \\ &= \frac{1}{\pi}\int_0^\pi \phi(t)\sin mt\cot\tfrac{1}{2}t\,dt + \frac{1}{\pi}\int_0^\pi \phi(t)\cos mt\,dt \\ &= \alpha_m+\beta_m+\gamma_m; \end{aligned}$$

where $\alpha_m = \dfrac{1}{\pi}\displaystyle\int_0^\epsilon \phi(t)\sin mt\cot\tfrac{1}{2}t\,dt$, $\beta_m = \displaystyle\int_\epsilon^\pi \phi(t)\sin mt\cot\tfrac{1}{2}t\,dt$,

and
$$\gamma_m = \frac{1}{\pi}\int_0^\pi \phi(t)\cos mt\,dt;$$

and ϵ denotes a fixed number in the interval $(0, \pi)$.

Employing an inequality given in I, § 435, we have

$$\left\{\sum_{m=0}^{m=n}|f_m(x)-f(x)|^2\right\}^{\frac{1}{2}} \leq \left(\sum_{m=0}^{m=n}|\alpha_m|^2\right)^{\frac{1}{2}} + \left(\sum_{m=0}^{m=n}|\beta_m|^2\right)^{\frac{1}{2}} + \left(\sum_{m=0}^{m=n}|\gamma_m|^2\right)^{\frac{1}{2}};$$

and we can estimate the values of the three expressions on the right-hand side separately.

We have $|\sin mt\cot\tfrac{1}{2}t| < m\,|t\cot\tfrac{1}{2}t| < Am$, where A is the maximum of $|t\cot\tfrac{1}{2}t|$ in $(0, \pi)$; therefore $\alpha_m < Am\displaystyle\int_0^\epsilon |\phi(t)|\,dt$. It follows that $\left\{\sum\limits_{m=0}^{m=n}|\alpha_m|^2\right\}^{\frac{1}{2}}$ is $(\Sigma m^2)^{\frac{1}{2}}\,o(\epsilon)$, or $n^{\frac{3}{2}}\,o(\epsilon)$, if x is a point at which

$$\int_0^\epsilon |\phi(t)|\,dt = o(\epsilon).$$

* *Comptes Rendus*, vol. CLVI (1913), p. 1307.

Again, $2\gamma_m$ is the coefficient of $\cos mt$ in the Fourier's cosine series which represents the function $\phi(t)$, the square of which is summable; it therefore follows from Parseval's theorem that $\sum_{m=0}^{m=\infty} |\gamma_m|^2 = O(1)$.

We have next to evaluate $\sum_{m=0}^{\infty} |\beta_m|^2$. By integration by parts we find that

$$\beta_m = -\frac{1}{\pi}\cot\tfrac{1}{2}\epsilon \,.\, \psi_m(\epsilon) + \frac{1}{2\pi}\int_\epsilon^\pi \psi_m(t)\operatorname{cosec}^2\tfrac{1}{2}t\,dt,$$

where $\psi_m(t)$ denotes $\int_0^t \phi(t)\sin mt\,dt$.

It follows, by employing the same inequality as before, that

$$\left[\sum_{m=1}^{m=n} |\beta_m|^2\right]^{\frac{1}{2}} \leqq \frac{1}{\pi}\cot\tfrac{1}{2}\epsilon\left[\sum_{m=1}^{m=n}\{\psi_m(\epsilon)\}^2\right]^{\frac{1}{2}} + \frac{1}{2\pi}\left[\sum_{m=1}^{m=n}\left\{\int_\epsilon^\pi \psi_m(t)\operatorname{cosec}^2\tfrac{1}{2}t\,dt\right\}^2\right]^{\frac{1}{2}}.$$

Now

$$\left\{\int_\epsilon^\pi \psi_m(t)\operatorname{cosec}^2\tfrac{1}{2}t\,dt\right\}^2 \leqq \int_\epsilon^\pi \operatorname{cosec}^2\tfrac{1}{2}t\,\{\psi_m(t)\}^2\,dt\int_\epsilon^\pi \operatorname{cosec}^2\tfrac{1}{2}t\,dt$$
$$\leqq \frac{4}{\epsilon}\int_\epsilon^\pi \operatorname{cosec}^2\tfrac{1}{2}t\,\{\psi_m(t)\}^2\,dt.$$

Hence we have

$$\left[\sum_{m=1}^{m=n} |\beta_m|^2\right]^{\frac{1}{2}} \leqq \frac{1}{\pi}\cot\tfrac{1}{2}\epsilon\left[\sum_{m=1}^{m=n}\{\psi_m(\epsilon)\}^2\right]^{\frac{1}{2}} + \frac{1}{\pi\sqrt{\epsilon}}\left[\int_\epsilon^\pi \operatorname{cosec}^2\tfrac{1}{2}t\sum_{m=1}^{m=n}\{\psi_m(t)\}^2\,dt\right]^{\frac{1}{2}}.$$

The Fourier's sine series of the odd function which has the value $\phi(t)$ in the interval $(0, \epsilon)$, and the value zero in the interval (ϵ, π), is

$$\frac{2}{\pi}\sum_{m=1}^{\infty}\sin mt\,.\int_0^\epsilon \phi(t)\sin mt\,dt$$

or

$$\frac{2}{\pi}\Sigma\,\psi_m(\epsilon)\sin mt.$$

It follows by Parseval's theorem that

$$\frac{2}{\pi}\sum_{m=1}^{\infty}\{\psi_m(\epsilon)\}^2 = \int_0^\epsilon \{\phi(t)\}^2\,dt,$$

and this can be shewn to be $o(\epsilon)$, almost everywhere. For we have

$$\int_0^\epsilon \{\phi(t)\}^2\,dt \leqq 2\int_0^\epsilon \{f(x+t)-f(x)\}^2\,dt + 2\int_0^\epsilon \{f(x-t)-f(x)\}^2\,dt$$

and

$$\int_0^\epsilon \{f(x+t)-f(x)\}^2\,dt = \int_0^\epsilon \{f(x+t)\}^2\,dt + \epsilon\{f(x)\}^2 - 2f(x)\int_0^\epsilon f(x+t)\,dt;$$

now
$$\int_0^\epsilon \{f(x+t)\}^2\,dt = \epsilon\,\{f(x)\}^2 + o(\epsilon)$$
and
$$\int_0^\epsilon f(x+t)\,dt = \epsilon f(x) + o(\epsilon),$$
almost everywhere, since (I, § 432) for any summable function $\phi(x')$, $\int_x^{x'} \{\phi(x') - \phi(x)\}\,dx'$ has a differential coefficient equal to zero for almost all values of x, and we may put $x' = x + t$, and $f(x)$, or $\{f(x)\}^2$, for $\phi(x)$. Hence
$$\int_0^\epsilon \{f(x+t) - f(x)\}^2\,dt = o(\epsilon),$$
for almost all values of x; similarly
$$\int_0^\epsilon \{f(x-t) - f(x)\}^2\,dt = o(\epsilon),$$
almost everywhere, and therefore
$$\int_0^\epsilon \{\phi(t)\}^2\,dt = o(\epsilon),$$
for almost all values of x.

In particular this relation holds at every point at which
$$\lim_{t\sim 0}\{f(x+t) + f(x-t)\}$$
is convergent, in which case the limit is $2f(x)$, by adjustment if necessary, of the value of $f(x)$.

We now have
$$\left[\sum_{m=1}^{m=\infty} |\beta_m|^2\right]^{\frac{1}{2}} = \frac{1}{\epsilon}\,o(\epsilon^{\frac{1}{2}}) + \frac{k}{\epsilon^{\frac{1}{2}}}\left[\int_\epsilon^\pi \operatorname{cosec}^2 \tfrac{1}{2}t\,.\,o(t)\,dt\right]^{\frac{1}{2}}$$
$$= \frac{1}{\epsilon^{\frac{1}{2}}}\,o(1) + \frac{k'}{\epsilon^{\frac{1}{2}}}\left(\log\frac{\pi}{\epsilon}\right)^{\frac{1}{2}},$$
where k, k' are fixed numbers.

Let $\epsilon = n^{-1}$, the right-hand side is then $o(n)$.

We now have, for almost every value of x,
$$\left[\sum_{m=0}^{m=n} |f_m(x) - f(x)|^2\right]^{\frac{1}{2}} < n^{\frac{1}{2}}\,o(1) + O(1) + o(n) = o(n),$$
and the theorem has thus been established.

The foregoing proof is founded upon the proof given* by Carleman of the following more general theorem:

If, at a particular point x, the relations
$$\int_0^\epsilon |f(x+t) + f(x-t) - 2f(x)|\,dt = o(\epsilon),$$
$$\int_0^\epsilon \{f(x+t) + f(x-t) - 2f(x)\}^2\,dt = O(\epsilon)$$

* *Proc. Lond. Math. Soc.* (2), vol. XXI (1923), p. 484. See also Sutton.

hold good, then $\lim_{n\sim\infty} \frac{1}{n+1}\sum_{m=0}^{m=n} |f_m(x) - f(x)|^k = 0$, *for every positive value of* k.

It has been shewn above that, if $\{f(x)\}^2$ is a summable function, the second relation holds for almost all values of x, $O(\epsilon)$ being in that case $o(\epsilon)$; but in the theorem it need only be assumed that $\{f(x)\}^2$ is summable in a neighbourhood of the point x.

EXTENSION AND GENERALIZATION OF PARSEVAL'S THEOREM

388. Some caution is requisite in the employment of Parseval's theorem in particular cases; it is always necessary to make sure that the conditions of one or other of the theorems given above are actually satisfied. It has not been proved that the existence of the integral $\int_{-\pi}^{\pi} f(x)\,g(x)\,dx$ is by itself sufficient for the validity of the theorem. For example, if $f(x)$ has the value zero at all points of a measurable set E, contained in $(-\pi, \pi)$, and $g(x)$ has the value zero at all points of the set which is complementary to E relatively to $(-\pi, \pi)$, the integral $\int_{-\pi}^{\pi} f(x)\,g(x)\,dx$ exists, and has the value zero; but, unless $f(x)$, $g(x)$ satisfy further conditions, it cannot be inferred that the series $\frac{1}{2}a_0a_0' + \sum_{n=1} (a_na_n' + b_nb_n')$ converges to zero. In the particular case however in which the set E consists of a finite set of intervals, it can be shewn that, subject to a certain condition, Parseval's theorem is still valid. The following theorem will be established:

If the summable functions $f(x)$, $g(x)$ *are such that* $f(x)$ *has the value zero at the points of a finite set of intervals contained in* $(-\pi, \pi)$, *and if* $g(x)$ *has the value zero at all points not in these intervals, and has bounded variation in sufficiently small neighbourhoods of the end-points of the intervals of the set, then* $\frac{1}{2}a_0a_0' + \sum_{n=1} (a_na_n' + b_nb_n')$ *converges to zero.*

It will be sufficient to take the case in which there is a single interval (α, β), for which the conditions of the theorem are satisfied. There exist intervals $(\alpha - h, \alpha + h)$, $(\beta - h', \beta + h')$ in which $g(x)$ has bounded variation, and we may assume that in intervals $(-\pi, \alpha + \zeta)$, $(\beta - \zeta, \pi)$, where ζ is less than both h and h', the convergence of $g_n(x)$ to $g(x)$ is bounded (see § 341), so that $|g_n(x)| < A_\zeta$. Let ζ be so chosen that $\int_{\alpha-\zeta}^{\alpha} |f(x)|\,dx < \epsilon$, $\int_{\beta}^{\beta+\zeta} |f(x)|\,dx < \epsilon$, where ϵ is a positive number, chosen arbitrarily. We have

$$\int_{-\pi}^{\pi} f(x)\,g_n(x)\,dx = \int_{-\pi}^{\alpha-\zeta} f(x)\,g_n(x)\,dx + \int_{\alpha-\zeta}^{\alpha} f(x)\,g_n(x)\,dx + \int_{\beta}^{\beta+\zeta} f(x)\,g_n(x)\,dx + \int_{\beta+\zeta}^{\pi} f(x)\,g_n(x)\,dx;$$

in the intervals $(-\pi, \alpha - \zeta)$ and $(\beta + \zeta, \pi)$, $g_n(x)$ converges uniformly to zero, since the intervals are interior to intervals in which $g(x)$ is of bounded variation, therefore the first and fourth integrals on the right-hand side are each numerically less than $\epsilon \int_{-\pi}^{\pi} |f(x)|\, dx$, provided n is not less than some integer n_ϵ. Also $g_n(x)$ converges boundedly to its limit in the intervals $(\alpha - \zeta, \alpha)$, $(\beta, \beta + \zeta)$, and thus the second and third integrals on the right-hand side are numerically less than $A_\zeta \epsilon$, for all values of n. We now have $\left|\int_{-\pi}^{\pi} f(x)\, g_n(x)\, dx\right| < 2\epsilon \int_{-\pi}^{\pi} |f(x)|\, dx + 2\epsilon A_\zeta$, for $n \geqq n_\epsilon$; and thus, since ϵ is arbitrary, and A_ζ is independent of ϵ, it has been shewn that

$$\lim_{n\sim\infty} \int_{-\pi}^{\pi} f(x)\, g_n(x)\, dx = 0,$$

which is equivalent to the result stated in the theorem.

From this theorem the following extension of Parseval's theorem may be deduced, which is capable of application in certain cases:

If $f(x)$, $\{g(x)\}^2$ are summable in the interval $(-\pi, \pi)$, and if the further conditions are satisfied that (1), *in some neighbourhood of a point c in the interval, $\{f(x)\}^2$ is summable, and* (2), *$g(x)$ is of bounded variation in $(-\pi, \pi)$, when a neighbourhood of the point c is excluded from the interval, then Parseval's theorem holds good for $\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$.*

It can easily be seen that the theorem can be extended to cases in which there are a finite number of such points c.

To prove the theorem, let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = 0$ in the interval $(c - \delta, c + \delta)$, and $f_2(x) = 0$ outside this interval; thus $f_1(x) = f(x)$ outside the interval, and $f_2(x) = f(x)$ in the interval. The interval can be so chosen that $\{f_2(x)\}^2$ is summable in $(-\pi, \pi)$.

Let $g(x)$ be expressed in a precisely similar manner as the sum of two functions $g_1(x)$ and $g_2(x)$. Any Fourier's coefficient for $f(x)$ or $g(x)$ is the sum of the corresponding Fourier's coefficients for $f_1(x)$ and $f_2(x)$, or for $g_1(x)$ and $g_2(x)$.

Parseval's theorem holds for $\int_{-\pi}^{\pi} f_1(x)\, g_1(x)\, dx$, because $f_1(x)$ is summable and $g_1(x)$ is of bounded variation. It holds for $\int_{-\pi}^{\pi} f_2(x)\, g_2(x)\, dx$ because both $f_2(x)$ and $g_2(x)$ have their squares summable. It holds for $\int_{-\pi}^{\pi} f_2(x)\, g_1(x)\, dx$, since $\{f_2(x)\}^2$, $\{g_1(x)\}^2$ are summable. By the last theorem it holds for $\int_{-\pi}^{\pi} f_1(x)\, g_2(x)\, dx$, since $f_1(x) = 0$ in the interval $(c - \delta, c + \delta)$, and $g_2(x)$ is zero at all points not in that interval, and has

bounded variation in neighbourhoods of the points $c-\delta, c+\delta$. It now follows, by addition, that $\frac{1}{2}a_0a_0' + \sum_{n=1} (a_n a_n' + b_n b_n')$ converges to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

389. The following theorem is an extension of the first theorem of § 388:

If the summable functions $f(x)$, $g(x)$ are such that $f(x)$ has the value zero at the points of a finite set of non-abutting intervals Δ, contained in $(-\pi, \pi)$, and if $g(x)$ has the value zero at all points not in those intervals, and is bounded in the neighbourhood of the end-points of the intervals, then the sum $(C, 1)$ of the series $\frac{1}{2}a_0a_0' + \sum_{n=1}^{\infty} (a_n a_n' + b_n b_n')$ is zero; and consequently, if the series is convergent, it converges to zero.

We may say that Parseval's theorem holds $(C, 1)$ for the two functions.

If (α_r, β_r) be one of the intervals of Δ, the points α_r, β_r have neighbourhoods in which $g(x)$ is bounded. Let $G_n(x)$ be the nth Cesàro partial sum of the Fourier's series corresponding to $g(x)$. It is known (§ 366) that, in any interval interior to an interval in which $g(x)$ is bounded, $G_n(x)$ is bounded with respect to (n, x). The integral of $f(x)\, G_n(x)$ over $(-\pi, \pi)$ may be expressed as the sum of integrals over the intervals

$$(-\pi, \alpha_1 - \zeta),\ (\alpha_1 - \zeta, \alpha_1),\ (\beta_1, \beta_1 + \zeta),\ (\beta_1 + \zeta, \alpha_2 - \zeta),\ (\alpha_2 - \zeta, \alpha_2),\ \ldots\ (\beta_r + \zeta, \pi).$$

We may choose ζ so small that $|G_n(x)|$ is bounded in the intervals $(\alpha_1 - \zeta, \alpha_1)$, $(\beta_1, \beta_1 + \zeta)$, $(\alpha_2 - \zeta, \alpha_2), \ldots$, and thus $|G_n(x)|$ is less than a fixed number A, through these intervals.

If ϵ be an arbitrarily chosen positive number, ζ may be so chosen, by diminishing it, if necessary, that the integral of $|f(x)|$ over each of these intervals is $< \epsilon$.

It follows that $\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx$ is numerically less than

$$\left|\int_{-\pi}^{\alpha_1 - \zeta} f(x)\, G_n(x)\, dx\right| + \left|\int_{\beta_1 + \zeta}^{\alpha_2 - \zeta} f(x)\, G_n(x)\, dx\right| + \ldots + \left|\int_{\beta_r + \zeta}^{\pi} f(x)\, G_n(x)\, dx\right| + 2rA\epsilon.$$

Now $\lim_{n \sim \infty} \int_{-\pi}^{\alpha_1 - \zeta} f(x)\, G_n(x)\, dx = 0$, since $(-\pi, \alpha_1 - \zeta)$ is interior to an interval in which $g(x)$ is bounded, and $G_n(x)$, $g_n(x)$ converge to zero. A similar statement applies to each of the other integrals. It follows that, if n is $\geqq$ an integer n_ϵ,

$$\left|\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx\right| < \epsilon\,(1 + 2rA)$$

and, since ϵ is arbitrary, $\lim_{n\sim\infty}\int_{-\pi}^{\pi} f(x)\, G_n(x)\, dx = 0$; thus the theorem is established.

From the above theorem the following theorem may be deduced:

If $f(x)$, $\{g(x)\}^2$ are summable in $(-\pi, \pi)$, and the closed set H of points of infinite discontinuity of $g(x)$ is such that no point of H is a point of non-summability of $\{f(x)\}^2$, then Parseval's theorem holds $(C, 1)$ for $f(x)\, g(x)$. In case the series is convergent, Parseval's theorem holds in its original form.

The closed set H may be enclosed in the interior of the intervals of a finite set Δ, so that Δ contains no point of non-summability of $\{f(x)\}^2$; for the set of all such points is closed, and therefore has a finite distance from the set H. Each end-point of an interval of Δ has a neighbourhood in which $g(x)$ is bounded.

Let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = 0$ in Δ, $f_2(x) = 0$ outside Δ, and $g(x) = g_1(x) + g_2(x)$, where $g_1(x) = 0$ in Δ, $g_2(x) = 0$ outside Δ; it follows that $\{f_2(x)\}^2$ is summable in the interval $(-\pi, \pi)$.

For $f_1(x)\, g_1(x)$, $f_1(x)$ is summable, and $g_1(x)$ is bounded; thus Parseval's theorem holds $(C, 1)$ for $f_1(x)\, g_1(x)$.

For $f_2(x)\, g_2(x)$, $\{f_2(x)\}^2$ and $\{g_2(x)\}^2$ are both summable, and thus Parseval's theorem holds for $f_2(x)\, g_2(x)$.

Similarly Parseval's theorem holds for $f_2(x)\, g_1(x)$.

For $f_1(x)\, g_2(x)$, we have $f_1(x) = 0$ in Δ, $g_2(x)$ is zero outside Δ, and is bounded in neighbourhoods of the end-points of the intervals of Δ; therefore, by the last theorem, Parseval's theorem holds $(C, 1)$ for $f_1(x)\, g_2(x)$.

The truth of the theorem now follows by addition.

390. We can now establish the following general theorem:

If $f(x)$, $g(x)$ are both summable in $(-\pi, \pi)$, and the set of points of non-summability of the functions $\{f(x)\}^2$, $\{g(x)\}^2$ be K_1, K_2 respectively; which are contained respectively in H_1, H_2 the closed sets of points of infinite discontinuity of $f(x)$, $g(x)$; then, if H_1 have no point in common with K_2, and H_2 no point in common with K_1, Parseval's theorem holds $(C, 1)$ for $f(x)\, g(x)$.

In accordance with the assumption made in this statement there is no point which is a point of non-summability both of $\{f(x)\}^2$ and $\{g(x)\}^2$, and at each point of non-summability of either function, the other function is bounded in the neighbourhood of the point.

Let H_2 be enclosed in the interior of intervals of a finite set Δ; then $g(x)$ is bounded in the neighbourhoods of the end-points of the intervals of Δ.

Let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = 0$ in Δ, and $f_2(x) = 0$ outside Δ, and let $g(x) = g_1(x) + g_2(x)$, where $g_1(x) = 0$ in Δ, and $g_2(x) = 0$ outside Δ; $\{f_2(x)\}^2$ is summable, because Δ can be so chosen as to contain no points of the closed set K_1.

Since $f_1(x)$ is summable, and $g_1(x)$ is bounded, Parseval's theorem holds $(C, 1)$ for $f_1(x)\, g_1(x)$. Since $\{f_2(x)\}^2$, $g_2(x)$ are summable, and no point of infinite discontinuity of $f_2(x)$ is a point of non-summability of $\{g_2(x)\}^2$, by the last theorem, Parseval's theorem holds $(C, 1)$ for $f_2(x)\, g_2(x)$. Since $f_2(x)$ is summable and $g_1(x)$ is bounded, Parseval's theorem holds $(C, 1)$ for $f_2(x)\, g_1(x)$. Since $f_1(x)$, $g_2(x)$ are summable, and $g_2(x)$ is bounded in the neighbourhoods of the end-points of the intervals Δ, Parseval's theorem holds $(C, 1)$ for $f_1(x)\, g_2(x)$. It now follows that Parseval's theorem holds $(C, 1)$ for $f(x)\, g(x)$.

The above theorems and those of § 389 are capable of generalization by taking into account M. Riesz' theorem given in § 399, in which powers of $|f(x)|$ other than the square are taken into account.

391. It has been shewn, in § 230, that $\int_{-\pi}^{\pi} f(x+t)\, g(t)\, dt$ is a continuous function of x if, either (1), $f(t)$ is summable and $g(t)$ is bounded, or (2), $|f(t)|^p$ and $|g(t)|^q$ are summable, where p and q are both positive, and such that $\frac{1}{p} + \frac{1}{q} = 1$; (2) includes the case in which $p = q = 2$. The functions $f(t)$, $g(t)$ are here supposed to be defined for all values of t so that they are periodic, with period 2π.

Let $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nt + b_n \sin nt)$, $\frac{1}{2}a_0' + \sum_{n=1} (a_n' \cos nt + b_n' \sin nt)$ be the Fourier's series corresponding to the periodic functions $f(t)$, $g(t)$.

Let us consider the function $\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t+x) + f(t-x)]\, g(t)\, dt$ which is an even periodic and continuous function of x, when one of the above conditions is satisfied. The coefficient of $\cos nx$ in its Fourier's series is

$$\frac{1}{2\pi^2}\int_{-\pi}^{\pi} \cos nx\, dx \int_{-\pi}^{\pi} [f(t+x) + f(t-x)]\, g(t)\, dt.$$

The same conditions that are satisfied by $f(t)$, $g(t)$ are satisfied by $|f(t)|$, $|g(t)|$, and thus, by the theorem of § 230, when the same conditions are satisfied $\int_{-\pi}^{\pi} |f(t+x)\, g(t)|\, dt$, $\int_{-\pi}^{\pi} |f(t-x)\, g(t)|\, dt$ are continuous functions of x, and therefore the repeated integral

$$\int_{-\pi}^{\pi} |\cos nx|\, dx \int_{-\pi}^{\pi} |f(t+x) + f(t-x)|\, |g(t)|\, dt$$

exists.

In accordance with the theorem of § 237, $\cos nx\,[f(t+x)+f(t-x)]\,g(t)$ is therefore summable in the domain of (t, x), and its repeated integrals are equal to one another. Therefore the coefficient of $\cos nx$ in the Fourier's series is

$$\frac{1}{2\pi^2}\int_{-\pi}^{\pi} g(t)\,dt \int_{-\pi}^{\pi} [f(t+x)+f(t-x)]\cos nx\,dx,$$

and this is equal to $\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)(a_n \cos nt + b_n \sin nt)\,dt$ or to $a_n a_n' + b_n b_n'$.

In a precisely similar manner it can be shewn that the coefficient of $\sin nx$ in the Fourier's series corresponding to the continuous odd function $\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(t+x)-f(t-x)]\,g(t)\,dt$ is $-a_n b_n' + b_n a_n'$. We have now obtained the following theorem*:

If $f(t)$, $g(t)$ be periodic summable functions, with period 2π, and (a_n, b_n), (a_n', b_n') be the Fourier's constants in the Fourier's series corresponding to them, then

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1} (a_n a_n' + b_n b_n')\cos nx$$

and

$$\sum_{n=1} (b_n a_n' - a_n b_n')\sin nx$$

are the Fourier's series corresponding to the continuous functions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(t+x)+f(t-x)\}\,g(t)\,dt,\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(t+x)-f(t-x)\}\,g(t)\,dt$$

respectively, provided either (1), *$g(t)$ is bounded, or* (2), *$|f(t)|^p$, $|g(t)|^q$ are summable, where p, q are positive numbers such that* $\frac{1}{p}+\frac{1}{q}=1$.

As regards the convergence of the Fourier's series in this theorem the following statements may be made:

If either (1), *$f(t)$ is summable and $g(t)$ is of bounded variation, or* (2), *$\{f(t)\}^2$ and $\{g(t)\}^2$ are both summable, the series*

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1} (a_n a_n' + b_n b_n')\cos nx,$$

$$\sum_{n=1} (b_n a_n' - a_n b_n')\sin nx$$

converge everywhere to the continuous functions which they represent.

If (3), *$|f(t)|^p$, $|g(t)|^q$ are summable, where $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, the Cesàro sums* $(C, 1)$ *of the two series are everywhere equal to the functions which the series represent, and for any value of x at which one of the series is convergent, that series converges to the value of the function.*

To prove the statement (1), we see by a change of variables that

$$\int_{-\pi}^{\pi} [f(t+x)\pm f(t-x)]\,g(t)\,dt = \pm\int_{-\pi}^{\pi} [g(t+x)\pm g(t-x)]\,f(t)\,dt;$$

* W. H. Young, *Proc. Roy. Soc.* vol. LXXXV (1911), p. 110.

if now $g(t)$ be of bounded variation, so also is $g(t+x) \pm g(t-x)$, considered as a function of x, and this property is preserved after multiplication by $f(t)$ and integration, since the property is clearly so preserved in the case of a monotone function. Therefore, when $g(t)$ is of bounded variation, the continuous functions of x which the Fourier's series represent are of bounded variation, and consequently the series converge everywhere to the values of those functions.

To prove the statement (2), we observe that the general term of either series does not exceed numerically $\frac{1}{2}(a_n^2 + b_n^2 + a_n'^2 + b_n'^2)$, which is the general term of a convergent series. This is seen from § 378, or by observing that if $f(x) = g(x)$, and $x = 0$, the series $\sum\limits_{n=1} (a_n^2 + b_n^2)$ is convergent, since it cannot oscillate. It follows that both series are absolutely convergent, and that the convergence is uniform, and they therefore converge to the values of the corresponding functions.

In case (3), the Cesàro sum of the series always exists and has the value of the function, since the functions are both everywhere continuous; and when either series converges, its sum is equal to the sum $(C, 1)$.

If we consider the point $x = 0$, we obtain the following extension of Fatou's form of Parseval's theorem (see § 378):

If $|f(x)|^p$, $|g(x)|^q$ are both summable in the interval $(-\pi, \pi)$, where p and q are positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, the series

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1} (a_n a_n' + b_n b_n')$$

converges $(C, 1)$ to the value $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$, and if the series be convergent, its sum has this value.

These theorems were given* by W. H. Young. An indication will be given in § 399 of a proof, due to M. Riesz, that the series in the last theorem is necessarily convergent, and thus that $\frac{1}{2}a_0 a_0' \sum\limits_{n=1} + (a_n a_n' + b_n b_n')$ converges, in the case specified, to $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$; the extension of Fatou's theorem being accordingly complete.

EXAMPLES

(1) The Fourier's coefficients corresponding to $f(x)$ being a_n, b_n, shew that

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \int_{-\pi}^{x} f(x)\, dx - \int_{-\pi}^{0} f(x)\, dx.$$

In § 360 it has been shewn that $\frac{1}{2}a_0' + \sum \frac{a_n \sin nx - b_n \cos nx}{n}$ converges uniformly to

* *Proc. Lond. Math. Soc.* (2), vol. XI (1912), p. 88.

$\int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x$. Taking the point $x=0$, $\frac{1}{2}a_0' - \sum_{n=1} \frac{b_n}{n}$ converges to $\int_{-\pi}^{0} f(x)\,dx$; therefore $\sum_{n=1} \frac{b_n}{n}$ converges to $\frac{1}{2}a_0' - \int_{-\pi}^{0} f(x)\,dx$, which is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \left\{\int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x\right\} - \int_{-\pi}^{0} f(x)\,dx$$

or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \int_{-\pi}^{x} f(x)\,dx - \int_{-\pi}^{0} f(x)\,dx.$$

(2*) If $0<q<1$, then, provided $f(x)$ has bounded variation in the neighbourhood of the point $x=0$,

$$\int_0^\infty x^{q-1} f_1(x)\,dx = \Gamma(q)\cos\tfrac{1}{2}q\pi \sum_{n=1}^{\infty} \frac{a_n}{n^q},$$

$$\int_0^\infty x^{q-1} f_2(x)\,dx = \Gamma(q)\sin\tfrac{1}{2}q\pi \sum_{n=1}^{\infty} \frac{b_n}{n^q},$$

where

$$f_1(x) = \tfrac{1}{2}[f(x)+f(-x)] - \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x)+f(-x)]\,dx,$$

$$f_2(x) = \tfrac{1}{2}[f(x)-f(-x)].$$

Shew that, if $\frac{1}{2}<q<1$, the result still holds if $\{f(x)\}^2$ is summable in the neighbourhood of the point $x=0$.

(3*) Prove that, provided $\{f(x)\}^2$ is summable in some neighbourhood of the point $x=0$, $\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\log(\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}x) f(x)\,dx = \sum_{n=1}^{\infty} \frac{a_n}{n}$. The Fourier's series corresponding to

$$\tfrac{1}{2}\log(\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x)$$

is $\sum_{n=1} \frac{\cos nx}{n}$. The function $\frac{1}{2}\log(\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}x)$ has its square summable in the interval $(-\pi,\pi)$, and it is of bounded variation, except in a neighbourhood of $x=0$. Thus, since $f(x)$ has its square summable in some neighbourhood of $x=0$, the conditions of the second theorem in § 388 are satisfied. Hence the result is obtained by applying Parseval's theorem to the two functions $f(x)$, $g(x)$. The necessary and sufficient conditions that, for any summable function, $\sum_{n=1} \frac{a_n}{n}$ should be convergent have been given† by Hardy and Littlewood.

(4*) Obtain expressions for

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2},\ \sum_{n=1}^{\infty}\frac{b_n}{n^2},\ \sum_{n=1}^{\infty}\frac{a_n}{n^3},\ \sum_{n=1}^{\infty}\frac{b_n}{n^3},\ \sum_{n=1}^{\infty}\frac{a_n}{n^{q+1}},\ \sum_{n=1}^{\infty}\frac{b_n}{n^{q+1}},\ \sum_{n=1}^{\infty}\frac{a_n}{n^{q+2}},\ \sum_{n=1}^{\infty}\frac{b_n}{n^{q+2}},$$

where $0<q<1$.

(5*) If $\frac{f(x)-f(-x)}{x}$ is summable in an interval which contains the point $x=0$, then

$$\sum_{n=1}^{\infty} b_n = \frac{1}{\pi}\int_0^\infty \frac{f(x)-f(-x)}{x}\,dx.$$

* See W. H. Young, *Proc. Roy. Soc.* (A), vol. LXXXV (1911), pp. 14–24, see also p. 415. There is an *hiatus* in the proof given (p. 19) of the result in Ex. 3, as the necessity for employing a theorem such as that in § 388 appears to have been overlooked. When the interval $(-\pi,\pi)$ is divided into two parts, there are four separate products to consider; and one of these requires the extension of Parseval's theorem given in § 388.

† *Math. Zeitschr.* vol. XIX (1924), p. 95.

392. We proceed to establish two theorems, the first of which may be regarded as a generalization of Parseval's theorem, and the second as a generalization of the Riesz-Fischer theorem (§ 379).

I. *If p be a number $\geqq 1$, and the function $f(x)$ be such that $|f(x)|^{1+\frac{1}{p}}$ is integrable (L) in the interval $(-\pi, \pi)$, and the Fourier's series corresponding to $f(x)$ be denoted by $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, then the series*

$$\left|\frac{a_0}{\sqrt{2}}\right|^{1+p} + \sum_{n=1}^{\infty} (|a_n|^{1+p} + |b_n|^{1+p})$$

is convergent, and its sum is

$$\leqq \frac{1}{\pi^p}\left\{\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}} dx\right\}^p.$$

II. *If, for the trigonometrical series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, the sum*

$$\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{n=1}^{\infty} (|a_n|^{1+\frac{1}{p}} + |b_n|^{1+\frac{1}{p}})$$

is convergent, for a value of p that is $\geqq 1$, then the trigonometrical series is a Fourier's series corresponding to a function $f(x)$, such that $|f(x)|^{1+p}$ is summable in $(-\pi, \pi)$; and the sum of the series of powers of the coefficients is

$$\geqq \frac{1}{\pi^{\frac{1}{p}}}\left\{\int_{-\pi}^{\pi} |f(x)|^{1+p} dx\right\}^{\frac{1}{p}}.$$

In case $p = 1$, the two theorems reduce to the equality

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

which is equivalent to Parseval's theorem. In this case, II is equivalent to the Riesz-Fischer theorem.

A slightly less general theorem than I is:

I′. *If $|f(x)|^q$ is integrable (L), where $1 < q \leqq 2$, then the series*

$$\sum_{n=1} (a_n^2 + b_n^2)^{\frac{q}{2(q-1)}}$$

is convergent.

This follows from I by taking account of the inequality

$$\left(\frac{a_n^2 + b_n^2}{2}\right)^{\frac{q}{2(q-1)}} \leqq \frac{(a_n^2)^{\frac{q}{2(q-1)}} + (b_n^2)^{\frac{q}{2(q-1)}}}{2}.$$

It follows from II, by taking account of the inequality

$$\left(\frac{a_n^2 + b_n^2}{2}\right)^{\frac{1}{2}q} \geqq \frac{|a_n^2|^{\frac{1}{2}q} + |b_n^2|^{\frac{1}{2}q}}{2},$$

that:

Let $g(x) = \phi_n(x)$, then $\int_{-\pi}^{\pi} \bar{f}(x)\,\phi_n(x)\,dx = \lim_{n\sim\infty}\int_{-\pi}^{\pi} f^{(n)}(x)\,\phi_n(x)\,dx$, for $n = 1, 2, 3, \ldots r$; and it follows that, for the function $\bar{f}(x)$, the r constants $\bar{c}_n$ satisfy the condition $\sum_{n=1}^{n=r} |\bar{c}_n|^{1+p} = 1$; hence we have

$$\int_{-\pi}^{\pi} |\bar{f}(x)|^{1+\frac{1}{p}}\,dx \geqq L.$$

From the two inequalities we find that $\int_{-\pi}^{\pi} |\bar{f}(x)|^{1+\frac{1}{p}}\,dx = L$, and $\bar{f}(x)$ is a function whose r constants $\bar{c}_1, \bar{c}_2, \ldots \bar{c}_r$ satisfy the condition

$$\sum_{n=1}^{n=r} |\bar{c}_n|^{1+p} = 1,$$

and thus $\bar{f}(x)$ cannot be a null-function, and we have consequently $L > 0$.

The existence of the minimal function $\bar{f}(x)$ having been established, it is clear that it is the minimal function for

$$\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}}\,dx \Big/ \left\{\sum_{n=1}^{n=r} |c_n|^{1+p}\right\}^{\frac{1}{p}}$$

when the condition $\sum_{n=1}^{n=r} |c_n|^{1+p} = 1$ is no longer imposed. For if $f(x)$ could be such that this ratio were $< L$, writing

$$\psi(x) = \frac{f(x)}{\left\{\sum_{n=1}^{n=r} |c_n|^{1+p}\right\}^{\frac{1}{1+p}}},$$

we should have, for the constants c_n' of $\psi(x)$,

$$\sum_{n=1}^{n=r} |c_n'|^{1+p} = 1, \text{ and } \int_{-\pi}^{\pi} |\psi(x)|^{1+\frac{1}{p}}\,dx < L,$$

which is impossible.

Let $$f(x) = \bar{f}(x) + \lambda h(x), \quad c_n = \bar{c}_n + \lambda c_n^{(h)},$$

where $h(x)$ is any chosen function, of type $[L^{1+\frac{1}{p}}]$, and $c_n^{(h)}$ are the constants corresponding to $h(x)$. The ratio must then satisfy the necessary condition that $\lambda = 0$ makes it a minimum, namely, that its differential coefficient with respect to λ vanishes when $\lambda = 0$; we thus have

$$\int_{-\pi}^{\pi} |\bar{f}(x)|^{\frac{1}{p}} \operatorname{sign}[\bar{f}(x)]\,h(x)\,dx = \int_{-\pi}^{\pi} |\bar{f}(x)|^{1+\frac{1}{p}}\,dx \,.\sum_{n=1}^{n=r} |\bar{c}_n|^{p} \,.\, c_n^{(h)} \operatorname{sign}[\bar{c}_n],$$

where sign $(z) = 1, -1$, or 0, according as z is positive, negative, or zero. If we give $h(x)$ the special values $\phi_n(x)$, where $n = 1, 2, 3, \ldots r$, successively, we obtain the values of the r constants C_n corresponding to the function

$$|\bar{f}(x)|^{\frac{1}{p}} \operatorname{sign}[\bar{f}(x)],$$

which we may denote by $F(x)$; the value of C_n is thus

$$\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}} dx \,.\, |\bar{c}_n|^p \operatorname{sign}[\bar{c}_n].$$

If we assign to $h(x)$ the value of a function which is orthogonal to all the functions $\phi_1(x), \phi_2(x), \ldots \phi_r(x)$; and for which therefore $c_n^{(h)} = 0$, for $n = 1, 2, 3, \ldots r$, we see that this function is also orthogonal to $F(x)$. Therefore all the Fourier constants, after the first r, corresponding to the function $F(x)$ vanish, and thus the Fourier's series for $F(x)$ converges everywhere, being a finite trigonometrical polynomial. It follows that, almost everywhere,

$$F(x) = |f(x)|^{\frac{1}{p}} \operatorname{sign}[f(x)] = \int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}} dx \,.\, \sum_{n=1}^{n=r} |c_n|^p \operatorname{sign}[\bar{c}_n]\, \phi_n(x);$$

and from this we have

$$\int_{-\pi}^{\pi} \{F(x)\}^2 dx = \int_{-\pi}^{\pi} |f(x)|^{\frac{2}{p}} dx = \sum_{n=1}^{n=r} |c_n|^{2p} \left\{\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}}\right\}^2.$$

Let $p_1 = 2p - 1$, then $1 + \dfrac{1}{p_1} = t\left(1 + \dfrac{1}{p}\right) + (1-t)\dfrac{2}{p}$, where $t = \dfrac{2(p-1)}{2p-1}$; we have then

$$\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p_1}} dx = \int_{-\pi}^{\pi} |f(x)|^{t\left(1+\frac{1}{p}\right)} |f(x)|^{\frac{2}{p}(1-t)} dx$$

$$\leqq \left\{\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}}\right\}^t \left\{\int_{-\pi}^{\pi} |f(x)|^{\frac{2}{p}} dx\right\}^{1-t}$$

On substitution of the value of $\int_{-\pi}^{\pi} |f(x)|^{\frac{2}{p}} dx$, obtained above, we have

$$\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}} dx \geqq \left\{\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p_1}} dx\right\}^{\frac{p_1}{p_1+1}} \Big/ \left\{\sum_{n=1}^{n=r} |\bar{c}_n|^{1+p_1}\right\}^{\frac{1}{p_1+1}};$$

the expression on the right-hand side is not less than the minimum of

$$\left\{\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p_1}} dx\right\}^{\frac{p_1}{1+p_1}}$$

for all functions for which the first r constants satisfy the condition

$$\sum_{n=1}^{n=r} |c_n|^{1+p_1} = 1.$$

Denoting the minimum of $\int_{-\pi}^{\pi} |f(x)|^{1+\frac{1}{p}} dx$, for all functions for which the first r constants c_n satisfy the condition $\sum_{n=1}^{n=r} |c_n|^{1+p} = 1$, by u_p, we have

$$u_p \geqq u_{p_1}^{\frac{p_1}{p_1+1}}.$$

Similarly it appears that

$$u_{p_1} \geqq u_{p_2}^{\frac{p_2}{p_2+1}}, \ldots u_{p_s} \geqq u_{p_1}^{\frac{p_s}{p_s+1}}; \text{ where } p_s = 2p_{s-1} - 1.$$

The constants c_n, for a function $f(x)$, are numerically $\leqq \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} | f(x) | \, dx$, which is

$$\leqq \frac{1}{\sqrt{\pi}} \left\{ \int_{-\pi}^{\pi} | f(x) |^{1+\frac{1}{p_s}} dx \right\}^{\frac{p_s}{p_s+1}} (2\pi)^{\frac{1}{p_s+1}};$$

and if $\bar{f}(x)$ be chosen to be the minimal function for all the functions of the prescribed type whose coefficients c_n satisfy the condition

$$\sum_{n=1}^{n=r} | c_n |^{1+p_s} = 1,$$

we have
$$\bar{c}_n \leqq \frac{1}{\sqrt{\pi}} (2\pi)^{\frac{1}{p_s+1}} u_{p_s}^{\frac{p_s}{p_s+1}}.$$

From this we have $\quad 1 = \sum_{n=1}^{n=r} | \bar{c}_n |^{1+p_s} \leqq 2\pi r u_{p_s}^{p_s} / \pi^{\frac{1+p_s}{2}}.$

Now p_s increases indefinitely with s, hence if $u_{p_s}/\sqrt{\pi}$ were less than 1, for sufficiently large values of s, this inequality would be impossible. It follows that $u_{p_s} \geqq \sqrt{\pi}$, for all values of s; and we have

$$u_p \geqq \left(\frac{u_{p_s}}{\sqrt{\pi}} \right)^{P_s} (\sqrt{\pi})^{P_s},$$

for every value of s, where P_s denotes $\frac{p_1 p_2 \ldots p_s}{(1+p_1)(1+p_2)\ldots(1+p_s)}$; it can thus be inferred that $u_p \geqq (\sqrt{\pi})^{\lim_{s\sim\infty} P_s}$, assuming that the formation of the sequence $\{p_s\}$ is continued indefinitely.

Now

$$1 - \frac{1}{p} = \frac{p_1 - 1}{p_1 + 1} = \frac{p_1}{p_1 + 1} \cdot \left(1 - \frac{1}{p_1}\right) = \frac{p_1}{p_1+1} \cdot \frac{p_2}{p_2+1} \left(1 - \frac{1}{p_3}\right) \ldots;$$

hence
$$\lim_{s\sim\infty} \frac{p_1 p_2 \ldots p_s}{(p_1+1)\ldots(p_s+1)} = 1 - \frac{1}{p},$$

and thus we have $u_p \geqq \pi^{\frac{1}{2} - \frac{1}{2p}}$; therefore

$$\int_{-\pi}^{\pi} | f(x) |^{1+\frac{1}{p}} dx \geqq \pi^{\frac{1}{2} - \frac{1}{2p}} \left\{ \sum_{n=1}^{n=r} | c_n |^{1+p} \right\}^{\frac{1}{p}},$$

where c_n are the first r constants, corresponding to $f(x)$.

If the constants c_n are expressed in terms of the ordinary Fourier's constants a_n and b_n, we obtain a_n or b_n by multiplying c_n by $1/\sqrt{\pi}$, except

that $a_0 = c_0\sqrt{\frac{2}{\pi}}$. We have therefore, since the above inequality holds for every value of r, the result of Theorem I, that

$$\left(\frac{a_0}{\sqrt{2}}\right)^{1+p} + \sum_{n=1}^{\infty}(|\,a_n\,|^{1+p} + |\,b_n\,|^{1+p})$$

converges to a value $\leq \frac{1}{\pi^p}\left\{\int_{-\pi}^{\pi}|\,f(x)\,|^{1+\frac{1}{p}}\right\}^p$.

393. In order to prove Theorem II, let $a_0, a_1, b_1, \ldots$ be such that the series

$$\left(\frac{a_0}{\sqrt{2}}\right)^{1+\frac{1}{p}} + \sum_{n=1}^{\infty}(|\,a_n\,|^{1+\frac{1}{p}} + |\,b_n\,|^{1+\frac{1}{p}}),$$

for a value of $p \geqq 1$, is convergent; and let $f_r(x)$ denote the sum

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{n=r}(a_n \cos nx + b_n \sin nx).$$

Let $A_0, A_1, B_1, \ldots$ be such that $\left(\frac{A_0}{\sqrt{2}}\right)^{1+p} + \sum_{n=1}^{\infty}(|\,A_n\,|^{1+p} + |\,B_n\,|^{1+p})$ is convergent, which will, in accordance with Theorem I, be the case if $A_0, A_1, B_1, \ldots$ are the Fourier's constants corresponding to a function $\phi(x)$, such that $|\,\phi(x)\,|^{1+\frac{1}{p}}$ is summable. We have then

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)f_r(x)\,dx\right| = \left|\tfrac{1}{2}a_0A_0 + \sum_{n=1}^{n=r}(a_nA_n + b_nB_n)\right|$$

$$\leq \left[\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{n=1}^{n=r}(|\,a_n\,|^{1+\frac{1}{p}} + |\,b_n\,|^{1+\frac{1}{p}})\right]^{\frac{p}{p+1}}$$

$$\times\left[\left(\frac{A_0}{\sqrt{2}}\right)^{1+p} + \sum_{n=1}^{n=r}(|\,A_n\,|^{1+p} + |\,B_n\,|^{1+p})\right]^{\frac{1}{1+p}}$$

Employing Theorem I, the expression in the second bracket on the right-hand side is

$$\leq \left[\frac{1}{\pi^p}\left\{\int_{-\pi}^{\pi}|\,\phi(x)\,|^{1+\frac{1}{p}}\,dx\right\}^p\right]^{\frac{1}{1+p}}$$

Assigning to $\phi(x)$ the value $|\,f_r(x)\,|^p \operatorname{sign}\{f_r(x)\}$, we obtain

$$\frac{1}{\pi}\left\{\int_{-\pi}^{\pi}|\,f_r(x)\,|^{p+1}\,dx\right\}^{\frac{1}{1+p}} \leq \frac{1}{\pi^{\frac{p}{1+p}}}\left[\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{n=1}^{n=r}(|\,a_n\,|^{1+\frac{1}{p}} + |\,b_n\,|^{1+\frac{1}{p}})\right]^{\frac{p}{p+1}}$$

Thus $\int_{-\pi}^{\pi}|\,f_r(x)\,|^{p+1}\,dx$ is, for all values of r, less than a fixed positive number. Accordingly the sequence $\{f_r(x)\}$ contains a sequence which

converges weakly, for the exponent $p+1$ (see § 175), to a function $f(x)$, which is such that

$$\int_{-\pi}^{\pi} f(x) \frac{\cos}{\sin} rx\,dx = \lim_{r\sim\infty} \int_{-\pi}^{\pi} f_r(x) \frac{\cos rx}{\sin rx}\,dx = \frac{a_r}{b_r}.$$

Therefore $a_0, a_1, b_1, \ldots$ are the Fourier's constants corresponding to the function $f(x)$; and in accordance with the properties of weak convergence, we have

$$\int_{-\pi}^{\pi} |f(x)|^{1+p}\,dx \leqq \underline{\lim}_{r\sim\infty} \int_{-\pi}^{\pi} |f_r(x)|^{1+p}\,dx.$$

It follows that

$$\int_{-\pi}^{\pi} |f(x)|^{1+p}\,dx \leqq \pi \left[\left| \frac{a_0}{\sqrt{2}} \right|^{1+\frac{1}{p}} + \sum_{n=1}^{n=\infty} \left(|a_n|^{1+\frac{1}{p}} + |b_n|^{1+\frac{1}{p}} \right) \right]^p$$

or $$\left| \frac{a_0}{\sqrt{2}} \right|^{1+\frac{1}{p}} + \sum_{n=1}^{\infty} \left(|a_n|^{1+\frac{1}{p}} + |b_n|^{1+\frac{1}{p}} \right) \geqq \frac{1}{\pi^p} \left\{ \int_{-\pi}^{\pi} |f(x)|^{1+p}\,dx \right\}^{\frac{1}{p}},$$

which is the result stated in Theorem II.

394. The following Lemma was (*loc. cit.*) established by W. H. Young:

If $g(x)$ is summable in the linear interval (a, b), and $f(x)$ is summable in every finite interval, $\int_a^b f(x+t)\,g(t)\,dt$ is a function of x which exists for almost all values of x in any finite interval, and is summable in such interval.

Since each of the functions $f(t)$, $g(t)$ can be expressed as the difference of two non-negative functions, it is clearly sufficient to prove the theorem for non-negative functions, $f(x+t)$, $g(t)$.

The repeated integral $\int_a^b dt \int_a^x f(t+u)\,g(t)\,du$ is equal to

$$\int_a^b [F(x+t) - F(a+t)]\,g(t)\,dt,$$

which exists, since $F(x+t)$, $F(a+t)$ are continuous functions of t, where $F(x)$ denotes $\int_a^x f(x)\,dx$. It follows, by applying a theorem in I, § 429, that $f(t+u)\,g(t)$ is summable in the domain $[a \leq u \leq x,\ a \leqq t \leqq b]$; consequently the repeated integral is equal to $\int_a^x du \int_a^b f(t+u)\,g(t)\,dt$. It follows that $\int_a^b f(t+u)\,g(t)\,dt$ exists for almost all values of u in a finite interval, and that it is a summable function of u.

395. To apply the above Lemma, let $f(x)$, $g(x)$ be summable periodic functions, of period 2π, and let a_n, b_n and a_n', b_n' be respectively their sets

of Fourier's coefficients. We have then, since the Fourier's series corresponding to $F(t+x)-\frac{1}{2}a_0(t+x)$ is

$$C+\sum_{n=1}\frac{a_n\sin n(t+x)-b_n\cos n(t+x)}{n},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}F(x+t)\,g(t)\,dt=C+\tfrac{1}{2}a_0a_0'x+\sum_{n=1}\frac{a_n\cos nx+b_n\sin nx}{n}b_n'$$
$$+\sum_{n=1}\frac{a_n\sin nx-b_n\cos nx}{n}a_n'.$$

The series on the right-hand side being the Fourier's series of an integral in x, we see that the differentiated series

$$\tfrac{1}{2}a_0a_0'+\sum_{n=1}(a_na_n'+b_nb_n')\cos nx-\sum_{n=1}(a_nb_n'-a_n'b_n)\sin nx$$

is the Fourier's series corresponding to the function $\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\,g(t)$, and the Theorem I, of § 392, may be applied to this function. We have the known inequality

$$\int_a^b|uvw|\,dt\leq\left\{\int_a^b|u|^{\frac{1}{\alpha}}dt\right\}^{\alpha}\left\{\int_a^b|v|^{\frac{1}{\beta}}dt\right\}^{\beta}\left\{\int_a^b|w|^{\frac{1}{\gamma}}dt\right\}^{\gamma},$$

where $\alpha+\beta+\gamma=1$, $\alpha>0$, $\beta>0$, $\gamma>0$. Let $\alpha=1-\frac{1}{\mu}$, $\beta=1-\frac{1}{\lambda}$, $\gamma=\frac{1}{\lambda}+\frac{1}{\mu}-1$, where $\lambda>1$, $\mu>1$; and let $|u|=|u|^{\lambda\alpha}$, $|v|=|\bar{v}|^{\mu\beta}$, $|w|=|u|^{1-\lambda\alpha}|v|^{1-\mu\beta}$; the inequality then becomes

$$\left|\int_a^b uv\,dt\right|\leq\int_a^b|uv|\,dt\leq\left\{\int_a^b|u|^{\lambda}|v|^{\mu}dt\right\}^{\frac{1}{\lambda}+\frac{1}{\mu}-1}$$
$$\left\{\int_a^b|u|^{\lambda}dt\right\}^{1-\frac{1}{\mu}}\left\{\int_a^b|\bar{v}|^{\mu}dt\right\}^{1-\frac{1}{\lambda}},$$

where $\lambda>1$, $\mu>1$, $\frac{1}{\lambda}+\frac{1}{\mu}-1>0$.

Now let it be assumed that $|f(x)|^{1+p}$, $|g(x)|^{1+q}$ are summable, for $1+p=\lambda$, $1+q=\mu$, $\frac{1}{1+p}+\frac{1}{1+q}>1$; if we put $b=\pi$, $a=-\pi$, $u=f(x+t)$, $\bar{v}=g(t)$, we have

$$\left|\int_{-\pi}^{\pi}f(x+t)\,g(t)\,dt\right|^{\frac{(1+p)(1+q)}{1-pq}}$$
$$\leq\left\{\int_{-\pi}^{\pi}|f(x+t)|^{1+p}|g(t)|^{1+q}dt\right\}$$
$$\left\{\int_{-\pi}^{\pi}|f(x+t)|\,dx\right\}^{\frac{q(1+p)}{1-pq}}\left\{\int_{-\pi}^{\pi}|g(t)|\,dx\right\}^{\frac{p(1+q)}{1-pq}}.$$

The expression on the right-hand side is a summable function of x, by applying the Lemma to the first factor; it follows that

$$\left|\int_{-\pi}^{\pi} f(x+t)\, g(t)\, dt\right|^{\frac{(1+p)(1+q)}{1-pq}}$$

is summable, provided $pq < 1$, $p > 0$, $q > 0$.

Applying the Theorem I, of § 392, we have then the following extension of Parseval's theorem:

If p, q be positive numbers such that $pq < 1$, $\frac{(1+p)(1+q)}{1-pq} \leqq 2$, the series

$$\left|\frac{a_0 a_0'}{\sqrt{2}}\right|^{k+1} + \sum_{n=1}^{\infty}\{|a_n a_n' + b_n b_n'|^{k+1} + |a_n b_n' - b_n a_n'|^{k+1}\}$$

converges to a sum

$$\leqq \frac{1}{\pi^k}\left[\int_{-\pi}^{\pi} dx \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, g(t)\, dt\right|^{1+\frac{1}{k}}\right]^k,$$

where $k = \frac{(1-pq)}{p+q+2pq}$, provided $|f(x)|^{1+p}$, $|g(x)|^{1+q}$ are summable, and the constants a_n, b_n are the Fourier's coefficients for $f(x)$, and a_n', b_n' those for $g(x)$.

In case $k = 1$, the relation is that of equality; thus for example, if $|f(x)|^{\frac{4}{3}}$, $|g(x)|^{\frac{4}{3}}$ are both summable, the series

$$\tfrac{1}{2}a_0^2 a_0'^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)(a_n'^2 + b_n'^2)$$

converges to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx \left|\int_{-\pi}^{\pi} f(x+t)\, g(t)\, dt\right|^2.$$

396. The following theorem has been given* by F. Riesz:

The necessary and sufficient condition that the trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$$

is the Fourier's series of a function $f(x)$ such that $|f(x)|^p$ is summable, where p is a positive number > 1, is that a positive number M should exist such that, for every value of n,

$$\left|\tfrac{1}{2}a_0 a_0' + \sum_{r=1}^{r=n}(a_r a_r' + b_r b_r')\right| \leqq M^{\frac{1}{q}}\left\{\int_{-\pi}^{\pi} |g_n(x)|^q\, dx\right\}^{\frac{1}{q}},$$

where $a_0', a_1', b_1', \ldots$ is an arbitrarily chosen set of constants. The constant M is independent of $n, a_0', a_1', b_1', \ldots$, and $g_n(x)$ denotes the sum

$$\tfrac{1}{2}a_0' + \sum_{r=1}^{r=n}(a_r' \cos rx + b_r' \sin rx), \text{ and } \frac{1}{q} + \frac{1}{p} = 1.$$

* *Math. Annalen*, vol. LXIX (1910), pp. 469–474.

It may be observed that the method of proof is such that the theorem holds good when any bounded set of normal orthogonal functions $\{\phi_n(x)\}$ is substituted for the special set

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\cos x, \quad \frac{1}{\sqrt{\pi}}\sin x, \ldots.$$

That the condition stated in the theorem is necessary follows at once from the inequality

$$\left|\int_{-\pi}^{\pi} f(x)\, g_n(x)\, dx\right| \leq \left\{\int_{-\pi}^{\pi} |f(x)|^p\, dx\right\}^{\frac{1}{p}} \left\{\int_{-\pi}^{\pi} |g_n(x)|^q\, dx\right\}^{\frac{1}{q}}.$$

In order to prove the sufficiency of the theorem, let

$$\tfrac{1}{2}a_0 a_0' + \sum_{r=1}^{r=n} (a_r a_r' + b_r b_r')$$

be denoted by U_n, and let the constants $a_0', a_1', b_1', \ldots a_n', b_n'$, be subjected to the condition $\int_{-\pi}^{\pi} |g_n(x)|^q\, dx = 1$. We proceed to determine the maximum value of $|U_n|$ for all values of the constants $a_0', a_1', \ldots b_n'$ such that $\int_{-\pi}^{\pi} |g_n(x)|^q\, dx = 1$. This maximum value must then satisfy the condition $|U_n| \leq M^{\frac{1}{q}}$. We equate to zero the partial differential coefficients with respect to a_r', of

$$|U_n|^q - \lambda \int_{-\pi}^{\pi} |g_n(x)|^q\, dx;$$

we have then

$$a_r . |U_n|^{q-1} . \operatorname{sign} U_n - \lambda \int_{-\pi}^{\pi} \cos rx\, |g_n(x)|^{q-1} . \operatorname{sign} g_n(x)\, dx = 0.$$

A similar equation holds for b_r. Taking these equations for $r = 0, 1, 2, \ldots n$; multiplying them by a_r', b_r' and adding, we find that $|U_n|^q - \lambda = 0$. We thus find that

$$a_r = U_n \int_{-\pi}^{\pi} \cos rx\, |g_n(x)|^{q-1} . \operatorname{sign} g_n(x)\, dx,$$

$$b_r = U_n \int_{-\pi}^{\pi} \sin rx\, |g_n(x)|^{q-1} . \operatorname{sign} g_n(x)\, dx.$$

Let $f_n(x)$ denote $\pi U_n . |g_n(x)|^{q-1} . \operatorname{sign} g_n(x)$; we have then

$$\int_{-\pi}^{\pi} |f_n(x)|^p\, dx = \pi^p |U_n|^p \int_{-\pi}^{\pi} |g_n(x)|^q\, dx \leq \pi^p M^{p-1}.$$

Also $\qquad a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(x) \cos rx\, dx, \quad b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(x) \sin rx\, dx.$

Assuming that n may have all integral values, the sequence $\{f_n(x)\}$ converges weakly, with exponent p, to a function $f(x)$, such that $|f(x)|^p$ is summable in the interval $(-\pi, \pi)$.

We have also

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \frac{\cos}{\sin} rx\,dx = \frac{1}{\pi}\lim_{n\sim\infty}\int_{-\pi}^{\pi} f_n(x)\frac{\cos}{\sin} rx\,dx = \frac{a_r}{b_r};$$

therefore $a_0, a_1, b_1, \ldots$ are the Fourier's constants for the function $f(x)$. The sufficiency of the theorem has now been established.

A proof of the Riesz-Fischer theorem (§ 379) may be obtained from the above theorem.

Taking $p = q = \frac{1}{2}$, we have,

$$\left\{\tfrac{1}{2}a_0 a_0' + \sum_{r=1}^{r=n}(a_r a_r' + b_r b_r')\right\}^2$$

$$\leq \left[\tfrac{1}{2}a_0^2 + \sum_{r=1}^{r=n}(a_r^2 + b_r^2)\right]\left[\tfrac{1}{2}a_0'^2 + \sum_{r=1}^{r=n}(a_r'^2 + b_r'^2)\right]$$

$$\leq \frac{1}{\pi}\left[\tfrac{1}{2}a_0^2 + \sum_{r=1}^{r=n}(a_r^2 + b_r^2)\right]\int_{-\pi}^{\pi} |g_n(x)|^2\,dx.$$

It follows that the condition in the above theorem is satisfied if

$$\tfrac{1}{2}a_0^2 + \sum_{r=1}(a_r^2 + b_r^2)$$

is convergent; and therefore a function $f(x)$ exists such that $|f(x)|^2$ is summable, and such that its Fourier constants are the numbers $a_0, a_1, b_1, \ldots$.

M. RIESZ' EXTENSION OF PARSEVAL'S THEOREM

397. It will be shewn that the condition that $\int_{-\pi}^{\pi} |f_n(x)|^p\,dx$ is a bounded function of n, where $f_n(x)$ denotes $\frac{1}{2}a_0 + \sum_{r=1}^{r=n}(a_r \cos rx + b_r \sin rx)$, and p is a number > 1, is a sufficient condition that the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$$

should be the Fourier's series corresponding to a function $f(x)$, such that $|f(x)|^p$ is summable in the interval $(-\pi, \pi)$.

In accordance with the theorem of § 176, a subsequence $\{f_{n_r}(x)\}$ of $\{f_n(x)\}$ exists, which converges weakly with exponent p to a function $f(x)$ such that $|f(x)|^p$ is summable. We have then

$$\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \lim_{r\sim\infty}\int_{-\pi}^{\pi} f_{n_r}(x)\,g(x)\,dx,$$

where $g(x)$ is any function such that $|g(x)|^q$ is summable.

Let $g(x) = \frac{\cos}{\sin} mx$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\cos}{\sin} mx\,dx = \lim_{r\sim\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f_{n_r}(x)\frac{\cos}{\sin} mx\,dx = \frac{a_m}{b_m};$$

and thus $f(x)$ is such that its Fourier constants are $a_0, a_1, b_1, \ldots$. Therefore the sufficiency of the condition has been established.

It is however possible to shew that the condition stated is necessary, and thus that:

The necessary and sufficient condition that $a_0, a_1, b_1, a_2, b_2, \ldots$ are the Fourier constants corresponding to a function $f(x)$, such that $|f(x)|^p$ is summable in $(-\pi, \pi)$, where p is a number > 1, is that $\int_{-\pi}^{\pi} |f_n(x)|^p dx$ should be a bounded function of n; where $f_n(x)$ denotes

$$\tfrac{1}{2}a_0 + \sum_{r=1}^{r=n} (a_r \cos rx + b_r \sin rx).$$

We have

$$f_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \frac{\sin \frac{1}{2}(2n+1)(t-x)}{\sin \frac{1}{2}(t-x)} dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sin n(t-x) \cot \frac{t-x}{2} dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cos n(t-x)\, dt;$$

hence $$f_n(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cos n(t-x)\, dt$$

may be expressed in the form

$$\frac{1}{2\pi} \cos nx \int_{-\pi}^{\pi} f(t) \sin nt \cot \frac{t-x}{2} dt - \frac{1}{2\pi} \sin nx \int_{-\pi}^{\pi} f(t) \cos nt \cot \frac{t-x}{2},$$

where the integrals $$\int_{-\pi}^{\pi} f(t) \frac{\cos}{\sin} nt \cot \frac{t-x}{2} dt$$

are taken to denote

$$\lim_{\epsilon \to 0} \left\{ \int_{-\pi}^{x-\epsilon} f(t) \frac{\cos}{\sin} nt \cot \frac{t-x}{2} dt + \int_{x+\epsilon}^{\pi} f(t) \frac{\cos}{\sin} nt \cot \frac{t-x}{2} dt \right\},$$

and are not necessarily L-integrals.

In connection with the theory of the series allied with a Fourier's series, an outline of a proof has been given* by M. Riesz that, if $|f(x)|^p$ is summable, for a value of p, > 1, and if $\bar{f}(x)$ denotes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cot \frac{t-x}{2} dt,$$

then $|\bar{f}(x)|^p$ is summable, and

$$\int_{\pi}^{\pi} |\bar{f}(x)|^p dx \leq M \int_{-\pi}^{\pi} |f(x)|^p dx,$$

where M is independent of the particular function $f(x)$ and depends only on p. This theorem will be assumed here. Applying it to the two functions $f(x) \cos nx$, $f(x) \sin nx$, we have

$$|f_n(x)| < k + |\bar{f}_1(x)| + |\bar{f}_2(x)|,$$

* *Comptes Rendus*, vol. CLXXVI (1924), p. 1464; and *Proc. Lond. Math. Soc.* (2), vol. XXII (1924), *Records*, p. iv.

where $k = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\, dt$, and where $\bar{f}_1(x)$, $\bar{f}_2(x)$ denote respectively the integrals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cot\frac{t-x}{2} \begin{matrix}\cos\\ \sin\end{matrix}\, nt\,dt.$$

It follows that $\int_{-\pi}^{\pi} |f_n(x)|^p\, dx$ does not exceed a fixed multiple of

$$2\pi k^p + \int_{-\pi}^{\pi} |\bar{f}_1(x)|^p\, dx + \int_{-\pi}^{\pi} |\bar{f}_2(x)|^p\, dx.$$

Employing M. Riesz' theorem, and the fact that k^p does not exceed a fixed multiple of $\int_{-\pi}^{\pi} |f(x)|^p\, dx$, it is now seen that

$$\int_{-\pi}^{\pi} |f_n(x)|^p\, dx \leqq \lambda \int_{-\pi}^{\pi} |f(x)|^p\, dx,$$

where λ depends only upon p. It has now been established that the condition in the theorem is necessary, as well as sufficient.

An earlier theorem, of a similar kind, in which $S_n(x)$ the nth Cesàro sum $(C, 1)$ was employed, instead of $f_n(x)$, has been given* by W. H. and G. C. Young.

398. The following theorem, stated† by M. Riesz, will now be established:

If $\int_{-\pi}^{\pi} |f(x)|^p\, dx$ *has a finite value, for a value of* p (> 1), *then*

$$\lim_{n\sim\infty}\int_{-\pi}^{\pi} |f(x) - f_n(x)|^p\, dx = 0,$$

where $f_n(x)$ *denotes a partial sum of the Fourier's series for* $f(x)$.

The proof of this theorem given here is due substantially to Littlewood.

We have (see § 365)

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} \{f(x+t) - f(x)\}\, \chi_n(t)\, dt,$$

where $\chi_n(t)$ satisfies the conditions

$$\chi_n(t) \geqq 0, \quad \int_{-\pi}^{\pi} \chi_n(t)\, dt = 1, \quad \lim_{n\sim\infty}\left\{\int_{-\pi}^{\pi} \chi_n(t)\, dt - \int_{-\delta}^{\delta} \chi_n(t)\, dt\right\} = 0.$$

Employing Hölder's inequality, we have

$$|S_n(x) - f(x)| \leqq \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, \{\chi_n(t)\}^{\frac{1}{p}} \{\chi_n(t)\}^{\frac{1}{q}}\, dt$$

$$\leqq \left[\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p\, \chi_n(t)\, dt\right]^{\frac{1}{p}} \left[\int_{-\pi}^{\pi} \chi_n(t)\, dt\right]^{\frac{1}{q}},$$

* *Quarterly Journal*, vol. XLIV (1913), p. 57; see also W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. XI (1912), p. 89.

† *Comptes Rendus*, vol. CLXXVI (1924), p. 1464.

or $\quad | S_n (x) - f(x) |^p \leqq \int_{-\pi}^{\pi} | f(x+t) - f(x) |^p \chi_n (t)\, dt \left[\int_{-\pi}^{\pi} \chi_n (t)\, dt \right]^{p-1};$

hence $$\int_{-\pi}^{\pi} | S_n (x) - f(x) |^p \, dx \leqq \int_{-\pi}^{\pi} \Phi (t) \chi_n (t)\, dt,$$

where $\Phi (t)$ denotes $\quad \displaystyle\int_{-\pi}^{\pi} | f(x+t) - f(x) |^p \, dx,$

which is a bounded function of t.

In order to prove the theorem, we can divide the integral on the right-hand side into three parts, over the intervals $(-\delta, \delta)$, $(-\pi, -\delta)$, (δ, π). Since $\Phi (t)$ is bounded, and the limit of the integral $\chi_n (t)$ over the intervals $(-\pi, -\delta)$, (δ, π) is zero, as $n \sim \infty$, we have only to consider

$$\int_{-\delta}^{\delta} \Phi (t) \chi_n (t)\, dt;$$

this is less than $M \displaystyle\int_{-\delta}^{\delta} \chi_n (t)\, dt$, where M is the maximum of $\Phi (t)$ in the interval $(-\delta, \delta)$, and thus the integral is less than M. It can be shewn that, by choosing δ sufficiently small, M becomes arbitrarily small.

We have in fact to prove that

$$\lim_{t \sim 0} \int_{-\pi}^{\pi} | f(x+t) - f(x) |^p \, dx = 0;$$

and this has been shewn in I, § 433, to hold in the case $p = 2$. For general values of $p\ (> 1)$, it has been shewn in § 173 that a continuous function $\phi (x)$ can be so determined that

$$\int_{-\pi}^{\pi} | f(x+t) - \phi (x+t) |^p \, dx < \epsilon, \text{ and } \int_{-\pi}^{\pi} | f(x) - \phi (x) |^p < \epsilon;$$

and since

$$\int_{-\pi}^{\pi} | f(x+t) - f(x) |^p \, dx \leqq 3^{p-1} \int_{-\pi}^{\pi} | f(x+t) - \phi (x+t) |^p \, dx$$
$$+ 3^{p-1} \int_{-\pi}^{\pi} | \phi (x+t) - \phi (x) |^p \, dx + 3^{p-1} \int_{-\pi}^{\pi} | \phi (x) - f(x) |^p,$$

we have $$\int_{-\pi}^{\pi} | f(x+t) - f(x) |^p \, dx \leqq 2\epsilon . 3^{p-1} + \epsilon,$$

provided t is sufficiently small. Therefore $\Phi (t)$ converges to zero with t, and consequently M is arbitrarily small, if δ be properly chosen. It has now been shewn that

$$\lim_{n \sim \infty} \int_{-\pi}^{\pi} | S_n (x) - f(x) |^p \, dx = 0.$$

If now $\gamma_n (x)$ denotes the sum of the first $2n + 1$ terms of the Fourier's series for $f(x) - S_{n_1} (x)$, where n_1 is such that

$$\int_{-\pi}^{\pi} | S_n (x) - f(x) |^p \, dx < \eta, \text{ for } n \geqq n_1,$$

we have $\int |\psi_n(x)|^p\,dx < A\eta$, where A depends only on p. But, for $n > n_1$, we have $f_n(x) = S_{n_1}(x) + \psi_n(x)$; hence

$$\int_{-\pi}^{\pi} |f(x) - f_n(x)|^p\,dx \leqq 2^{p-1}\int_{-\pi}^{\pi} |f(x) - S_{n_1}(x)|^p\,dx + 2^{p-1}\int_{-\pi}^{\pi} |\psi_m(x)|^p\,dx$$
$$\leqq 2^{p-1}\eta + 2^{p-1}A\eta.$$

Since η is arbitrary, it follows that $\lim_{n\sim\infty}\int_{-\pi}^{\pi} |f(x) - f_n(x)|^p\,dx = 0$. It follows from the theorem that, if $|f(x)|^p$ is summable in $(-\pi, \pi)$, then

$$\lim_{m\sim\infty,\, n\sim\infty}\int_{-\pi}^{\pi} |f_m(x) - f_n(x)|^p\,dx = 0.$$

Kolmogoroff has shewn* that, for any summable function $f(x)$,

$$\lim_{n\sim\infty}\int_{-\pi}^{\pi} |f(x) - f_n(x)|^{1-\epsilon}\,dx = 0,$$

where $0 < \epsilon < 1$.

399. The theorem of § 398 may now be employed to prove that:

If $|f(x)|^p$, $|g(x)|^q$ *are both summable in* $(-\pi, \pi)$, *where* p, q *are positive numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then Parseval's theorem*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0 a_0' + \sum_{n=1}^{\infty}(a_n a_n' + b_n b_n')$$

holds for the pair of functions $f(x)$, $g(x)$.

This result may be proved as follows. Since

$$\left|\int_{-\pi}^{\pi} f(x)\{g(x) - g_n(x)\}\,dx\right| \leqq \left\{\int_{-\pi}^{\pi} |f(x)|^p\,dx\right\}^{\frac{1}{p}} \left\{\int_{-\pi}^{\pi} |g(x) - g_n(x)|^q\right\}^{\frac{1}{q}},$$

we see from § 398 that

$$\left|\int_{-\pi}^{\pi} f(x)\{g(x) - g_n(x)\}\,dx\right| \leqq \eta\left\{\int_{-\pi}^{\pi} |f(x)|^p\,dx\right\}^{\frac{1}{p}},$$

provided n is sufficiently large. Since η is arbitrary, we have

$$\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \lim_{n\sim\infty}\int_{-\pi}^{\pi} f(x)\,g_n(x)\,dx,$$

which is equivalent to Parseval's theorem.

SYSTEMS OF FOURIER'S CONSTANTS

400. If $\{\lambda_n\}$ be a sequence of numbers, the question has been investigated what conditions the sequence $\{\lambda_n\}$ must satisfy in order that the series

$$\tfrac{1}{2}a_0\lambda_0 + \sum_{n=1}\lambda_n(a_n \cos nx + b_n \sin nx)$$

may be a Fourier's series whenever the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$$

* *Fundamenta Math.* vol. VII (1925), p. 28.

is a Fourier's series. An account of these investigations will be given here; and it will be seen that an answer which contains a characterization of the sequences has been obtained (see § 405) to the question, which may be stated as that of the determination of all sequences $\{\lambda_n\}$, each one of which has the property of converting every set $a_0, a_1, b_1, a_2, b_2, \ldots$ of Fourier constants, by multiplication, into another set $\lambda_0 a_0, \lambda_1 a_1, \lambda_1 b_1, \lambda_2 a_2, \lambda_2 b_2, \ldots$ of Fourier constants.

In connection with this matter, properties of certain trigonometrical series said to be allied with Fourier's series present themselves. If

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

be a trigonometrical series, the series $\sum\limits_{n=1} (a_n \sin nx - b_n \cos nx)$ is said to be *the trigonometrical series allied with, or conjugate to, the first series.* The series allied with a Fourier's series is not itself necessarily a Fourier's series. For example, it will be shewn later that the series $\sum\limits_{n=2} \frac{\cos nx}{\log n}$ is a Fourier's series; the allied series $\sum\limits_{n=2} \frac{\sin nx}{\log n}$ is however not a Fourier's series. This is seen from the fact that the integrated series $\sum\limits_{n=2} \frac{\cos nx}{n \log n}$ is divergent at the point $x = 0$, and cannot therefore converge to an integral (see § 360).

It has been proved in § 395, that, if

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx), \quad \tfrac{1}{2}a_0' + \sum_{n=1} (a_n' \cos nx + b_n' \sin nx)$$

be any two Fourier's series, then

$$\tfrac{1}{2}a_0 a_0' + \sum_{n=1} \{(a_n a_n' + b_n b_n') \cos nx - (a_n b_n' - a_n' b_n) \sin nx\}$$

is also a Fourier's series. Taking $a_n' = \lambda_n$, $b_n' = 0$, we have the following property:

If $\lambda_1 \cos x + \lambda_2 \cos 2x + \ldots + \lambda_n \cos nx + \ldots$ be a Fourier's series, the coefficients form a sequence $\{\lambda_n\}$ which has the property of converting by multiplication of the terms any set whatever $a_1, b_1, a_2, b_2, \ldots$ of Fourier constants into a new set $\lambda_1 a_1, \lambda_1 b_1, \lambda_2 a_2, \lambda_2 b_2, \ldots$ of Fourier constants.

If we take $a_n' = 0$, $b_n' = \lambda_n$, it is seen that $\sum\limits_{n=1} \lambda_n (a_n \sin nx - b_n \cos nx)$ is a Fourier's series; we thus obtain the following result:

If $\sum\limits_{n=1} \lambda_n \sin nx$ is a Fourier's series, the series

$$\sum_{n=1} (a_n \sin nx - b_n \cos nx)$$

allied with a Fourier's series $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ is converted by multiplication of the terms by λ_n into a Fourier's series

$$\sum_{n=1} \lambda_n (a_n \sin nx - b_n \cos nx),$$

whether or not the allied series be a Fourier's series.

W. H. Young, to whom the above theorems are due, has combined them* into the following statement:

If $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ *be the Fourier's series for a summable function* $f(x)$, *and* $\{a_n'\}$, $\{b_n'\}$ *be the coefficients in any other Fourier's series, then*

$$\sum_{n=1} a_n' (a_n \cos nx + b_n \sin nx), \quad \sum_{n=1} b_n' (a_n \sin nx - b_n \cos nx)$$

are both Fourier's series, whether the allied series is a Fourier's series or not.

401. With a view to the generalization of the result contained in § 400, the following Lemma, which is a generalization of the Lemma given in § 394, will be required:

If $f(x)$ *be summable in every finite interval, and* $g(x)$ *be a function which has bounded variation in the interval* (a, b), *the Lebesgue-Stieltjes integral* $\int f(x+t)\, dg(t)$ *taken over the interval* (a, b), *of* t, *exists for almost all values of* x *in* (a, b), *and is a summable function of* x.

This theorem† was given by W. H. Young.

Since the function $f(x)$ may be expressed as the difference of two non-negative summable functions, and $g(t)$ may be expressed as the difference of two positive monotone non-diminishing functions, it is clearly sufficient to prove the theorem for the case in which $f(x+t) \geqq 0$, and $g(t)$ is a positive monotone non-diminishing function.

As in I, § 445, let ξ have the value of $g(t)$ at any point t at which $g(t)$ is continuous, and let it have the set of values in the interval

$$(g(t-0),\ g(t+0))$$

at a point t at which $g(t)$ is discontinuous.

Denoting $\int_a^x f(x)\, dx$ by $F(x)$, let $F(x+t) = \Phi(x, \xi)$, and

$$f(x+t) = \phi(x, \xi);$$

we then have

$$\int_a^b [F(x+t) - F(a+t)]\, dg(t) = \int_\alpha^\beta [\Phi(x, \xi) - \Phi(a, \xi)]\, d\xi,$$

where $\alpha = g(a)$, $\beta = g(b)$. These integrals are equal to $\int_\alpha^\beta d\xi \int_a^x \phi(u, \xi)\, du$, which therefore exists for every value of x, since the first integral exists, as $F(x+t)$, $F(a+t)$ are continuous. Since $\phi(u, \xi)$ is a non-negative function it follows, from a theorem given in I, § 429, that $\phi(u, \xi)$ is summable over the domain of (u, ξ), and therefore the order of integration may be reversed without changing the value of the repeated integral. It follows

* *Proc. Lond. Math. Soc.* (2), vol. x (1911), p. 351, where another proof of this theorem is given.

† *Proc. Roy. Soc.* vol. LXXXVIII (1913), p. 563.

that $\int_a^x du \int_a^\beta \phi(u,\xi)\, d\xi$ exists and is equal to $\int_a^b [F(x+t) - F(a+t)]\, dg(t)$; hence $\int_a^\beta \phi(u,\xi)\, d\xi$, or $\int_a^b f(u,t)\, dg(t)$, exists for almost all values of u, and is a summable function of u.

402. In order to apply the theorem, let the Fourier's series which converges to the function $g(t)$, of bounded variation, be

$$\tfrac{1}{2}A_0 + \sum_{n=1} \frac{\alpha_n \sin nt - \beta_n \cos nt}{n},$$

of which the differentiated series is $\sum\limits_{n=1} (\alpha_n \cos nt + \beta_n \sin nt)$. We assume that $g(t)$ is periodic, of period 2π, so that $g(\pi) = g(-\pi)$. Let $f(x)$ be a summable function, to which corresponds the Fourier's series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx);$$

and let

$$\tfrac{1}{2}A_0 + \sum_{n=1} (A_n \cos nx + B_n \sin nx)$$

be the Fourier's series corresponding to the function $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, dg(t)$, or $\frac{1}{\pi}\int_a^\beta \phi(x,\xi)\, d\xi$. We have then

$$A_n = \frac{1}{\pi^2}\int_{-\pi}^{\pi} \cos nx \int_a^\beta \phi(x,\xi)\, d\xi, \quad B_n = \frac{1}{\pi^2}\int_{-\pi}^{\pi} \sin nx \int_a^\beta \phi(x,\xi)\, d\xi;$$

and since $\phi(x,\xi)$ is summable over the domain of (x,ξ), it follows that $\phi(x,\xi)\cos nx$, and $\phi(x,\xi)\sin nx$ are both summable over that domain. We have therefore

$$A_n = \frac{1}{\pi^2}\int_a^\beta d\xi \int_{-\pi}^{\pi} \phi(x,\xi)\cos nx\, dx;$$

and since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sum_{n=1}^{\infty} \{a_n \cos n(x+t) + b_n \sin n(x+t)\}\, dx$$
$$= \pi(a_n \cos nt + b_n \sin nt),$$

we have

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (a_n \cos nt + b_n \sin nt)\, dg(t)$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} n(a_n \sin nt - b_n \cos nt)\, g(t)\, dt$$
$$= a_n \alpha_n + b_n \beta_n.$$

Similarly, we have $B_n = -(a_n\beta_n - b_n\alpha_n)$; and thus the following generalization of the theorem of § 400 has been obtained:

If $g(x)$ be a function of bounded variation such that the series obtained by differentiating the Fourier's series corresponding to $g(x)$ is

$$\sum_{n=1} (\alpha_n \cos nx + \beta_n \sin nx),$$

then, if $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ *be any Fourier's series whatever, the series* $\sum_{n=1} \{(a_n \alpha_n + b_n \beta_n) \cos nx + (a_n \beta_n - b_n \alpha_n) \sin nx\}$ *is a Fourier's series.*

It may be observed that the coefficients α_n, β_n can be simply expressed as Stieltjes integrals with regard to the function $g(x)$.

We have $\frac{\alpha_n}{n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin nx\, dx$, then remembering that $g(\pi) = g(-\pi)$, since $g(x)$, $\sin nx$ are both of bounded variation, we find by integration by parts (see I, § 376) that $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, dg(x)$. Similarly it can be shewn that $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, dg(x)$.

If we take $g(x)$ to be an odd function, we have $\beta_n = 0$; and thus:

If the constants λ_n *are such that the series* $\sum_{n=1} \frac{\lambda_n \sin nx}{n}$ *is a Fourier's series corresponding to a function of bounded variation, then if*

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

is a Fourier's series, the series $\sum_{n=1} \lambda_n (a_n \cos nx + b_n \sin nx)$ *is a Fourier's series.*

If, on the other hand, we take $g(x)$ to be an even function, we have $\alpha_n = 0$; and thus:

If the constants λ_n *are such that* $\sum_{n=1} \frac{\lambda_n \cos nx}{n}$ *is the Fourier's series corresponding to a function of bounded variation, and the series*

$$\sum_{n=1} (a_n \sin nx - b_n \cos nx)$$

is allied with a Fourier's series, the series

$$\sum_{n=1} \lambda_n (a_n \sin nx - b_n \cos nx)$$

is a Fourier's series, whether $\sum_{n=1} (a_n \sin nx - b_n \cos nx)$ *is a Fourier's series or not.*

403. The following theorem will be established:

If $\{\lambda_n\}$ *be a sequence such that* $\Delta\lambda_n > 0$, $\Delta^2\lambda_n > 0$, *for* $n = 0, 1, 2, 3, \ldots$, *and* $\lim_{n\sim\infty} \lambda_n = 0$, *the series* $\frac{1}{2}\lambda_0 + \sum_{n=1} \lambda_n \cos nx$ *is the Fourier's series of a non-negative summable function.*

The series converges uniformly in any interval (ϵ, X), where

$$0 < \epsilon < X < 2\pi,$$

(see § 24 Ex.) to a value $f(x)$. Hence $\int_\epsilon^X f(x)\,dx$ is the sum of the series $\frac{1}{2}\lambda_0(X-\epsilon)+\sum\limits_{n=1}\frac{\lambda_n}{n}(\sin nX-\sin n\epsilon)$. It is known that $\sum\limits_{n=1}\frac{\lambda_n}{n}\sin nx$ converges uniformly in the interval $(-\pi, \pi)$, since $\sum\limits_{n=1}^{n=N}\frac{\sin nx}{n}$ is bounded, and therefore its sum-function is continuous.

It follows that $\lim\limits_{\epsilon\sim 0}\int_\epsilon^X f(x)\,dx$ exists, and is the sum of the series, for $\epsilon=0$.

We have

$$2\sin\tfrac{1}{2}x\left(\tfrac{1}{2}\lambda_0+\sum_1^m\lambda_n\cos nx\right)=\sin\tfrac{1}{2}x\,.\,\Delta\lambda_0+\sin\tfrac{3}{2}x\,.\,\Delta\lambda_1+\dots$$
$$+\sin\frac{2m-1}{2}x\,.\,\Delta\lambda_{m-1}+\lambda_m\sin\frac{2m+1}{2}x;$$

hence

$$4\sin^2\tfrac{1}{2}x\left(\tfrac{1}{2}\lambda_0+\sum_{n=1}^m\lambda_n\cos nx\right)=(1-\cos x)\,\Delta^2\lambda_0+(1-\cos 2x)\,\Delta^2\lambda_1$$
$$+\dots+(1-\cos\overline{m-1}\,x)\,\Delta^2\lambda_{m-2}+(1-\cos mx)\,\Delta\lambda_{m-1}$$
$$+(\cos x-\cos\overline{m+1}\,x)\,\lambda_m.$$

The expression on the right-hand side is greater than

$$(1-\cos mx)\,\Delta\lambda_{m-1}+(\cos x-\cos\overline{m+1}\,x)\,\lambda_m,$$

and this converges to zero, as $m\sim\infty$; it follows that $\frac{1}{2}\lambda_0+\sum\limits_{n=1}\lambda_n\cos nx$ converges to a non-negative value. Since $f(x)\geqq 0$, $\int_0^X f(x)\,dx$ is an absolutely convergent integral, that is $f(x)$ is summable in the interval (o, X), and the series $\sum\limits_1^n\lambda_n\cos nx$ is such that the integrated series converges to an integral, therefore the series is a Fourier's series.

These theorems are due* to W. H. Young, who has further given the following theorem:

If $\Delta\lambda_n\geqq 0$, $\lim\limits_{n\sim\infty}\lambda_n=0$, *and the series* $\sum\limits_{n=1}\frac{\lambda_n}{n}$ *is convergent, then the series* $\Sigma\lambda_n\sin nx$ *is the Fourier's series corresponding to a function which, for positive values of* x, *has a finite lower boundary, and for negative values of* x *a finite upper boundary.*

This theorem may be proved in a manner similar to that given above for the case of the cosine series.

It follows from this theorem and the last theorem in § 402, that:

If $\{\lambda_n\}$ *be such that* $\Delta\lambda_n\geqq 0$, $\lim\limits_{n\sim\infty}\lambda_n=0$, *and* $\sum\limits_{n=1}\frac{\lambda_n}{n}$ *is convergent,*

* *Proc. Lond. Math. Soc.* (2), vol. XII (1912), p. 41.

then the series $\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx)$ *allied with any Fourier's series* $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *is converted by means of the factors* $\{\lambda_n\}$ *into a Fourier's series* $\sum_{n=1} \lambda_n (a_n \sin nx - b_n \cos nx)$.

For example $\sum_{n=2} \frac{a_n \sin nx - b_n \cos nx}{(\log n)^{1+\epsilon}}$, where $\epsilon > 0$, is a Fourier's series.

It has been shewn* by Szidon that the conditions $\Delta\lambda_n > 0$, $\lim_{n\sim\infty} \lambda_n = 0$ are not sufficient to ensure that the series $\frac{1}{2}\lambda_0 + \sum_{n=1} \lambda_n \cos nx$ is a Fourier's series. He has also shewn that the series is necessarily a Fourier's series if the series $\sum_{n=1} |\Delta\lambda_n \log n|$ is convergent, but that in this result $\log n$ cannot be replaced by a number c_n, for which $\Delta^2 c_n < 0$, and

$$\lim_{n\sim\infty} \frac{c_n}{\log n} = 0.$$

It follows that $\frac{1}{2}\lambda_0 + \sum_{1}^{n} \lambda_n \cos nx$ is a Fourier's series provided λ_n converges monotonely to zero, and $\sum_{n=1} \frac{\lambda_n}{n}$ is convergent. Also, if $\sum_{n=1} \lambda_n \sin nx$, where λ_n converges monotonely to zero, is a Fourier's series, so also is

$$\tfrac{1}{2}\lambda_0 + \sum_{n=1} \lambda_n \cos nx.$$

EXAMPLES

(1) The series $\sum_{n=1} \frac{\cos nx}{n^k}$, $\sum_{n=1} \frac{\sin nx}{n^k}$, where $k > 0$, are both Fourier's series. The series $\sum_{n=1} \frac{\cos nx}{n^{k+1}}$, $\sum_{n=1} \frac{\sin nx}{n^{k+1}}$ converge to functions which are indefinite L-integrals, and consequently are of bounded variation.

If $\lambda_n = \frac{1}{n^k}$, we have $\lim_{n\sim\infty} \lambda_n = 0$, and $\Delta\lambda_n > 0$; also since the function $\frac{1}{y^k} - \frac{1}{(y+1)^k}$ diminishes as y increases, we have $\Delta\lambda_n > \Delta\lambda_{n+1}$, or $\Delta^2\lambda_n > 0$. Further, $\sum \frac{\lambda_n}{n}$ is convergent. Thus λ_n satisfies the conditions of both the theorems. Therefore $\sum \frac{\cos nx}{n^k}$, $\sum \frac{\sin nx}{n^k}$ are both Fourier's series; and the integrated series $\sum \frac{\sin nx}{n^{k+1}}$, $\sum \frac{\cos nx}{n^{k+1}}$ represent functions which are integrals.

(2†) If $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ is a Fourier's series, then

$$\sum_{n=1} \frac{1}{n^k}(a_n \cos nx + b_n \sin nx), \quad \sum_{n=1} \frac{1}{n^k}(a_n \sin nx - b_n \cos nx)$$

are also Fourier's series, where $k > 0$.

This follows from the theorem of § 402, employing the result of Ex. 1.

* *Math. Zeitschr.* vol. x (1921), p. 126.

† See W. H. Young, *Proc. Roy. Soc.* vol. LXXXV (1911), p. 417, where applications and extensions of this theorem will be found.

(3) If $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ is a Fourier's series, so also is

$$\sum_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n},$$

and also

$$\sum_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n\,(\log\log n)}.$$

If $\lambda_n = \frac{1}{\log n}$, we have $\Delta\lambda_n > 0$. If $u = \frac{1}{\log y} - \frac{1}{\log (y+1)}$, we find that $\frac{du}{dy} < 0$, for $y > 1$; hence u diminishes as y increases, and therefore $\Delta^2 \frac{1}{\log n} > 0$. It follows that the series $\sum_{n=2} \frac{\cos nx}{\log n}$ is a Fourier's series. Applying the theorem of § 400, we see that

$$\sum_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n} \text{ is a Fourier's series.}$$

The second result stated can be proved in a similar manner.

404. In view of the theorem of § 402, it is desirable to possess a criterion which will decide the question whether a given trigonometrical series is obtainable by differentiation term by term of the Fourier's series corresponding to a function of bounded variation. This criterion is supplied by the following theorem* due to W. H. Young:

The necessary and sufficient condition that the trigonometrical series $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ *should be the series obtained by term by term differentiation of the Fourier's series of a function of bounded variation is that* $\int_{-\pi}^{\pi} | S_n (x) | \, dx$ *should be bounded, where* $S_n (x)$ *denotes the arithmetic mean of the first n partial sums of the given series.*

First, it will be proved that the condition is sufficient. Both

$$\int_{-\pi}^{x} | S_n (x) | \, dx \quad\text{and}\quad \int_{-\pi}^{x} \{| S_n (x) | + S_n (x)\} \, dx$$

are bounded functions of (n, x), and the integrands are non-negative. To each of them the theorem given in § 223 may therefore be applied. A sequence of integers can be so determined that for this sequence the first integral describes a convergent sequence. In this sequence another sequence is contained for which the second integral describes a convergent sequence. Therefore a sequence $\{n_p\}$ of integers exists such that both the integrals, and therefore their difference $\int_{-\pi}^{x} S_n (x) \, dx$, describes a convergent sequence. By the theorem in § 223, the limits of both integrals are functions of bounded variation; therefore, the limit $\lim_{p\sim\infty} \int_{-\pi}^{x} S_{n_p} (x) \, dx$ is a function $g (x)$, of bounded variation. We thus have

$$\begin{aligned} g (x) &= \lim_{p\sim\infty} \int_{-\pi}^{x} \left\{ \sum_{\iota=1}^{\iota=n_p} \left(1 - \frac{\iota - 1}{n_p}\right) (a_\iota \cos \iota x + b_\iota \sin \iota x) \right\} dx \\ &= \lim_{p\sim\infty} \sum_{\iota=1}^{\iota=n_p} \left(1 - \frac{\iota - 1}{n_p}\right) \left(\frac{a_\iota \sin \iota x - b_\iota \cos \iota x + (-1)^\iota b_\iota}{\iota}\right). \end{aligned}$$

* *Proc. Roy. Soc.* vol. LXXXVIII (1913), p. 572.

Since $\sum_{\iota=1}^{\iota=n_p}\left(1-\frac{\iota-1}{n_p}\right)\left(\frac{a_\iota \sin \iota x - b_\iota \cos \iota x + (-1)\ b_\iota}{\iota}\right)$ is a function bounded in (p, x), we may integrate term by term after multiplication by $\cos mx$, the result being equal to $\int_{-\pi}^{\pi} g(x) \cos mx dx$; thus

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \cos mx dx = \lim_{p\sim\infty}\left(-1+\frac{m-1}{n_p}\right)\frac{b_m}{m} = -\frac{b_m}{m}.$$

In a similar manner we find that $\frac{1}{\pi}\int_{-\pi}^{\pi} g(x) \sin mx dx = \frac{a_m}{m}$; hence the Fourier's series corresponding to $g(x)$ is $C + \sum_{m=1} \frac{a_m \sin mx - b_m \cos mx}{m}$; and thus the differentiated series is $\sum_{n=1} (a_m \cos mx + b_m \sin mx)$. Hence the condition has been proved to be sufficient.

To prove that the condition is necessary, we assume that

$$\sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

is the differentiated series of the Fourier's series corresponding to a function $g(x)$, of bounded variation. Since $g(x)$ is the difference of two monotone increasing functions, and the Cesàro mean $S_n(x)$ may be expressed as the difference of two corresponding Cesàro means, it is sufficient to prove the necessity of the condition for the case in which $g(x)$ is monotone increasing. In this case, we have, since $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt dg(t)$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt dg(t)$, $S_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left\{\frac{\sin \frac{1}{2}n(x-t)}{\sin \frac{1}{2}(x-t)}\right\}^2 dg(t)$, and thus $S_n(x)$ is non-negative, and $\int_{-\pi}^{\pi} | S_n(x) | dx = \int_{-\pi}^{\pi} S_n(x) dx$.

Hence we have $\int_{-\pi}^{\pi} S_n(x) dx = T_n(x) - T_n(-\pi)$, where $T_n(x)$ is the Cesàro partial sum of the Fourier's series from which the given series is obtained by differentiating term by term. Since $T_n(x)$ is bounded, with respect to (n, x), it follows that $\left|\int_{-\pi} S_n(x) dx\right|$ is bounded; and thus the necessity of the condition has been proved.

The following theorem is also of interest:

The necessary and sufficient condition that a given trigonometrical series should be the Fourier's series of a bounded function is that $| S_n(x) |$ *should be bounded with respect to* (n, x).

Since $S_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left\{\frac{\sin \frac{1}{2}n(x-t)}{\sin \frac{1}{2}(x-t)}\right\}^2 f(t) dt$, we have $| S_n(x) | <$ a fixed number, when $f(t)$ is bounded. Therefore the condition is necessary.

If $S_n(x)$ is bounded with respect to (n, x), the partial Cesàro sums of the integrated series form a sequence which oscillates continuously and homogeneously. Hence a sequence of these partial Cesàro sums can be found which converges to an integral (see § 222). Thus

$$\int_{-\pi}^{x} f(x)\,dx = \lim_{p\sim\infty} \int_{-\pi}^{x} S_{n_p}(x)\,dx,$$

where $f(x)$ is a bounded function. Multiplying both sides by $\cos mx$, or by $\sin mx$, and integrating term by term, it is seen that the integrated series is a Fourier's series having $\int_{-\pi}^{x} f(x)\,dx$ for its corresponding function, that is, the integrated series is the Fourier's series of the integral of a bounded function, from which the sufficiency of the condition follows.

405. If, in the theorem of § 492, we suppose the function $g(x)$, of bounded variation, to be an odd function, we see that, if $\sum\limits_{n=1} \frac{\lambda_n}{n} \sin nx$ is the Fourier's series which represents $g(x)$, then, if

$$\tfrac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx)$$

be any Fourier's series whatever, the series $\sum \lambda_n (a_n \cos nx + b_n \sin nx)$ is also a Fourier's series. It has been shewn by* Szidon that all sequences $\{\lambda_n\}$ which have this property must be such that $\sum\limits_{n=1} \frac{\lambda_n}{n} \sin nx$ is the Fourier's series of a function with bounded variation.

The general theorem may be stated as follows:

It is necessary and sufficient, in order that a sequence $\{\lambda_n\}$ of numbers may have the property that $\sum\limits_{n=1} \lambda_n (a_n \cos nx + b_n \sin nx)$ is a Fourier's series, provided a_n, b_n are any set of numbers whatever such that

$$\sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

is a Fourier's series, that $\sum\limits_{n=1} \lambda_n \cos nx$ is the series obtained by differentiating the Fourier's series which represents an odd function of bounded variation.

CONVERGENCE FACTORS FOR FOURIER'S SERIES

406. The existence of certain factors $\{\lambda_n\}$ which have the property of converting, by multiplication, any Fourier's series $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ into a series $\sum\limits_{n=1} \lambda_n (a_n \cos nx + b_n \sin nx)$ which is almost everywhere convergent will now be considered. A system of factors which have this property may be termed a set of convergence factors.

* *Math. Zeitschr.* vol. x (1921), p. 121. See also Steinhaus, *ibid.* vol. v (1919), p. 186.

It was first established by W. H. Young that

$$\lambda_n = \frac{1}{(\log n)^{2+\delta}}, \quad \lambda_n = \frac{1}{\log n\,(\log\log n)^{2+\delta}},$$

where $\delta > 0$, are examples of such factors; thus, for example, that the series $\sum\limits_{n=2} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{2+\delta}}$ converges almost everywhere.

It was afterwards proved* by W. H. Young that the factors

$$\frac{1}{n^\delta}, \quad \frac{1}{(\log n)^{1+\delta}}, \quad \frac{1}{\log n\,(\log\log n)^{1+\delta}},$$

are such convergence factors, and that they are also convergence factors for the allied series $\sum\limits_{n=1} (a_n \sin nx - b_n \cos nx)$; both series becoming, on the introduction of the factors, Fourier's series.

Lastly, it was proved by G. H. Hardy† that $\frac{1}{\log n}$ is a convergence factor for all Fourier's series, and thus that $\sum\limits_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n}$ converges almost everywhere.

It was shewn by Hardy (*loc. cit.*) that *if* $s_n(x)$ *be a partial sum of the Fourier's series* $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$, *then* $s_n(x) = o(\log n)$, *for almost every value of* x.

If $\phi(t)$ denote $f(x+t) + f(x-t) - 2\phi(t)$, it is only necessary to shew that, η being some positive number, $\int_0^\eta \phi(t) \frac{\sin nt}{t}\, dt = o(\log n)$. First, consider the integral $\int_0^{\frac{1}{n}} \phi(t) \frac{\sin nt}{t}\, dt$, that is numerically less than $n \int_0^{\frac{1}{n}} |\phi(t)|\, dt$, which is $o(1)$ almost everywhere, that is, at every point x at which $\frac{1}{\lambda}\int_0^\lambda |\phi(t)|\, dt$ converges to zero, as λ does so. Next consider the integral $\int_{\frac{1}{n}}^\eta \phi(t) \frac{\sin nt}{t}\, dt$, which is numerically less than $\int_{\frac{1}{n}}^\eta \frac{|\phi(t)|}{t}\, dt$, which becomes, on integration by parts, $\frac{1}{\eta}\Phi(\eta) - n\Phi\left(\frac{1}{n}\right) + \int_{\frac{1}{n}}^\eta \frac{\Phi(t)}{t^2}\, dt$, where $\Phi(t)$ denotes $\int_0 |\phi(t)|\, dt$. This is equal almost everywhere to

$$O(1) + o(1) + A_n(\eta) \log(n\eta),$$

where $|A_n(\eta)| < K(\eta)$, for all values of $n\,(> 1/\eta)$, and $K(\eta)$ converges with η to zero. Therefore, we have $\overline{\lim\limits_{n\sim\infty}} \left|\frac{s_n(x)}{\log n}\right| \leq K(\eta)$; and consequently $s_n(x) = o(\log n)$, for almost all values of x.

* *Comptes Rendus*, vol. CLV (1912), p. 1480. See also *Proc. Roy. Soc.* vol. LXXXVIII (1912), p. 179.

† *Proc. Lond. Math. Soc.* (2), vol. XII (1913), p. 365.

407. If $\{\alpha_n\}$ is a monotone sequence of positive numbers which converges to zero, and satisfies the two conditions that $\alpha_n = O\left(\frac{1}{\log n}\right)$ and that the series $\Sigma\, \Delta\alpha_n \,.\, \log n$ is convergent, then the series

$$\Sigma\, \alpha_n\,(a_n \cos nx + b_n \sin nx)$$

is convergent almost everywhere. For

$$\sum_1^n \alpha_n\,(a_n \cos nx + b_n \sin nx) = \sum_{r=1}^{r=n-1} s_r(x)\,\Delta\alpha_r + s_n(x)\,\alpha_n,$$

and since $|\, s_n(x)\,\alpha_n\,| = O\left(\frac{1}{\log n}\right) o(\log n) = o(1)$, almost everywhere, we see that the series $\sum\limits_{n=2} \alpha_n\,(a_n \cos nx + b_n \sin nx)$ converges almost everywhere.

The condition is satisfied by any of the values

$$\alpha_n = \frac{1}{(\log n)^{1+\delta}},\quad \alpha_n = \frac{1}{\log n\,(\log\log n)^{1+\delta}},\ \ldots,\ \text{where } \delta > 0.$$

Therefore these values of α_n provided sets of convergence factors for any Fourier's series. That the new series so formed are Fourier's series follows from the fact that the series $\Sigma \frac{\cos nx}{(\log n)^{1+\delta}}$, $\Sigma \frac{\cos nx}{\log n\,(\log\log n)^{1+\delta}}, \ldots$ are Fourier's series (see § 403).

408. It will now be proved that:

If $\frac{1}{2}a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx)$ *be any Fourier's series, the series* $\sum\limits_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n}$ *is a Fourier's series which converges almost everywhere.*

It has been shewn in § 403, Ex. 3, that $\sum\limits_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n}$ is a Fourier's series, and consequently it is summable $(C, 1)$ almost everywhere. The condition that, for a value of x for which it is summable $(C, 1)$, it should be convergent is that $\sum\limits_{\nu=2}^{\nu=n} \nu \,.\, \frac{a_\nu \cos \nu x + b_\nu \sin \nu x}{\log \nu} = o(n)$, see Ex. 6, § 6. To shew that this condition is satisfied, we have, when $s_n(x) = o(\log n)$,

$$\sum_{\nu=2}^{\nu=n} \nu \,.\, \frac{a_\nu \cos \nu x + b_\nu \sin \nu x}{\log \nu} = \sum_{\nu=2}^{\nu=n-1} s_\nu(x)\,\Delta \frac{\nu}{\log \nu} + \frac{n s_n(x)}{\log n}.$$

The series on the right-hand side is numerically less than $A + n\epsilon$, where ϵ is arbitrary, and A depends only on ϵ. This is seen by taking the summation in $(2, m-1)$ and $(m, n-1)$ separately. Therefore $\sum\limits_{\nu=2}^{\nu=n} \nu \,.\, \frac{a_\nu \cos \nu x + b_\nu \sin \nu x}{\log \nu} = o(n)$, and consequently the condition that the series $\sum\limits_{n=2} \frac{a_n \cos nx + b_n \sin nx}{\log n}$ should be convergent is satisfied

for almost all values of x. The corresponding theorem that the factors $\frac{1}{\log n}$ converts the associated series $\sum_{n=2} (a_n \sin nx - b_n \cos nx)$ into a series which converges almost everywhere is also known* to hold good.

409. It is not known whether the preceding theorem is the best of its kind; that is whether there exists a set of convergence factors $\{\lambda_n\}$ such that $\lambda_n \log n$ diverges as $n \sim \infty$. But if we restrict the Fourier's series to be such as correspond to functions of which the squares are summable, the factors $\lambda_n = \frac{1}{(\log n)^{\frac{1}{2}+p}}$, where p is any positive number, are convergence factors, and thus, if $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$ be any Fourier's series for which $\sum_{n=1} (a_n{}^2 + b_n{}^2)$ is convergent, the series

$$\sum_{n=2} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{\frac{1}{2}+p}}, \quad (p > 0)$$

converges almost everywhere. This is a consequence of the following theorem due† to A. Kolmogoroff and G. Seliverstoff:

If $\sum_{n=2} (a_n{}^2 + b_n{}^2) (\log n)^{1+\epsilon}$ $(\epsilon > 0)$ *is convergent, then*

$$\sum_{n=2} (a_n \cos nx + b_n \sin nx)$$

converges almost everywhere.

Let $\sum_{p=1}^{p=n} (a_p \cos px + b_p \sin px)$ be denoted by $S_n(x)$, and let

$$\sum_{p=1}^{p=\lambda(x)} (a_p \cos px + b_p \sin px)$$

be denoted by $S_{\lambda(x)}(x)$, where $\lambda(x)$ is any measurable function which takes only the set of values 1, 2, 3, ... n. It will first be shewn that

$$\left| \int_{-\pi}^{\pi} S_{\lambda(x)}(x)\,dx \right| \leq \left\{ C \log n \,.\, \sum_{p=1}^{p=n} (a_p{}^2 + b_p{}^2) \right\}^{\frac{1}{2}},$$

where C is an absolute constant.

We have

$$\left| \int_{-\pi}^{\pi} S_{\lambda(x)}(x)\,dx \right| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} S_n(x')\,dx' \int_{-\pi}^{\pi} \sum_{p=1}^{p=\lambda(x)} \cos p(x - x')\,dx' \right|$$

$$\leq \left\{ \frac{1}{\pi} \int_{-\pi}^{\pi} \{S_n(x')\}^2\,dx' \,.\, \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \sum_{p=1}^{p=\lambda(x)} \cos p(x - x')\,dx \right]^2 dx' \right\}^{\frac{1}{2}}$$

by employing Schwarz's inequality.

* See Plessner's tract, "Zur Theorie der konjugierten trigonometrischen Reihen" (1923), Giessen, p. 33.

† *Comptes Rendus*, vol. CLXXVIII (1924), p. 303.

Now

$$\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}\sum_{p=1}^{\lambda(x)}\cos p\,(x-x')\,dx\right]^2 dx'$$

$$=\int_{-\pi}^{\pi}dx'\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{p=1}^{p=\lambda(x)}\cos p\,(x-x')\,.\sum_{p=1}^{p=\lambda(y)}\cos p\,(y-x')\,dx\,dy$$

$$=\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\sum_{p=1}^{p=\lambda(x,y)}\int_{-\pi}^{\pi}\cos p\,(x-x')\cos p\,(y-x')\,dx',$$

where $\lambda(x, y)$ denotes the smaller of the two integers $\lambda(x)$, $\lambda(y)$; and the expression on the right-hand side is equal to

$$\pi\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}\sum_{p=1}^{p=\lambda(x,y)}\cos p\,(x-y)\,dy,$$

as is seen by carrying out the integration with respect to x'.

We have further

$$\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}\sum_{p=1}^{p=\lambda(x,y)}\cos p\,(x-y)\,dy$$

$$=\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}\frac{1}{2}\left[\frac{\sin\dfrac{2\lambda(x,y)+1}{2}(x-y)}{\sin\frac{1}{2}(x-y)}-1\right]dy;$$

the second part of this double integral is $-2\pi^2$. Let A be the part of the rectangle $(-\pi, -\pi; \pi, \pi)$ for which $\frac{1}{2}|x-y|$ is $<\dfrac{\pi}{2n+1}$ or $>\pi-\dfrac{\pi}{2n+1}$, and let B denote the remaining part. The portion of the double integral of $\sin\dfrac{2\lambda(x,y)+1}{2}(x-y)\Big/\sin\frac{1}{2}(x-y)$, taken over A, is less than $(2n+1)$ area of A, and this $=O(1)$. The double integral taken over B is less than $\displaystyle\int_{(B)}\left|\operatorname{cosec}\frac{x-y}{2}\right|d(x,y)$, and this, on changing the variables to

$$\xi=\tfrac{1}{2}(x-y),\quad \eta=\tfrac{1}{2}(x+y),$$

is less than a fixed multiple of

$$\int_{\frac{\pi}{2n+1}}^{\pi-\frac{\pi}{2n+1}}\operatorname{cosec}\xi\,d\xi,$$

or of $\left[\log\tan\frac{1}{2}\xi\right]_{\frac{\pi}{2n+1}}^{\pi-\frac{\pi}{2n+1}}$, which is $-\frac{1}{2}\log\tan\dfrac{\pi}{2(2n+1)}$, or $O(\log n)$.

It has now been proved that

$$\left|\int_{-\pi}^{\pi}S_{\lambda(x)}(x)\,dx\right|<\left[O(\log n)\,.\sum_{p=1}^{p=n}(a_p^2+b_p^2)\right]^{\frac{1}{2}}.$$

Let $\{u(n)\}$ be a sequence of increasing positive numbers such that $\sum\limits_{n=1} u(n)(a_n^2+b_n^2)$ converges to a sum A, and that $\sum\limits_{n=1}\dfrac{1}{nu(n)}$ converges

to N. Let $S_{p,l}(x) = \sum_{q=2^{2^p}}^{l} (a_q \cos qx + b_q \sin qx)$, where $2^{2^p} < l \leq 2^{2^{p+1}}$; we then have, from what has been proved above,

$$\left|\int_{-\pi}^{\pi} S_{p,l(x)}(x)\,dx\right| < \{C \log 2^{2^{p+1}} A_p\}^{\frac{1}{2}} < C'(2^p A_p)^{\frac{1}{2}},$$

where A_p denotes $\sum_{q=2^{2^p}}^{q=2^{2^{p+1}}} (a_q^2 + b_q^2)$, and $l(x)$ is a measurable function having only integral values $2^{2^p}+1, 2^{2^p}+2, \dots 2^{2^{p+1}}$. Let $l(x)$ be so chosen that, at each point x, $|S_{p,l(x)}(x)|$ has the maximum value, that is the value of the partial remainder $R_{2^{2^p},m}$, of the Fourier's series, which has the numerically greatest value, for $m = 1, 2, 3, \dots, 2^{2^{p+1}} - 2^{2^p}$; and let $\Phi_p(x)$ be the value of $S_{p,l(x)}$, when $l(x)$ is so chosen; we have then

$$\left|\int_{-\pi}^{\pi} \Phi_p(x)\,dx\right| < C'(2^p A_p)^{\frac{1}{2}}.$$

It can be shewn that the series $\sum_{p=1} (2^p A_p)^{\frac{1}{2}}$ is convergent; for

$$2^p A_p = \frac{2^p}{u(2^{2^p})} \sum_{q=2^{2^p}}^{q=2^{2^{p+1}}} (a_q^2 + b_q^2)\, u(2^{2^p}),$$

from which we have

$$2(2^p A_p)^{\frac{1}{2}} < \frac{2^p}{u(2^{2^p})} + \sum_{q=2^{2^p}}^{q=2^{2^{p+1}}} (a_q^2 + b_q^2)\, u(2^{2^p})$$

$$< \frac{2^p}{u(2^{2^p})} + \sum_{q=2^{2^p}}^{q=2^{2^{p+1}}} u(q)(a_q^2 + b_q^2).$$

Also
$$\frac{2^p}{u(2^{2^p})} < 2 \sum_{r=2^{p-1}}^{r=2^p} \frac{1}{u(2^r)} < 4 \sum_{n=2^{2^{p-1}}}^{n=2^{2^p-1}} \frac{1}{nu(n)}.$$

It follows that $2\sum_{p=p_1} (2^p A_p)^{\frac{1}{2}} < 8N + 2A$, and thus the series $\Sigma(2^p A_p)^{\frac{1}{2}}$ is convergent. Let us choose p so large that $\sum_{m=p}^{\infty} (2^m A_m)^{\frac{1}{2}} < \frac{\epsilon^2}{C'}$, then we have

$$\sum_{s=p}^{s=p'} \left|\int_{-\pi}^{\pi} \Phi_s(x)\,dx\right| < \epsilon^2.$$

If we denote by $\psi_{p,p'}(x)$ the maximum, for each value of x, of the absolute values of the partial remainders of the series

$$\Sigma(a_n \cos nx + b_n \sin nx),\ R_{2^p,m},\ \text{for } m = 1, 2, 3, \dots 2^{p'} - 2^p,$$

we have $\int_{-\pi}^{\pi} \psi_{p,p'}(x)\,dx < \epsilon^2$. It follows that, in a set of points of measure $> 2\pi - \epsilon$, we have $\psi_{p,p'}(x) < \epsilon$. Let there be assigned to ϵ the values $\epsilon_1, \epsilon_2, \epsilon_3, \dots$ in a diminishing sequence of numbers, such that $\sum_{r=1} \epsilon_r$ converges to an assigned positive number η. Let $p_1, p_2, p_3, \dots$ be the values

of p corresponding to $\epsilon_1, \epsilon_2, \ldots$; when $p = p_1$, we take $p' = p_2$; when $p = p_2$, we take $p' = p_3$, and so on. We then see that in a set of points of measure $> 2\pi - \eta$, which is the set common to all the sets of measures $> 2\pi - \epsilon_1$, $2\pi - \epsilon_2, \ldots$, employed in the above reasoning, we have $| R_{2^{p_1}, m}(x) | < \eta$, for all values $p_1 + 1$, $p_1 + 2, \ldots$ of m. Denote this set of points by E_{p_1}, where $m(E_{p_1})$ is $> 2\pi - \eta$. By assigning to η a set of values $\eta_1, \eta_2, \ldots$ which are diminishing, and such that $\sum\limits_{n=1}^{\infty} \eta_1 = \zeta$, where ζ is an arbitrarily chosen number, we see that there exists a set F_ζ, of measure $> 2\pi - \zeta$, such that in this set $| R_{2^{p_1}, m}(x) | < \eta_1$, $| R_{2^{p_2}, m}(x) | < \eta_2, \ldots$ for all values of $m > 2^{p_1}, > 2^{p_2}, \ldots$.

It follows that in the set F_ζ the series is uniformly convergent; and since ζ is arbitrary, it follows that the series $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ is almost everywhere convergent.

The condition that $\sum\limits_{n=1} \dfrac{1}{nu(n)}$ should be convergent is satisfied by $u(n) = (\log n)^{1+\epsilon}$, where $\epsilon > 0$. Thus the theorem stated above is established.

We might also take $u(n) = \log n (\log \log n)^{1+\epsilon}$, in which case it appears that, if $\sum\limits_{n=4} \log n (\log \log n)^{1+\epsilon} (a_n^2 + b_n^2)$ is convergent for a value of ϵ that is > 0, the series $\sum\limits_{n=4} (a_n \cos nx + b_n \sin nx)$ converges almost everywhere. By continuing this scale we have a series of theorems of which the one stated above is the first.

Thus *if* $\sum (a_n^2 + b_n^2)$ *is convergent, the series* $\sum \dfrac{a_n \cos nx + b_n \sin nx}{(\log n)^{\frac{1}{2}} (\log \log n)^{\frac{1}{2}+k}}$ *is convergent almost everywhere, when* $k > 0$.

POISSON'S METHOD OF SUMMATION

410. One of the most important, both intrinsically and historically, of the conventional sums of a Fourier's series is that which was first employed by Poisson.

If $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ be a Fourier's series, let the sum $\frac{1}{2}a_0 + \sum\limits_{r=1}^{r=n} h^n (a_n \cos nx + b_n \sin nx)$, where $|h| < 1$, be denoted by $P(x, n, h)$.

If $\lim\limits_{h \sim 1} \lim\limits_{n \sim \infty} P(x, n, h)$ exists, it may be defined to be the Poisson sum of the Fourier's series. The ordinary sum, when it exists, is

$$\lim_{n \sim \infty} \lim_{h \sim 1} P(x, n, h);$$

and in accordance with the general mode of introducing conventional sums,

referred to in §§ 44–46, the Poisson sum is such a conventional sum. The Poisson sum of the series is accordingly defined to be

$$\lim_{h\sim 1}\left[\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\,h^n\right],$$

whenever the limit exists.

The condition of consistency must be established, that, when both the Poisson sum and the ordinary sum exist, they have the same value. Since

$$|\,a_n\cos nx+b_n\sin nx\,|$$

is bounded, the limit $\lim\limits_{n\sim\infty} P(x, n, h)$ exists for every value of h such that $|\,h\,|<1$. In accordance with Abel's theorem (§ 126), when, for a particular value of x, the Fourier's theorem is convergent, the Poisson sum

$$\lim_{h\sim 1}\lim_{n\sim\infty} P(x, n, h)$$

exists, and has the value to which the Fourier's series converges. It was however assumed by Poisson and by many subsequent writers that the converse of this always holds good. Thus, by Poisson and his followers, a proof of the convergence of Fourier's series which is now regarded as wanting in rigour was given, which depended upon the ascertainment of the Poisson sum, and the assumption that the Fourier's series necessarily converges to the same limit.

An important application of the theory of the Cesàro summation of Fourier's series may be made in this connection. It has been shewn in § 368 that a Fourier's series is summable $(C, 1)$ almost everywhere in the interval $(-\pi, \pi)$, and that in particular its Cesàro sum is $f(x)$ at any point of continuity of $f(x)$, and is $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point of ordinary discontinuity. If we now apply the theorem of Frobenius (§ 128), we see that, for any value of x for which the Fourier's series is summable $(C, 1)$, the limit, as $h\sim 1$, of the sum of the corresponding power-series, in powers of h, exists, and has the same value as the sum $(C, 1)$ of the Fourier's series. It thus follows that, for any such value of x, the Poisson sum exists, and is equal to the Cesàro sum $(C, 1)$ of the Fourier's series.

It has accordingly been established that:

For any Fourier's series, the Poisson sum exists almost everywhere, and has the value $f(x)$; and it is equal to $f(x)$, or to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point of continuity or of ordinary discontinuity of the function.

If we consider the class of Fourier's series for which $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$, Littlewood's theorem (§ 132) may be applied to prove that the Fourier's series converges to the Poisson sum, wherever the latter

exists; this has been shewn above to be the case almost everywhere, and in particular at every point of continuity or of ordinary discontinuity of the function. It follows that the series converges to $f(x)$ almost everywhere, and that, at a point of ordinary discontinuity, it converges to

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Another proof of this result will be given in § 414.

If we shew, by means of any independent investigation, that in case $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$, the Poisson sum exists almost everywhere, Littlewood's theorem (§ 132) enables us to infer the convergence of the Fourier's series at every point at which the Poisson sum exists. Such an independent investigation, in the case $a_n = o\left(\frac{1}{n}\right)$, $b_n = o\left(\frac{1}{n}\right)$, was given* by Fatou, who also shewed that, in this case, the Fourier's series converges almost everywhere.

If $f(x)$ have bounded variation in $(-\pi, \pi)$, then $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$, and the Poisson sum exists everywhere. It follows then, by applying Littlewood's theorem, that the Fourier's series converges everywhere to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$. Thus, in the case of such functions, the hiatus in the older proofs of convergence of Fourier's series by means of Poisson's sum is filled up.

Interesting properties of the Poisson sum have been given† by Gross.

411. The definition of the Poisson sum is applicable to the case of any trigonometrical series, which is not necessarily a Fourier's series, provided the series $\frac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)\, h^n$ is convergent when $|h| < 1$.

Applying the extension of the theorem of Frobenius, given in § 128 (3), we see that, such a trigonometrical series, when it is summable (C, r) for some value of r $(\geqq 0)$, at a point x, is also summable by Poisson's method, at the same point. That this is the case was first proved‡ by Hölder, in the case in which r is a positive integer.

It thus appears that Poisson summation is at least as general as Cesàro summation of a trigonometrical series.

If $f(x)$ be not summable in $(-\pi, \pi)$, but have a Denjoy integral in that interval, it has been shewn (§ 376) that the corresponding generalized Fourier's series is summable almost everywhere in each integral contiguous to the set H, of points of non-summability of $f(x)$. In case the set H has measure zero, and in particular when it has an HL-integral, the series is summable $(C, 1)$ almost everywhere in $(-\pi, \pi)$, and it is almost

* *Acta Math.* vol. xxx (1906), p. 379.

† *Wien. Ber.* vol. cxxiv, Abt. II*a* (1915), p. 1024.

‡ *Math. Annalen*, vol. xxxiii (1882), p. 246.

everywhere equal to $f(x)$; we have thus the following extension of the above result:

If $f(x)$ have a D-integral, the Poisson sum of the corresponding generalized Fourier's series exists, and is equal to $f(x)$, almost everywhere in each interval contiguous to the set H, of points of non-summability of $f(x)$. When $m(H) = 0$, as is the case when $f(x)$ has an HL-integral, the Poisson sum exists, and is equal to $f(x)$, almost everywhere in the interval. In either case it exists everywhere in an interval of continuity of $f(x)$, provided the continuity at the end-points is on both sides.

412. The limit of the sum $P(x, n, h)$, as $n \sim \infty$, is equivalent to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \sum_{r=1}^{\infty}\frac{1}{\pi}h^r\int_{-\pi}^{\pi} f(x')\cos r(x'-x)\,dx',$$

or to
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\frac{1-h^2}{1-2h\cos(x'-x)+h^2}\,dx'.$$

Thus the Poisson sum, when it exists, is given by

$$\frac{1}{2\pi}\lim_{h\sim 1}\int_{-\pi}^{\pi}\frac{1-h^2}{1-2h\cos(x'-x)+h^2}f(x')\,dx'.$$

The value of this limit was studied* by Schwarz in two memoirs. He considered the case, more general than that with which we are here concerned, in which x varies as well as h; he confined his attention however to the case in which $f(x)$ is either continuous, or else has only a finite set of discontinuities. A more complete discussion of questions connected with Poisson's integral has been given† by Fatou. An evaluation of the limit will be given here, by an application of the general method developed in Chapter VI.

Let x be any point of the interval $(-\pi+\epsilon, \pi-\epsilon)$; this will be taken as the set G to which x belongs. If $\mu < \epsilon$, the positive function

$$\frac{1-h^2}{1-2h\cos(x'-x)+h^2}$$

is less than
$$\frac{1}{(1-h)^2+4h\sin^2\dfrac{x'-x}{2}}$$

or than $\dfrac{1}{4h_1}\operatorname{cosec}^2\dfrac{\mu}{2}$, provided $|x'-x| \geqq \mu$, $1 > h > h_1$, where μ is a positive number $< \epsilon$. In order to apply the theorem of § 290, we may suppose $n = (1-h)^{-1}$, so that $n \sim \infty$, as $h \sim 1$. The first condition of the general convergence theorem of § 279 is accordingly satisfied. To shew that the second condition is satisfied, we have

$$\int_{\alpha_1}^{\beta_1}\frac{1-h^2}{1-2h\cos(x'-x)+h^2}\,dx' < \frac{1-h^2}{(1-h)^2+4h\sin^2\frac{1}{2}\mu}(\beta_1-\alpha_1),$$

* *Math. Abhandlungen*, vol. II, pp. 144, 175.

† *Acta Math.* vol. XXX (1906). See also Plessner's tract quoted on p. 626.

where (α_1, β_1) is any interval in $(-\pi, \pi)$, provided x is not interior to the interval $(\alpha_1 - \mu, \alpha_1 + \mu)$, and belongs to G. This converges to zero, uniformly for all such intervals, as $h \sim 1$; thus the second condition is satisfied; and therefore $\int_{-\pi}^{x-\mu} \frac{1-h^2}{1-2h\cos(x'-x)+h^2} f(x')\,dx'$, and the similar integral with the limits $x+\mu, \pi$, converge to zero uniformly for all values of x belonging to G.

We have also

$$\lim_{n\sim\infty} \frac{1}{2\pi} \int_{-\mu}^{\mu} \frac{1-h_n{}^2}{1-2h_n\cos(x'-x)+h_n{}^2}\,dx'$$

$$= \lim_{n\sim\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-h_n{}^2}{1-2h_n\cos(x'-x)+h_n{}^2}\,dx' = 1.$$

It follows, from the theorem of § 292, that the limit of Poisson's integral is

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

at any point of ordinary discontinuity. Also it converges uniformly in any interval in which $f(x)$ is continuous, the continuity at the end-points of the interval being assumed to be on both sides.

In order to apply the theorem of § 295, we observe that

$$\lim_{n\sim\infty} \frac{1-h_n{}^2}{1-2h_n\cos t+h_n{}^2} = 0, \text{ for } t \neq 0;$$

also $t \,.\, \frac{1-h_n{}^2}{1-2h_n\cos t+h_n{}^2}$ has, in the interval $(0, \mu)$ the total variation

$$(1-h_n{}^2)\int_0^{\mu} \left\{\frac{1}{1-2h_n\cos t+h_n{}^2} - \frac{t\,.\,2h_n\sin t}{(1-2h_n\cos t+h_n{}^2)^2}\right\} dt$$

which is less than

$$2\int_0^{\mu} \frac{1-h_n{}^2}{1-2h_n\cos t+h_n{}^2}\,dt - \frac{\mu(1-h_n{}^2)}{1-2h_n\cos\mu+h_n{}^2},$$

or than 2π, which is independent of n. It thus appears that the integral converges to $f(x)$ at every point at which $\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt$ has, at $t=0$, a differential coefficient of value zero. It has thus been shewn that:

The Poisson sum of the Fourier's series corresponding to the summable function $f(x)$ is $f(x)$ at every point at which

$$\lim_{t\sim 0} \frac{1}{t}\int_0^t \{f(x+t)+f(x-t)-2f(x)\}\,dt = 0;$$

and this is the case almost everywhere in the interval $(-\pi, \pi)$. The Poisson sum is $\tfrac{1}{2}\{f(x+0)+f(x-0)\}$ at any point of ordinary discontinuity. The Poisson summation converges uniformly in any interval in which the function is continuous, the continuity at the end-points being assumed to be on both sides.

413. The following theorem, which is due to Fatou, is of importance in the general theory:

If the summable function $f(x)$ has, at a point α, a finite differential coefficient, and $F'(x, h)$ denote the sum of the series

$$\sum_{n=0}^{\infty} n\,(-a_n \sin nx + b_n \cos nx)\, h^n, \quad (h < 1),$$

then at the point α, $\qquad \lim\limits_{h\sim 1} F'(\alpha, h) = f'(\alpha).$

Without loss of generality we may take $\alpha = 0$. If $f(x) = 1$, in the interval $(-\pi, \pi)$, the theorem is obviously true, since all the coefficients of the Fourier's series vanish, except a_0. The theorem is also easily verified at the point 0 for the function $f(x) = x$. Writing

$$\phi(x) = f(x) - xf'(o) - f(o),$$

we have $\phi(0) = 0$, $\phi'(0) = 0$, and it is sufficient to prove the theorem for this function $\phi(x)$.

We have since

$$F(x, h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-h^2}{1-2h\cos(\theta - x) + h^2}\,\phi(\theta)\,d\theta,$$

$$\lim_{h\sim 1} F'(o, h) = \frac{1}{2\pi}\lim_{h\sim 1}\int_{-\pi}^{\pi} \frac{(1-h^2)\,2h\sin\theta}{\{1-2h\cos\theta + h^2\}^2}\,\phi(\theta)\,d\theta,$$

and it is sufficient to shew that this limit has the value 0, that of $\phi'(o)$. Writing H for $\dfrac{1-h^2}{1-2h\cos\theta+h^2}$, we have

$$\int_{-\pi}^{\pi} \frac{(1-h^2)\,2h\sin\theta}{\{1-2h\cos\theta+h^2\}^2}\,\phi(\theta)\,d\theta$$
$$= \int_{-\pi}^{\pi} H\left\{\frac{2h\sin\theta}{1-2h\cos\theta+h^2} - \frac{\sin\theta}{1-\cos\theta}\right\}\phi(\theta)\,d\theta + \int_{-\pi}^{\pi} H\,\frac{\phi(\theta)}{\tan\frac{1}{2}\theta}\,d\theta.$$

Since $\lim \dfrac{\phi(\theta)}{\tan\frac{1}{2}\theta} = 2\lim\dfrac{\phi(\theta)}{\theta} = 0$, the function $\dfrac{\phi(\theta)}{\tan\frac{1}{2}\theta}$ is summable in $(-\pi, \pi)$, and it has the limit 0 at $\theta = 0$; therefore, by § 412,

$$\lim_{h\sim 1}\int_{-\pi}^{\pi} H\,\frac{\phi(\theta)}{\tan\frac{1}{2}\theta}\,d\theta = 0.$$

The other integral may be written in the form

$$-\int_{-\pi}^{\pi} H\,\frac{(1-h)^2}{1-2h\cos\theta+h^2}\,\frac{\phi(\theta)}{\tan\frac{1}{2}\theta}\,d\theta,$$

and this is numerically less than

$$\int_{-\pi}^{\pi} H\left|\frac{\phi(\theta)}{\tan\frac{1}{2}\theta}\right|d\theta.$$

This has the limit 0, as $h \sim 1$, for a reason similar to the case already discussed. The theorem has now been established.

414. The following theorem will be established:

If $a_n = O\left(\frac{1}{n}\right)$, $b_n = O\left(\frac{1}{n}\right)$, the necessary and sufficient condition that the series $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ should converge at the point x to the value $f(x)$ is that the sum of the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{\sin nh}{nh}$$

should converge, as $h \sim 0$, to $f(x)$.

The given series is a Fourier's series, since $\sum\limits_{n=1} (a_n^2 + b_n^2)$ is convergent. The condition in the theorem may be stated in the form that $\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)\,dt$ should converge to $f(x)$. This condition is satisfied for almost all values of x (see I, § 432).

The theorem was given* by Hardy and Littlewood; it is a generalization of an earlier theorem due† to Fatou, which applies to the case in which

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right).$$

In order to prove that the condition in the theorem is sufficient, it is seen from Lebesgue's theorem in § 370 that, when the condition is satisfied at the point x, the Fourier's series is, at that point, summable $(C, 2)$. Consequently, since na_n, nb_n are bounded, it follows at once, by employing the theorem of § 54, that the series is convergent at the point.

In order to prove that the condition is necessary, denoting

$$a_n \cos nx + b_n \sin nx$$

by A_n, we may assume, without loss of generality, that $f(x) = 0$, $A_0 = 0$, $|nA_n| < 1$. Assuming that the Fourier's series converges to zero at the point x, we write

$$\Phi = \sum_{n=1}^{n=\mu} A_n \frac{\sin nh}{nh} + \sum_{n=\mu+1}^{n=m} A_n \frac{\sin nh}{nh} + \sum_{n=m}^{\infty} A_n \frac{\sin nh}{nh} \equiv \Phi_1 + \Phi_2 + \Phi_3.$$

Having assigned a positive integer k (> 2), let m and h be such that

$$k - 1 < mh \leq k,$$

then
$$|\Phi_3| < \frac{1}{h}\sum_{m=n+1}^{\infty} \frac{1}{n^2} < \frac{1}{mh},$$

and therefore $|\Phi_3| < \frac{2}{k}$. If the terms of $\sum\limits_{n=1}^{n=m} \frac{\sin nh}{nh}$ be grouped, their order being preserved, so that all the terms in any group are of the same sign, opposite to the sign of the terms in the neighbouring groups,

* *Proc. Lond. Math. Soc.* (2), vol. XVIII (1919), p. 228. A simplification of the proof, due to M. Riesz, is given in *Proc. Lond. Math. Soc.* (2), vol. XXII (1924), *Records*, p. xviii.

† *Acta Math.* vol. XXX (1906), pp. 345, 385.

the number of groups depends only on the value of mh, and does not exceed a number $K(k)$, since, for a fixed value of k, the number remains less than a fixed number, whatever be the values of m and h. Let ϵ be an arbitrarily chosen positive number, and let μ be so chosen that

$$\left|\sum_{n=1}^{n=\mu} A_n\right| < \frac{\epsilon}{K(k)}, \quad \left|\sum_{n=\nu}^{n=\nu'} A_n\right| < \frac{\epsilon}{K(k)},$$

for all values of ν and ν' such that $\mu \leqq \nu < \nu'$.

We now see that $|\Phi_2|$ is less than $\frac{\epsilon}{K(k)}$ multiplied by the number of groups in $\sum_{n=m+1}^{n=\mu} \frac{\sin nh}{nh}$, and this is less than ϵ. Also

$$|\Phi_1| < \left|\sum_{n=1}^{n=\mu} A_n\right| + \epsilon < 2\epsilon,$$

provided $0 < h \leq h_1$, where h_1 depends on μ and ϵ. We now have

$$|\Phi| < \frac{2}{k} + 3\epsilon, \text{ for } 0 < h \leq h_1;$$

and since $\frac{2}{k}$ and ϵ are both arbitrarily small, we have $\lim_{h\sim 0} \Phi = 0$. The necessity of the condition has now been established.

The theorem established may be stated as follows:

If na_n, nb_n are bounded, the necessary and sufficient condition that the series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$ converges for a given value of x is that the function $g(x)$ defined as $\frac{1}{2}a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$ shall have a differential coefficient $g'(x)$ at the given point; and then $g'(x)$ is the sum to which the given series converges.

If a_n, b_n be changed into nb_n, $-na_n$, the theorem may be stated as follows:

If $a_n = 0\left(\frac{1}{n^2}\right)$, $b_n = 0\left(\frac{1}{n^2}\right)$, the necessary and sufficient condition that the series $\sum_{n=1} n(-a_n \sin nx + b_n \cos nx)$ converges for a given value of x is that the function $g(x)$ defined as $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ shall have a differential coefficient $g'(x)$ at the given point; and then $g'(x)$ is the sum to which the given series converges.

APPROXIMATE REPRESENTATION OF FUNCTIONS BY FINITE TRIGONOMETRICAL SERIES

415. If the function $f(x)$, defined for the interval $(-\pi, \pi)$, be continuous in the interval (α, β), contained in $(-\pi, \pi)$, including the end-points α, β, it has been seen, in § 412, that Poisson's integral converges to the value $f(x)$, uniformly in the interval (α, β), as h converges to the value 1.

Therefore, a value h_1, of h, may be chosen, corresponding to an arbitrarily fixed positive number ϵ, so that $f(x)$ differs from the sum of the convergent series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \sum_{n=1}^{\infty} h_1^n \left\{\cos nx \,.\, \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'dx' + \sin nx \,.\, \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx'\right\}$$

by less than $\frac{1}{2}\epsilon$, for all values of x in (α, β). Since the series converges uniformly for all values of x, an integer m may be so fixed that the remainder of the series after the mth term is numerically less than $\frac{1}{2}\epsilon$, for all values of x. In this manner we obtain* *a finite trigonometrical series*

$$\tfrac{1}{2}A_0 + (A_1\cos x + B_1 \sin x) + \ldots + (A_m \cos mx + B_m \sin mx),$$

the sum of which differs from $f(x)$ by less than ϵ, for every value of x in the interval (α, β) in which $f(x)$ is continuous.

This mode of approximate representation of $f(x)$, in the interval (α, β), is clearly not unique, because the values of the function in that part of $(-\pi, \pi)$ which is not in (α, β) may be altered in any manner, subject only to the integrability of $f(x)$ in $(-\pi, \pi)$, and the continuity of $f(x)$ at the points α, β.

In the above finite series, each of the circular functions can be expanded in powers of x, and the result rearranged as a power-series, of which the sum consequently differs from $f(x)$ by less than ϵ, for all values of x in (α, β). Since the power-series is uniformly convergent, we thus obtain a proof of Weierstrass' theorem, already established in § 159, that a finite polynomial $P(x)$ can be determined, such that $|f(x) - P(x)| < 2\epsilon$, for all values of x in (α, β); the number ϵ being arbitrarily chosen.

Another method†, not involving the use of Poisson's integral, may be employed to determine an approximate representation of a function $f(x)$, continuous in (α, β), by means of finite trigonometrical series. Choose l, so that $-l < \alpha < \beta < l$. As in § 159, a continuous polygonal line can be constructed, such that its ordinate, for each point x in (α, β), differs from $f(x)$ by less than $\frac{1}{2}\epsilon$. The polygonal line may be extended to the whole interval $(-l, l)$, so as to be a continuous polygonal line for the whole interval, and to be such that its ordinates at the points $x = l, -l$ are equal to one another. In virtue of Dirichlet's theory of Fourier's series, the polygonal line may be represented, for the whole interval $(-l, l)$, by a Fourier's series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right);$$

* See Picard's *Traité d'Analyse*, 2nd ed. vol. I, p. 275.

† Volterra, *Rendiconti del Circolo mat. di Palermo*, vol. XI (1897), p. 83.

and, by the theorem of § 333, this series converges uniformly in $(-l, l)$ to the value of the polygonal function. The sum of the Fourier's series differs from $f(x)$ by less than $\frac{1}{2}\epsilon$, at every point of (α, β). The integer m may be so chosen that the sum of the terms for $n > m$ is less than $\frac{1}{2}\epsilon$, for all values of x in (α, β), on account of the uniform convergence. Therefore the finite series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{n=m}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

has the required property, that its sum differs from $f(x)$ by less than ϵ, for all values of x in (α, β). This method may be applied, in the same manner as in the case of the preceding one, to prove Weierstrass' theorem relating to the approximate representation of a continuous function by a finite polynomial.

416. Let $f(x)$ be a function such that both $f(x)$ and $\{f(x)\}^2$ possess Lebesgue integrals in the interval $(-\pi, \pi)$; and let $s_m(x)$ denote the sum of a finite trigonometrical series

$$\tfrac{1}{2}A_0 + \sum_{n=1}^{n=m}(A_n \cos mx + B_n \sin mx).$$

Let us consider the integral

$$I_m = \int_{-\pi}^{\pi}\{f(x) - s_m(x)\}^2 dx.$$

We find that

$$I_m = \int_{-\pi}^{\pi}\{f(x)\}^2 dx + \pi\left[\frac{1}{2}\left\{A_0 - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\right\}^2 \right.$$
$$\left. + \sum_{n=1}^{n=m}\left\{A_n - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right\}^2 + \sum_{n=1}^{n=m}\left\{B_n - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right\}^2\right]$$
$$- \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi} f(x)\,dx\right\}^2 - \frac{1}{\pi}\sum_{n=1}^{n=m}\left[\left\{\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right\}^2\right.$$
$$\left. + \left\{\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right\}^2\right].$$

If I_m be regarded as a quadratic function of

$$A_0, A_1, B_1 \dots A_m, B_m,$$

it is clear that the value of I_m will be least, when

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$
$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

for $n = 1, 2, 3, \dots m$; i.e. when A_0, A_n, B_n are the Fourier's coefficients corresponding to the function $f(x)$. These values of A_0, A_n, B_n are therefore such that the finite trigonometrical series gives the best approximation

to the value of $f(x)$, in accordance with the standard of the method of least squares. The following theorem has been now established:

If $f(x)$ be defined for the interval $(-\pi, \pi)$, and be such that both the function itself, and its square, possess Lebesgue integrals in the interval, then the values of the $2m+1$ constants $A_0, A_1, B_1, \ldots A_m, B_m$, which are such that*

$$\int_{-\pi}^{\pi}\left[f(x) - \tfrac{1}{2}A_0 - \sum_{n=1}^{n=m}(A_m \cos mx + B_m \sin mx)\right]^2 dx$$

has the smallest value, are the Fourier's coefficients corresponding to the function $f(x)$.

The minimum value of the integral I_m is

$$\int_{-\pi}^{\pi}\{f(x)\}^2 dx - \pi\left[\tfrac{1}{2}a_0^2 + \sum_{n=1}^{n=m}(a_n^2 + b_n^2)\right],$$

where a_0, a_n, b_n denote the Fourier's constants corresponding to the function $f(x)$. It follows that this difference is essentially positive, whatever value m may have, and therefore the series $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is necessarily convergent. It has been shewn, in § 378, that the series converges to the value $\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2 dx$. An attempt was made by Harnack† to establish this fact directly, and to found thereon a theory of the convergence of Fourier's series.

It follows, from the above result, that the series $\sum_1^{\infty} a_n^2$, $\sum_1^{n} b_n^2$ are both convergent, and therefore that $\lim_{n=\infty} a_n = 0$, $\lim_{n=\infty} b_n = 0$, which has already been established in § 334, independently of the assumption here made. that $\{f(x)\}^2$ is integrable in $(-\pi, \pi)$.

THE DIFFERENTIATION OF FOURIER'S SERIES

417. In general, the series obtained by differentiating a convergent Fourier's series is not convergent, as may, for example, be seen in the case of the series $\sum_1^{\infty}\frac{1}{n}\sin nx$; neither is the series so obtained necessarily the Fourier's series corresponding to $f'(x)$.

Let $f(x)$ be a bounded function, continuous except for a finite number of ordinary discontinuities; let it also be assumed that $f'(x)$ has a Lebesgue integral in $(-\pi, \pi)$, and that, if it have points of infinite discontinuity, such points form a reducible set. This is consistent with there being a set of points of zero measure at which $f'(x)$ has no definite value. At the

* This theorem was given by Toepler, in a somewhat less general form, see *Wiener Anzeigen*, vol. XIII (1876).

† See two articles in the *Math. Annalen*, vol. XVII (1880), p. 123, and vol. XIX (1882), pp. 254, 526.

points of discontinuity of $f(x)$, we may regard $f'(x)$ as undefined. We have then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \left[\frac{1}{n\pi} f(x)\sin nx\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$$

$$= \frac{1}{n\pi}\left[-\Sigma\{f(\alpha+0)-f(\alpha-0)\}\sin n\alpha\right] - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx,$$

the summation Σ referring to the finite number of points α of ordinary discontinuity of $f(x)$ in the interior of $(-\pi, \pi)$. In a similar manner, we find that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{n\pi}\left[(-1)^n\{f(-\pi+0)-f(\pi-0)\} + \Sigma\{f(\alpha+0)-f(\alpha-0)\}\cos n\alpha\right]$$
$$+ \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx.$$

Also

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{\pi}\left[f(\pi-0)-f(-\pi+0)-\Sigma\{f(\alpha+0)-f(\alpha-0)\}\right].$$

If then, the Fourier's coefficients for the functions $f(x)$, $f'(x)$ be denoted by a_0, a_n, b_n, and a_0', a_n', b_n' respectively, we have

$$a_0' = \frac{1}{\pi}\left[f(\pi-0)-f(-\pi+0)\right] - \frac{1}{\pi}\Sigma\{f(\alpha+0)-f(\alpha-0)\},$$

$$a_n' = nb_n - \frac{1}{\pi}\left[(-1)^n\{f(-\pi+0)-f(\pi-0)\} + \Sigma\{f(\alpha+0)-f(\alpha-0)\}\cos n\alpha\right],$$

$$b_n' = -na_n - \frac{1}{\pi}\Sigma\{f(\alpha+0)-f(\alpha-0)\}\sin n\alpha.$$

In particular, if $f(x)$ be continuous in the interval $(-\pi, \pi)$, so that the function obtained by extending $f(x)$ beyond the interval, in accordance with the rule $f(x) = f(x \pm 2\pi)$, is continuous except at the points $-\pi, \pi$, we have

$$a_0' = \frac{1}{\pi}\{f(\pi) - f(-\pi)\}, \quad a_n' = nb_n + \frac{(-1)^n}{\pi}\{f(\pi)-f(-\pi)\},$$

$b_n' = -na_n$. Unless $f(\pi) = f(-\pi)$, the Fourier's series corresponding to $f'(x)$ is not obtained by term by term differentiation of the Fourier's series for $f(x)$. Even when this condition is satisfied, no assertion can in general be made as to the convergence of the Fourier's series for $f'(x)$. We have thus obtained the following theorem:

If $f(x)$ be continuous in $(-\pi, \pi)$, and if $f(-\pi) = f(\pi)$, and $f'(x)$ have a Lebesgue integral, and have at most a reducible set of points of infinite discontinuity, the Fourier's series for $f'(x)$, whether it converge or not, is obtained by the term by term differentiation of that corresponding to $f(x)$.

If it be known that $f'(x)$ has limited derivatives at any point, or if

$$\lim_{h=+0} \frac{f'(x+h)-f'(x+0)}{h}, \quad \lim_{h=+0} \frac{f'(x-h)-f'(x-0)}{-h}$$

are definite, or are indeterminate between finite limits of indeterminacy, then, in accordance with Theorem (*a*), of § 342, the Fourier's series for $f'(x)$ converges at the point x.

418. In case the function $f(x)$ have derivatives $f'(x), f''(x), \ldots$ of any number of orders, and $f(x), f'(x), f''(x), \ldots$ are all bounded and continuous in $(-\pi, \pi)$, except at a finite number of points at which they have ordinary discontinuities, the coefficients a_n, b_n may be expressed in a form which exhibits these discontinuities.

At a point α, of discontinuity of $f(x)$, the function $f'(x)$ may be regarded as undefined, the values of $f'(\alpha+0), f'(\alpha-0)$ being

$$\lim_{h=+0} \frac{f(\alpha+h)-f(\alpha+0)}{h}, \quad \lim_{h=+0} \frac{f(\alpha-h)-f(\alpha-0)}{-h}$$

respectively. A similar remark applies to the higher differential coefficients.

We find, by integrating twice by parts,

$$a_n = -\frac{1}{n\pi} \Sigma \{f(\alpha+0)-f(\alpha-0)\} \sin n\alpha$$
$$-\frac{1}{n^2\pi} \Sigma \{f'(\beta+0)-f'(\beta-0)\} \cos n\beta - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx\,dx,$$

$$b_n = \frac{1}{n\pi} \Sigma \{f(\alpha+0)-f(\alpha-0)\} \cos n\alpha$$
$$-\frac{1}{n^2\pi} \Sigma \{f'(\beta+0)-f'(\beta-0)\} \sin n\beta - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \sin nx\,dx,$$

where $-\pi$ is now included among the points α of discontinuity of $f(x)$, and amongst β, the points of discontinuity of $f'(x)$. The points α in general occur amongst the points β.

We may proceed, by further* integration by parts, to express a_n and b_n in a series proceeding by powers of $1/n$, the coefficients of which involve the measures of discontinuity of the functions at the points $\alpha, \beta, \ldots$.

Conversely, if the Fourier's coefficients for $f(x)$ are given in the forms

$$a_n = \frac{1}{n} \Sigma A \sin n\alpha + \frac{1}{n^2} \Sigma B \cos n\beta + \ldots,$$

$$b_n = -\frac{1}{n} \Sigma A \cos n\alpha + \frac{1}{n^2} \Sigma B \sin n\beta - \ldots,$$

* See Stokes, "On the critical values of the sums of periodic series," *Math. and Phys. Papers*, vol. I, where this investigation is carried out in detail, and the resulting formulae for the differentiation of Fourier's series are applied to physical problems.

so that the Fourier's series has for its general term

$$\frac{1}{n}\Sigma A\sin n(\alpha - x) + \frac{1}{n^2}\Sigma B\cos n(\beta - x) + \dots,$$

we have

$$f(\alpha+0) - f(\alpha-0) = -\pi A,\quad f'(\beta+0) - f'(\beta-0) = -\pi B, \dots.$$

Thus the points of discontinuity, and the measures of discontinuity, of $f(x)$, $f'(x)$, ..., are determined when a_n, b_n are exhibited as series proceeding according to powers of $1/n$.

419. The following further theorems* relating to the differentiation of trigonometrical series will be stated:

If the trigonometrical series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

converge for a particular value c *of* x, *and if the series*

$$\sum_{n=1}^{\infty}(-na_n\sin nx + nb_n\cos nx),$$

obtained by term by term differentiation, converge uniformly in an interval (α, β) *which contains the point* c *in its interior, then the original series converges uniformly in* (α, β), *and the function* $f(x)$ *represented by it has, throughout the interval, a differential coefficient represented by the derived series.*

If the series $$\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

converge for a particular value c *of* x, *which is not zero or a multiple of* π, *and if* $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, *then throughout an interval* (α, β) *which contains the point* c *in its interior, but does not include the point* 0, *or* $k\pi$, *where* k *is any integer, the series converges uniformly, and the function* $f(x)$ *represented by it will have a differential coefficient* $f'(x)$ *given by*

$$2\sin x\,.\,f'(x) = \sum_{n=0}^{\infty}\{[(n-1)a_{n-1} - (n+1)a_{n+1}]\cos nx$$
$$+ [(n-1)b_{n-1} - (n+1)b_{n+1}]\sin nx\},$$

where $a_{-1} = b_{-1} = a_0 = b_0 = 0$, *provided this last series converges uniformly in the interval* (α, β).

For a function $f(x)$ which possesses differential coefficients of all orders in the interval $(-\pi, \pi)$, it is not in general possible to obtain representations of all these differential coefficients by means of successive term by term differentiation of the Fourier's series which represents $f(x)$.

* See Bôcher's "Introduction to the theory of Fourier's series," *Annals of Math.* (2), vol. VII (1906), p. 120. The second theorem is substantially due to Lerch, *Annales sc. de l'école normale* (3), vol. XII (1895), p. 351.

The following theorem, due to Borel*, gives the means of obtaining the requisite representation of such functions:

Having given a function $f(x)$ which has differential coefficients of all orders throughout the interval $(-\pi, \pi)$, the function can be represented by means of a series of the type

$$\sum_{n=0}^{\infty} (A_n x^n + a_n \cos nx + b_n \sin nx);$$

and the differential coefficients of $f(x)$, of all orders, are represented by the series obtained by successive term by term differentiation of this series. All the series so obtained converge uniformly in the interval $(-\pi, \pi)$.

GENERAL EXAMPLES

(1) The trigonometrical series

$$b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots$$

is uniformly convergent in any interval not containing the point $x=0$, or any point $x = \pm 2k\pi$, (k integral), if $\lim_{n\to\infty} b_n = 0$, and if also $\sum_{n=1}^{\infty} |b_n - b_{n+1}|$ be convergent. For

$$2 \sin \tfrac{1}{2}x \,.\, s_n(x) = b_1 \cos \tfrac{1}{2}x - \sum_{r=1}^{n-1} (b_r - b_{r+1}) \cos \tfrac{1}{2}(2r+1)x - b_n \cos \tfrac{1}{2}(2n+1)x,$$

whence the result follows. It suffices† for the convergence of the series that $\lim_{n\to\infty} b_n = 0$, and that also $b_n \geqq b_{n+1}$, for all values of n greater than some fixed value m; the convergence is then as before uniform in any interval which does not contain $x=0$ or $x = \pm 2k\pi$, for any integral value of k.

The series $\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots$ may similarly be shewn to converge uniformly in any interval not containing $x=0$, or any point $\pm 2k\pi$, if $\lim_{n\to\infty} a_n = 0$, and if also $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ be convergent. If $\lim_{n=\infty} a_n = 0$, and $a_n \geqq a_{n+1}$, for $n > m$, the series† converges as before for all values of x, except 0 or $\pm 2k\pi$.

(2) Let $f(x)$‡ be a function, of period 2π, bounded and measurable in any interval which does not contain the point $x=0$, or any point $x=2k\pi$; but let $f(x)$ not satisfy these conditions in the neighbourhood of $x=0$. Let it be assumed, (1), that $|f(x)+f(-x)|$ is integrable, in $(0, \pi)$, (2), that $\lim_{x=0} \{xf(x)\} = 0$, and (3), that $xf(x)$ has its Fourier's series convergent at the point $x=0$. The coefficients a_n, b_n, for the function $f(x)$, then exist, and $\lim_{n=\infty} a_n = 0$. Also it follows from (3) that $\lim_{n=\infty} b_n = 0$. For this last condition is equivalent to

$$\lim_{n=\infty} \int_{-\pi}^{\pi} f(x') \tan \tfrac{1}{2}x' \frac{\sin \frac{1}{2}(2n+1)x'}{\sin \frac{1}{2}x'} dx' = 0,$$

which holds if $f(x) \tan \frac{1}{2}x$ have its Fourier's series convergent at $x=0$; and $f(x)\tan\frac{1}{2}x$ may clearly be replaced by $xf(x)$.

* See the *Leçons sur les fonctions de variables réelles*, p. 68, where this theorem is proved.

† Schlömilch, *Compendium d. höheren Analysis*, vol. I, § 40.

‡ See Fatou, *Comptes Rendus*, vol. CXLII (1906), p. 765.

It can now be seen easily that

$$\int_{-\pi}^{\pi} \frac{\sin \frac{1}{2}(2n+1)(x-x')}{\sin \frac{1}{2}(x-x')} f(x')\,dx'$$

has the limit 0, when n is indefinitely increased, on condition that the integral is interpreted as having its Cauchy principal value in the neighbourhood of $x'=0$. When the conditions (1), (2), (3) are satisfied, the necessary and sufficient condition that the series should converge to $f(x)$ is that that function which $=f(x)$ in the neighbourhood of the point x, and is elsewhere zero, should be representable by a Fourier's series.

Let $$f(x) = \frac{\sin \frac{\pi}{x}}{x \log \frac{1}{x} \log\log \frac{1}{x}}, \text{ where } 0 < x \leqq a < e^{-e},$$

and let $f(x)+f(-x)=0$. This function satisfies conditions (1), (2), (3), and is representable by a series

$$a_1 \sin \frac{\pi x}{a} + a_2 \sin \frac{2\pi x}{a} + \dots .$$

$|f(x)|$ is not integrable, although $f(x)$ is so; thus the series is a generalized Fourier's series.

(3) The convergent series* $\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$ represents a function which is not integrable (L) or (D), in an interval containing the point $x=0$. The series $\sum_{n=2}^{\infty} \frac{-\cos nx}{n \log n}$ is not convergent.

(4) In the series $\sum_{1}^{\infty} \sin (n!\, \pi x)$, the coefficients do not become indefinitely small, and therefore the series is not a Fourier's series. The series converges, however, for all rational values of x; it also converges for an infinite number of irrational values, for example, for $x=\sin 1$, $\cos 1$, $2/e$, and for multiples of these values; also for odd multiples of e. This example is due to Riemann, and the series has been considered in detail by Genocchi†.

(5) Consider the series $\sum_{n=0}^{\infty} c_n \cos n^2 x$, $\sum_{n=1}^{\infty} c_n \sin n^2 x$, where $c_0, c_1, c_2, \dots$ are positive numbers, and such that $\lim_{n\to\infty} c_n = 0$, but such that $\sum_{n=1}^{\infty} c_n$ is divergent. The points of convergence, and the points of divergence, of these series both form everywhere-dense sets. These series have been treated‡ in detail by Hardy and Littlewood.

(6) The function $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(nx)$, where (nx) denotes the excess of nx over the nearest integer, and where $(nx)=0$ when nx is half an odd integer, is not integrable in accordance with Riemann's definition. Riemann has however given the series

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{S_\theta [-(-1)^\theta]}{n} \sin 2n\pi x,$$

as representing $f(x)$; where the summation S_θ refers to all the factors θ, of n.

* See Fatou, *Comptes Rendus*, vol. CXLII (1906), p. 767.

† *Atti di Torino*, vol. X (1875), p. 985.

‡ *Acta Math.* vol. XXXVII (1914), p. 222.

RIEMANN'S THEORY OF TRIGONOMETRICAL SERIES

420. After the fundamental investigation of Dirichlet, in which sufficient conditions were obtained for the convergence of the Fourier's series corresponding to a given function, the next great advance in the theory was made by Riemann*, in his celebrated memoir on the representation of a function by means of trigonometrical series. This memoir formed the point of departure, on which much of the subsequent development of the theory depended. An account of Riemann's theory, in a modified form, with some later developments, will be given here.

Denoting the series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$ by

$$A_0(x) + A_1(x) + A_2(x) + \dots + A_n(x) + \dots,$$

where $\qquad A_0(x) = \frac{1}{2}a_0, \quad A_n(x) = a_n \cos nx + b_n \sin nx,$

it is assumed in general that $\lim_{n\sim\infty} (a_n \cos nx + b_n \sin nx) = 0$, for each value of x in a given interval. It was proved later by Cantor that this assumption implies that $\lim_{n\sim\infty} a_n = 0$, and $\lim_{n\sim\infty} b_n = 0$. In some parts of Riemann's investigations it is sufficient to make the wider assumption that

$$| a_n \cos nx + b_n \sin nx |$$

is bounded for all values of n, and of x in some prescribed interval. It is not assumed that the coefficients necessarily have the form of the coefficients in a Fourier's series; so that the theory refers primarily to trigonometrical series in general.

Riemann's method of investigation depends essentially upon his introduction of a special method of treatment of the series, which leads to a conventional definition of the sum of the trigonometrical series. This conventional sum of the series, which may be spoken of as its sum (R), is equal to the ordinary sum of the series at any point x at which the latter exists, but the sum (R) may exist for a point x for which the series is not convergent.

If we take the series

$$A_0(x) + A_1(x)\left(\frac{\sin h}{h}\right)^2 + A_2(x)\left(\frac{\sin 2h}{2h}\right)^2 + \dots + A_n(x)\left(\frac{\sin nh}{nh}\right)^2 + \dots,$$

where $| A_n(x) | < k$, for all values of n and x, and denote its sum-function at the point x by $S(x, h)$, this sum-function having a unique value, for each value of h (> 0), since the series $\sum \frac{| A_n(x) |}{n^2}$ is convergent, then if $\lim_{h\sim 0} S(x, h) = S(x)$, the function $S(x)$ may be termed the Riemann sum-

* "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe." This memoir, originally written in 1854 as a thesis, was published in the *Abhandlungen d. K. Ges. d. Wissensch. zu Göttingen*, vol. XIII. See also Riemann's *Gesammelte Werke*, 2nd ed. p. 227.

function of the series $A_0(x) + A_1(x) + \dots$. This function $S(x)$ may, at a particular point x, have a definite value, or it may have an upper value $\bar{S}(x)$, and a lower value $\underline{S}(x)$; where

$$\bar{S}(x) = \overline{\lim_{h\sim 0}}\, S(x, h), \quad \underline{S}(x) = \underline{\lim_{h\sim 0}}\, S(x, h).$$

Thus $S(x)$ is the repeated limit

$$\lim_{h\sim 0} \lim_{n\sim\infty} \left[A_0(x) + A_1(x)\left(\frac{\sin h}{h}\right)^2 + \dots + A_n(x)\left(\frac{\sin nh}{nh}\right)^2\right],$$

whereas the ordinary sum-function of the series is

$$\lim_{n\sim\infty} \lim_{h\sim 0} \left[A_0(x) + A_1(x)\left(\frac{\sin h}{h}\right)^2 + \dots + A_n(x)\left(\frac{\sin nh}{nh}\right)^2\right].$$

It is in accordance with a frequent mode of procedure in defining a conventional value of a repeated limit, to regard it as the repeated limit when the order of the successive limits is reversed (see § 46).

Riemann introduced the continuous function $F(x)$ represented by the series

$$C + C'x + \tfrac{1}{2}A_0x^2 - A_1(x) - \frac{1}{2^2}A_2(x) - \dots - \frac{1}{n^2}A_n(x) - \dots,$$

which certainly exists when $|A_n(x)|$ is bounded for all values of n and of x, in a given interval. For if $|A_n(x)| < k$, the series

$$A_1(x) + \frac{1}{2^2}A_2(x) + \dots + \frac{1}{n^2}A_n(x) + \dots$$

converges uniformly in the given interval, and thus has its sum-function continuous. This is in particular the case, in any interval whatever, when $\lim_{n\sim\infty} a_n = \lim_{n\sim\infty} b_n = 0$. It is easily seen by substitution that, for the function $F(x)$ so defined, we have

$$\frac{F(x+2h) + F(x-2h) - 2F(x)}{4h^2} = A_0 + A_1\left(\frac{\sin h}{h}\right)^2 + \dots + A_n\left(\frac{\sin nh}{nh}\right)^2 + \dots.$$

It is convenient to define the generalized second differential coefficient of a function $\phi(x)$ at a point x, as $\lim_{h\sim 0} \dfrac{\phi(x+h) + \phi(x-h) - 2\phi(x)}{h^2}$. This may have a definite value $\mathscr{D}^2\phi(x)$, or it may have upper and lower values $\overline{\mathscr{D}}^2\phi(x)$, $\underline{\mathscr{D}}^2\phi(x)$.

It thus appears that the Riemann sum-function of the series

$$A_0(x) + A_1(x) + A_2(x) + \dots$$

is $\mathscr{D}^2F(x)$, where $F(x)$ denotes the continuous function

$$C + C'x + \tfrac{1}{2}A_0x^2 - A_1(x) - \frac{1}{2^2}A_2(x) - \dots - \frac{1}{n^2}A_n(x) - \dots.$$

It has been shewn* by Rajchman, that if $a_n = o(1)$, $b_n = o(1)$, and the trigonometrical series is summable (R), at a point x, it is also summable $(C, 3)$, at the same point.

Rajchman has also given† the following relations between the upper and lower Riemann and Poisson sums of a trigonometrical series for which $a_n = o(1)$, $b_n = o(1)$; and $P(r, x)$ denotes

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, r^n, \quad \underline{\mathscr{D}}^2 F(x) \leq \overline{\lim_{r\sim 1}}\, P(r, x),$$
$$\underline{\lim_{r\sim 1}}\, P(r, x) \leq \overline{\mathscr{D}}^2 F(x).$$

This theorem is stated by Zygmund‡ to hold provided only $F(x)$ is everywhere continuous. Rajchman and Zygmund have considered§ the relation of the Cesàro summation with a generalization of Riemann summation.

421. Riemann's first theorem, in a generalized form, consists of three parts, and may be stated as follows:

Theorem I. *Having given the trigonometrical series* $A_0 + \sum_{n=1} A_n(x)$, *where* $A_n(x)$ *denotes* $a_n \cos nx + b_n \sin nx$, *and* A_0 *denotes* $\tfrac{1}{2}a_0$, *for which* $a_n = o(1)$, $b_n = o(1)$, *there exists a continuous function defined by*

$$F(x) = C + C'x + \tfrac{1}{2}A_0 x^2 - \sum_{n=1}^{\infty} \frac{1}{n^2} A_n(x),$$

which has the following properties:

(1) *For any value of x for which the given series*

$$A_0 + A_1(x) + A_2(x) + \ldots + A_n(x) + \ldots$$

converges to the value $f(x)$, $\mathscr{D}^2 F(x)$ *has the definite value* $f(x)$. *Moreover if, at the point x, the given series has upper and lower sum-functions* $\bar{f}(x)$, $\underline{f}(x)$, *both* $\overline{\mathscr{D}}^2 F(x)$ *and* $\underline{\mathscr{D}}^2 F(x)$ *lie in the interval formed by the two numbers*

$$\tfrac{1}{2}\{\bar{f}(x) + \underline{f}(x)\} + \frac{\lambda}{2}\{\bar{f}(x) - \underline{f}(x)\},$$

where λ is some fixed number.

This property holds also if it is only assumed that $A_n(x)$ is bounded with respect to (n, x) in some neighbourhood of the point x, and thus that $F(x)$ exists in such neighbourhood.

The second part of the statement was first given substantially by Du Bois-Reymond.

(2) *For any value of x whatever*

$$\lim_{h\sim 0} \frac{F(x+2h) + F(x-2h) - 2F(x)}{2h} = 0.$$

It is unnecessary that the given trigonometrical series should converge at the point x.

* *Comptes Rendus de la soc. des sciences de Varsovie*, vol. XI (1918), p. 116. See also *Fundamenta Math.* vol. III (1922), p. 287.

† *Prace Matem.-fiz.* vol. XXX (1919), and *Comptes Rendus*, vol. CLXXVII (1923), p. 492.

‡ *Comptes Rendus*, vol. CLXXVII (1923), p. 523. § *Bulletin de l'Acad. Polonaise* (1925), p. 69.

This has as its consequence that, at each point x, $F(x)$ has its derivatives symmetrical as regards the right and left of the point, so that

$$D^+ F(x) = D^- F(x), \quad D_+ F(x) = D_- F(x).$$

(3) *If (b, c) be any interval, and if $\lambda(x)$ and its differential coefficient $\lambda'(x)$ are continuous in (b, c), and vanish at b and c, and if $\lambda''(x)$ be summable and everywhere finite in (b, c), then $\mu^2 \int_b^c F(x)\,\lambda(x) \cos \mu(x-a)\,dx$ converges to zero, as $\mu \sim \infty$, uniformly for all values of a. It is here necessary that $a_n = o(1)$, $b_n = o(1)$.*

Riemann himself restricted $\lambda''(x)$ to be a continuous function possessing only a finite set of maxima and minima in the interval, and he does not mention the uniform convergence for all values of a.

In the above statement it is not absolutely necessary that $\lambda''(x)$ should everywhere exist and be finite; more generally it is sufficient that $\lambda''(x)$ should be summable and that $\lambda'(x)$ should satisfy the condition of being an indefinite L-integral.

422. The part I (1) of the theorem has already been established in § 157.

In order to prove I (2), that, whether the series $\Sigma A_n(x)$ converges or not, so long as $\lim\limits_{n \sim \infty} A_n(x) = 0$, for each fixed x, $\dfrac{\nabla^2 F(x)}{2h}$ converges to zero, as $h \sim 0$. we divide the terms of the series $A_0 + \sum\limits_{n=1}^{\infty} A_n(x) \left(\dfrac{\sin nh}{nh}\right)^2$ into three parts.

The first part is $A_0 + \sum\limits_{n=1}^{n=m} A_n(x) \left(\dfrac{\sin nh}{nh}\right)^2$, where m is a fixed number so chosen that $|A_n(x)| < \epsilon$, for $n > m$. The limit of this sum, which we denote by Σ, is the finite number $A_0 + \sum\limits_{n=1}^{m} A_n(x)$, when $h \sim 0$.

The second part is taken to be $\sum\limits_{m+1}^{s} A_n(x) \left(\dfrac{\sin nh}{nh}\right)^2$, where

$$sh < \pi \leq (s+1)h;$$

the sum of these terms is numerically less than $\dfrac{\epsilon\pi}{h}$. The third part

$$\sum_{s+1}^{\infty} A_n(x) \left(\frac{\sin nh}{nh}\right)^2$$

is numerically less than $\dfrac{\epsilon}{h^2} \sum\limits_{s+1}^{\infty} \dfrac{1}{n^2}$, or than $\dfrac{\epsilon}{sh^2}$.

We have now $\qquad \left|\dfrac{\nabla^2 F(x)}{2h}\right| < 2h\Sigma + 2\pi\epsilon + \dfrac{2\epsilon}{sh}.$

Since sh converges to π, Σ_h is a finite number, and ϵ is arbitrary, we have

$$\lim_{h \sim 0} \frac{\nabla^2 F(x)}{2h} = 0.$$

With a view to proving Riemann's theorem I (3), let $\Phi(x)$ denote the periodic function $F(x) - C - C'x - \frac{1}{2}Ax^2$, or $-\sum_{n=1}^{\infty} \frac{1}{n^2} A_n(x)$. The following theorem, somewhat more general than is necessary for the special purpose, will be first established:

If $\psi(x)$ be such that it has a continuous differential coefficient $\psi'(x)$ in the interval (b, c), and $\psi(x)$, $\psi'(x)$ both vanish at b and c, and if also $\psi''(x)$ is summable in (b, c) and is such that $\psi'(x)$ is its indefinite integral, then

$$\mu^2 \int_b^c \psi(x)\, \Phi(x + t) \cos \mu (x - a)\, dx$$

converges to zero, as $\mu \sim \infty$, uniformly for all values of a and t.

It is easily seen that

$$\Phi(x + t) = -\sum_{n=1}^{\infty} \frac{1}{n^2} A_n(t) \cos nx - \sum_{n=1}^{\infty} \frac{1}{n^2} B_n(t) \sin nx,$$

where $B_n(t)$ denotes $b_n \cos nt - a_n \sin nt$. Denoting $\sqrt{a_n^2 + b_n^2}$ by c_n, where $\lim_{n\sim\infty} c_n = 0$, we may write $c_n \cos(nt - \beta_n)$, $-c_n \sin(nt \mp \beta_n)$, for $A_n(t)$ and $B_n(t)$ respectively, and thus

$$\Phi(x + t) = -\sum_{n=1}^{\infty} \frac{c_n}{n^2} \cos(nx + nt - \beta_n).$$

Since this series converges uniformly, we have to consider the expression

$$\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \int_b^c \psi(x) \cos(nx + nt - \beta_n) \cos \mu (x - a)\, dx.$$

We find by two integrations by parts, which are valid since $\psi(x)$, $\psi'(x)$ are indefinite integrals of $\psi'(x)$, $\psi''(x)$ respectively, that

$$\int_b^c \psi(x) \cos(kx - \alpha_k)\, dx = -\frac{1}{k^2} \int_b^c \psi''(x) \cos(kx - \alpha_k)\, dx,$$

where k is any number, and α_k may depend upon k, and upon any other parameters. The absolute value of the expression on the right-hand side may be written in the form $\frac{1}{k^2}\eta_k$, where, in accordance with the theorem of § 334, η_k converges to zero, as $k \sim \infty$, uniformly for all values of the parameters upon which α_k depends. Also $\eta_k < \bar{\eta}$, for all values of k, where $\bar{\eta}$ is a fixed number.

We have now to consider the expression

$$\tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \int_b^c \psi(x) \cos\{(\mu + n)x - \beta_n'\}\, dx$$
$$+ \tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \int_b^c \psi(x) \cos\{(\mu - n)x - \beta_n''\}\, dx,$$

where β_n', β_n'' depend upon a, μ, n, and t. The absolute value of this expression is less than

$$\tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \frac{\eta_{m+n}}{(\mu+n)^2} + \tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \frac{\eta_{|\mu-n|}}{(\mu-n)^2}.$$

If m be so chosen that $|\eta_{m+n}| < \delta$, for $n = 1, 2, 3, \ldots$; and such value of m may be chosen independently of the values of a and t, it appears that the first part of this expression is less than $\frac{1}{2}\delta \sum_{n=1}^{\infty} \frac{|c_n|}{n^2}$, or than $P\delta$, where P is some fixed number independent of δ.

Denoting by $[\frac{1}{2}\mu]$, $[\mu]$ the integers next less than $\frac{1}{2}\mu$, μ respectively, the expression $\frac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{|c_n|}{n^2} \frac{\eta_{|\mu-n|}}{(\mu-n)^2}$ may be divided into four parts. The first part contains those terms of the series for which n is taken from 1 to $[\frac{1}{2}\mu]$. This part is less than $2 \sum_{n=1}^{[\frac{1}{2}\mu]} \frac{|c_n|}{n^2} \eta_{\mu-n}$; and μ may be taken so great that $\eta_k < \delta$, for $k \geqq \mu - [\frac{1}{2}\mu]$; it then follows that this expression is less than $P'\delta$, where P' is some fixed number independent of δ and μ.

The second part contains those terms of the series for which n is taken from $[\frac{1}{2}\mu] + 1$ to $[\mu] - 1$; we may assume that μ is taken so great that $c_n < \delta$, for $n \geqq [\frac{1}{2}\mu] + 1$. This part is now seen to be numerically less than $\frac{1}{2}\delta\mu^2 \frac{\bar{\eta}}{[\frac{1}{2}\mu]^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$, or than $P''\delta$, where P'' is a fixed number independent of δ and μ.

We next take the two terms for which n has the values $[\mu]$, $[\mu] + 1$, both of which are fixed multiples of δ. In case μ is an integer, the term corresponding to $\mu = n$ may be omitted in the original expression, the corresponding term being $c_\mu \left|\int_b^c \psi(x) \cos \beta_\mu'' \, dx\right|$, which is less than a fixed multiple of δ. In any case the two terms together are less than $P'''\delta$, where P''' is independent of δ and μ.

The last part to consider contains those terms for which n has all values $> [\mu] + 1$.

This part is numerically less than $\frac{1}{2}\delta\mu^2 \frac{\bar{\eta}}{[\mu]^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$, or than $P^{\text{iv}}\delta$, where P^{iv} is independent of δ and μ.

It has now been shewn that the whole expression is, for sufficiently large values of μ, less than a fixed multiple of δ. Since δ is arbitrary, the theorem has been established.

It will next be shewn that

$$\mu^2 \int_b^c \psi(x) \{C + C'(x+t) + \tfrac{1}{2}A_0(x+t)^2\} \cos \mu (x - a) \, dx$$

converges to zero, as $\mu \sim \infty$, uniformly for all values of a, and uniformly

for all values of t in a finite interval, the function $\psi(x)$ satisfying the same conditions as before. The expression $C + C'(x+t) + \frac{1}{2}A_0(x+t)^2$ may be rearranged in the form $\alpha_t + \beta_t x + \frac{1}{2}A_0 x^2$, where α_t and β_t are quadratic functions of t, and therefore are bounded for all values of t in a finite interval. We have then to consider

$$\mu^2 \int_b^c \psi(x)\,(\alpha_t + \beta_t x + \tfrac{1}{2}A_0 x^2) \cos\mu(x-a)\,dx.$$

It can be verified that

$$\mu^2(\alpha_t + \beta_t x + \tfrac{1}{2}A_0 x^2)\cos\mu(x-a)$$
$$= -\frac{d^2}{dx^2}\left[\left\{\alpha_t - \frac{3A_0}{\mu^2} + \beta_t x + \tfrac{1}{2}A_0 x^2\right\}\cos\mu(x-a) - 2(\beta_t + A_0 x)\frac{\sin\mu(x-a)}{\mu}\right];$$

hence, on integrating twice by parts, the expression takes the form

$$\int_b^c \left[2(\beta_t + A_0 x)\frac{\sin\mu(x-a)}{\mu}\psi''(x) - \left\{\alpha_t - \frac{3A_0}{\mu^2} + \beta_t x + \tfrac{1}{2}A_0 x^2\right\}\cos\mu(x-a)\,\psi''(x)\right]dx.$$

Since the integrals $\int_b^c f(x) \begin{matrix}\cos\\ \sin\end{matrix} \mu(x-a)\,dx$, where $f(x)$ is any summable function, are numerically arbitrarily small, provided μ has a sufficiently large value, and since $|\beta_t|$, $|\alpha_t|$ are less than fixed numbers independent of the particular value of t, it follows that the integral converges to zero, uniformly for all values of t in a fixed finite interval, and uniformly for all values of a.

Combining this result with the theorem already established, we obtain the following theorem, which contains Riemann's theorem I (3) as the particular case which arises when t has the single value zero:

If $\psi(x)$ be such that it has a continuous differential coefficient $\psi'(x)$ in the interval (b, c), and $\psi(x)$, $\psi'(x)$ both vanish at b and c, and if also $\psi''(x)$ is summable in (b, c) and has $\psi'(x)$ for its indefinite integral, then

$$\mu^2 \int_b^c F(x+t)\,\psi(x)\cos\mu(x-a)\,dx$$

converges to 0, as $\mu \sim \infty$, uniformly for all values of t in a finite interval, and uniformly for all values of a.

423. Riemann's second theorem is concerned with conditions under which a trigonometrical series may exist of which the sum (R) shall have the values of a prescribed function. The theorem may be stated as follows:

Theorem II. *If $f(x)$ be a function, of period 2π, defined for every value of x, necessary and sufficient conditions that a trigonometrical series*

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx),$$

where β_n', β_n'' depend upon a, μ, n, and t. The absolute value of this expression is less than

$$\tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \frac{\eta_{m+n}}{(\mu+n)^2} + \tfrac{1}{2}\mu^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2} \frac{\eta_{|\mu-n|}}{(\mu-n)^2}.$$

If m be so chosen that $|\eta_{m+n}| < \delta$, for $n = 1, 2, 3, \ldots$; and such value of m may be chosen independently of the values of a and t, it appears that the first part of this expression is less than $\tfrac{1}{2}\delta \sum\limits_{n=1}^{\infty} \frac{|c_n|}{n^2}$, or than $P\delta$, where P is some fixed number independent of δ.

Denoting by $[\tfrac{1}{2}\mu]$, $[\mu]$ the integers next less than $\tfrac{1}{2}\mu$, μ respectively, the expression $\tfrac{1}{2}\mu^2 \sum\limits_{n=1}^{\infty} \frac{|c_n|}{n^2} \frac{\eta_{|\mu-n|}}{(\mu-n)^2}$ may be divided into four parts. The first part contains those terms of the series for which n is taken from 1 to $[\tfrac{1}{2}\mu]$. This part is less than $2 \sum\limits_{n=1}^{[\frac{1}{2}\mu]} \frac{|c_n|\,\eta_{\mu-n}}{n^2}$; and μ may be taken so great that $\eta_k < \delta$, for $k \geqq \mu - [\tfrac{1}{2}\mu]$; it then follows that this expression is less than $P'\delta$, where P' is some fixed number independent of δ and μ.

The second part contains those terms of the series for which n is taken from $[\tfrac{1}{2}\mu] + 1$ to $[\mu] - 1$; we may assume that μ is taken so great that $c_n < \delta$, for $n \geqq [\tfrac{1}{2}\mu] + 1$. This part is now seen to be numerically less than $\tfrac{1}{2}\delta\mu^2 \frac{\bar{\eta}}{[\frac{1}{2}\mu]^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, or than $P''\delta$, where P'' is a fixed number independent of δ and μ.

We next take the two terms for which n has the values $[\mu]$, $[\mu] + 1$, both of which are fixed multiples of δ. In case μ is an integer, the term corresponding to $\mu = n$ may be omitted in the original expression, the corresponding term being $c_\mu \left| \int_b \psi(x) \cos \beta_\mu''\, dx \right|$, which is less than a fixed multiple of δ. In any case the two terms together are less than $P'''\delta$, where P''' is independent of δ and μ.

The last part to consider contains those terms for which n has all values $> [\mu] + 1$.

This part is numerically less than $\tfrac{1}{2}\delta\mu^2 \frac{\bar{\eta}}{[\mu]^2} \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, or than $P^{\text{iv}}\delta$, where P^{iv} is independent of δ and μ.

It has now been shewn that the whole expression is, for sufficiently large values of μ, less than a fixed multiple of δ. Since δ is arbitrary, the theorem has been established.

It will next be shewn that

$$\mu^2 \int_b^c \psi(x) \{C + C'(x+t) + \tfrac{1}{2}A_0 (x+t)^2\} \cos \mu (x - a)\, dx$$

converges to zero, as $\mu \sim \infty$, uniformly for all values of a, and uniformly

for all values of t in a finite interval, the function $\psi(x)$ satisfying the same conditions as before. The expression $C + C'(x+t) + \frac{1}{2}A_0(x+t)^2$ may be rearranged in the form $\alpha_t + \beta_t x + \frac{1}{2}A_0 x^2$, where α_t and β_t are quadratic functions of t, and therefore are bounded for all values of t in a finite interval. We have then to consider

$$\mu^2 \int_b^c \psi(x)\,(\alpha_t + \beta_t x + \tfrac{1}{2}A_0 x^2)\cos\mu(x-a)\,dx.$$

It can be verified that

$$\mu^2(\alpha_t + \beta_t x + \tfrac{1}{2}A_0 x^2)\cos\mu(x-a)$$
$$= -\frac{d^2}{dx^2}\left[\left\{\alpha_t - \frac{3A_0}{\mu^2} + \beta_t x + \tfrac{1}{2}A_0 x^2\right\}\cos\mu(x-a) - 2(\beta_t + A_0 x)\frac{\sin\mu(x-a)}{\mu}\right];$$

hence, on integrating twice by parts, the expression takes the form

$$\int_b^c \left[2(\beta_t + A_0 x)\frac{\sin\mu(x-a)}{\mu}\psi''(x) - \left\{\alpha_t - \frac{3A_0}{\mu^2} + \beta_t x + \tfrac{1}{2}A_0 x^2\right\}\cos\mu(x-a)\,\psi''(x)\right]dx.$$

Since the integrals $\int_b^c f(x)\,{\cos \atop \sin}\,\mu(x-a)\,dx$, where $f(x)$ is any summable function, are numerically arbitrarily small, provided μ has a sufficiently large value, and since $|\beta_t|$, $|\alpha_t|$ are less than fixed numbers independent of the particular value of t, it follows that the integral converges to zero, uniformly for all values of t in a fixed finite interval, and uniformly for all values of a.

Combining this result with the theorem already established, we obtain the following theorem, which contains Riemann's theorem I (3) as the particular case which arises when t has the single value zero:

If $\psi(x)$ be such that it has a continuous differential coefficient $\psi'(x)$ in the interval (b, c), and $\psi(x)$, $\psi'(x)$ both vanish at b and c, and if also $\psi''(x)$ is summable in (b, c) and has $\psi'(x)$ for its indefinite integral, then

$$\mu^2 \int_b^c F(x+t)\,\psi(x)\cos\mu(x-a)\,dx$$

converges to 0, as $\mu \sim \infty$, uniformly for all values of t in a finite interval, and uniformly for all values of a.

423. Riemann's second theorem is concerned with conditions under which a trigonometrical series may exist of which the sum (R) shall have the values of a prescribed function. The theorem may be stated as follows:

Theorem II. *If $f(x)$ be a function, of period 2π, defined for every value of x, necessary and sufficient conditions that a trigonometrical series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}(a_n\cos nx + b_n \sin nx),$$

such that $a_n = o(1)$, $b_n = o(1)$ ***exists, of which*** $f(x)$ ***is the sum (R), and which, at every point of convergence, converges to the value*** $f(x)$, ***are the following:***

(1) ***That a continuous function*** $F(x)$ ***should exist, such that, for all values of*** x, $\mathscr{D}^2 F(x) = f(x)$.

(2) ***That, if*** b, c ***be any two numbers,***

$$\mu^2 \int_b^c F(x) \cos \mu (x - a) \lambda(x)\, dx$$

should converge to the limit zero, as μ ***is indefinitely increased, where*** $\lambda(x)$ ***is any function such that*** $\lambda'(x)$ ***exists in*** (b, c) ***and*** $\lambda''(x)$ ***exists and is summable, with*** $\lambda'(x)$ ***for its indefinite integral; and such that*** $\lambda(x)$, $\lambda'(x)$ ***both vanish at*** b ***and*** c.

It will be observed that the theorem makes no assertion as to the convergence of the trigonometrical series at any particular point, neither does it assert that the series is a Fourier's series.

That (1) and (2) are necessary conditions has been already established; it therefore remains to prove their sufficiency.

Let $\phi(x)$ denote $F(x + 2\pi) - F(x)$, then, from the condition (1), it follows that $D^2\phi(x) = 0$, for all values of x.

Applying Schwarz's theorem (I, § 272) to the function $\phi(x)$ in any finite interval, it follows that $\phi(x)$ must be a linear function of x. It thus appears that A_0 and C' can be so determined that $F(x) - C'x - \frac{1}{2}A_0 x^2$ is periodic, and of period 2π.

The condition (2) holds, not only for $F(x)$, by hypothesis, but also if $F(x)$ be replaced by $C'x + \frac{1}{2}A_0x^2$, as has been proved in § 422. Denoting by $\psi(x)$ the periodic function $F(x) - C'x - \frac{1}{2}A_0x^2$, it follows that

$$\lim_{\mu \sim \infty} \mu^2 \int_b^c \psi(x) \cos \mu (x - a) \lambda(x)\, dx = 0.$$

Writing x' instead of a, taking $b < -\pi$, $c > \pi$, and also taking $\lambda(x) = 1$ in the interval $(-\pi, \pi)$, we have

$$\lim_{\mu \sim \infty} \left[\mu^2 \int_{-\pi}^{\pi} \psi(x) \cos \mu (x - x')\, dx + \mu^2 \int_b^{-\pi} \psi(x) \lambda(x) \cos \mu (x - x')\, dx \right.$$
$$\left. + \mu^2 \int_{\pi}^{c} \psi(x) \lambda(x) \cos \mu (x - x')\, dx \right] = 0.$$

Taking μ to have the integral value n, we have then

$$\lim_{n \sim \infty} \left[n^2 \int_{-\pi}^{\pi} \psi(x) \cos n (x - x')\, dx + n^2 \int_{b+2\pi}^{c} \psi(x) \lambda_1(x) \cos n (x - x')\, dx \right] = 0;$$

where $\lambda_1(x) = \lambda(x)$ in the interval (π, c), and $\lambda_1(x) = \lambda(x - 2\pi)$ in the interval $(b + 2\pi, \pi)$ of x. The function $\lambda_1(x)$ satisfies the conditions in (2), for the interval $(b + 2\pi, c)$; hence we have

$$\lim_{n \sim \infty} n^2 \int_{b+2\pi}^{c} \psi(x) \lambda_1(x) \cos n (x - x')\, dx = 0,$$

and therefore also

$$\lim_{n\sim\infty} n^2 \int_{-\pi}^{\pi} \psi(x) \cos n(x-x')\, dx = 0.$$

Now let

$$C \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(x)\,dx, \quad -\frac{a_n}{n^2} \equiv \frac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\cos nx\,dx, \quad -\frac{b_n}{n^2} \equiv \frac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\sin nx\,dx,$$

so that
$$-\frac{A_n(x')}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi}\psi(x)\cos n(x-x')\,dx,$$

where $A_n(x')$ denotes $a_n \cos nx' + b_n \sin nx'$. It has been shewn that $\lim_{n\sim\infty} A_n(x') = 0$; and it follows that the Fourier's series

$$C - \frac{A_1(x')}{1^2} - \frac{A_2(x')}{2^2} - \dots - \frac{A_n(x')}{n^2} - \dots$$

is uniformly convergent, and therefore converges to the sum-function $\psi(x')$.

The series $\frac{1}{2}A_0 + A_1(x) + A_2(x) + \dots$, where $A_1, A_2, \dots$ have been determined as above, is the required trigonometrical series. Its sum (R) is the function $f(x)$, and if at any point x it is convergent, its ordinary sum at that point is $f(x)$. It will be observed that the theorem provides a method of determining the series, when $f(x)$ is prescribed, and is such that the function $F(x)$ satisfying the conditions (1) and (2) exists and can be determined. The Fourier's series corresponding to $F(x)$ can be then determined, and the required series is found by differentiation of that series twice.

Generalizations of Riemann's Theorems I and II have been given*, with an indication of the proofs, by Kogbetliantz.

424. Riemann's third theorem, which is here given in a simplified form, expresses a necessary and sufficient condition that a trigonometrical series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, for which $a_n \sim 0, b_n \sim 0$, as $n \sim \infty$, should be convergent at a particular point x. The theorem may be stated as follows:

Theorem III. *Let ϵ be an arbitrarily chosen positive number less than $\frac{1}{2}\pi$, and let $\rho(t)$ be a function defined in the interval $(-2\epsilon, 2\epsilon)$ of t, which has a bounded third differential coefficient $\rho'''(t)$. Let $\rho(t)$ have the value 1 in the whole interval $(-\epsilon, \epsilon)$, and the value 0 at the points $-2\epsilon, 2\epsilon$. Then the necessary and sufficient condition that the series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, such that $a_n \sim 0$, $b_n \sim 0$, as $n \sim \infty$, may be convergent at the point x is that*

$$\frac{1}{2\pi}\int_{-2\epsilon}^{2\epsilon} F(t+x)\,\rho(t)\,\frac{d^2}{dt^2}\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt$$

should converge to a definite limit, as $n \sim \infty$.

Let $\rho(t)$ be continued, by the rules that it is periodic, of period 2π, and that it vanishes in the two intervals $(-\pi, -2\epsilon)$, $(2\epsilon, \pi)$.

* *Comptes Rendus*, vol. CLXXVII (1923), p. 674.

It will be observed that $\rho(\pm 2\epsilon) = \rho'(\pm 2\epsilon) = \rho''(\pm 2\epsilon) = 0$ and that $\rho(\pm \epsilon) = 1$, $\rho'(\pm \epsilon) = \rho''(\pm \epsilon) = 0$, in virtue of the conditions to which $\rho(t)$ has been subjected.

Denoting the periodic function $F(t) - C't - \frac{1}{2}A_0 t^2$ by $\psi(t)$, we have

$$A_1(x) + A_2(x) + \ldots + A_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x+t) \sum_1^n (-n^2 \cos nt)\, dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(x+t) \frac{d^2}{dt^2}\frac{\sin \mu t}{\sin \frac{1}{2}t}\, dt,$$

where μ denotes $\frac{1}{2}(2n+1)$. Let $\lambda(t)$ denote $1 - \rho(t)$, then $\lambda(t)$ has similar properties as regards its differential coefficients to those of $\rho(t)$. The expression on the right-hand side may be put in the form

$$\frac{1}{2\pi}\int_{-2\epsilon}^{2\epsilon} \psi(x+t)\rho(t)\frac{d^2}{dt^2}\frac{\sin \mu t}{\sin \frac{1}{2}t}\, dt + \frac{1}{2\pi}\int_{\epsilon}^{2\pi-\epsilon} \psi(x+t)\lambda(t)\frac{d^2}{dt^2}\frac{\sin \mu t}{\sin \frac{1}{2}t}\, dt.$$

The second of these integrals may be written as

$$\frac{1}{2\pi}\int_{\epsilon}^{2\pi-\epsilon} \psi(x+t)\lambda_1(t)\sin \mu t\, dt + \frac{\mu}{\pi}\int_{\epsilon}^{2\pi-\epsilon} \psi(x+t)\lambda_2(t)\cos \mu t\, dt$$
$$- \frac{\mu^2}{2\pi}\int_{\epsilon}^{2\pi-\epsilon} \psi(x+t)\lambda_3(t)\sin \mu t\, dt,$$

where $\qquad \lambda_1(t) \equiv \lambda(t)\dfrac{d^2}{dt^2}\operatorname{cosec}\frac{1}{2}t, \quad \lambda_2(t) \equiv \lambda(t)\dfrac{d}{dt}\operatorname{cosec}\frac{1}{2}t,$

and $\qquad \lambda_3(t) \equiv \lambda(t)\operatorname{cosec}\frac{1}{2}t.$

Since $\operatorname{cosec}\frac{1}{2}t$ does not vanish in the interval $(\epsilon, 2\pi - \epsilon)$, it is clear that $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ all satisfy the conditions to which $\psi(x)$ is subjected in the theorem of § 422. It follows that the whole expression converges to zero, as μ is indefinitely increased. Hence the necessary and sufficient condition for the convergence of the series at the point x is that

$$\lim_{n\sim\infty}\int_{-2\epsilon}^{2\epsilon} \psi(x+t)\rho(t)\frac{d^2}{dt^2}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt = 0.$$

To shew that

$$\lim_{n\sim\infty}\int_{-2\epsilon}^{2\epsilon} \{C'(t+x) + \tfrac{1}{2}A_0(t+x)^2\}\rho(t)\frac{d^2}{dt^2}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt = 0,$$

we observe that by two partial integrations the integral is found to be equal to

$$\int_{-2\epsilon}^{2\epsilon} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\frac{d^2}{dt^2}[\rho(t)\{C'(t+x) + \tfrac{1}{2}A_0(t+x)^2\}]\, dt;$$

and since $\rho(t)$, $\rho'(t)$, $t\rho'(t)$, $\rho''(t)$, $t\rho''(t)$, $t^2\rho''(t)$ are all summable functions in the interval $(-2\epsilon, 2\epsilon)$ and all vanish in the remainder of the interval $(-\pi, \pi)$ and have bounded differential coefficients, they are all represented by convergent Fourier's series in the interval $(-2\epsilon, 2\epsilon)$. It follows that the integral converges to a definite limit, dependent on x, as $n \sim \infty$; and it is seen at once that this convergence is uniform for all values of x in a prescribed finite interval, since $\rho(0) = 1$, $\rho'(0) = 0$, $\rho''(0) = 0$.

It has now been shewn that the series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

converges to a definite limit at the point x, if

$$\lim_{n\sim\infty} \frac{1}{2\pi}\int_{-2\epsilon}^{2\epsilon} F(t+x)\,\rho(t)\,\frac{d^2}{dt^2}\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt$$

does so, and that this condition is a necessary one.

Riemann himself assumed that the function $\rho(t)$ had the value 1 only at the single point 0. The definition of $\rho(t)$ as having the value 1 in a whole interval $(-\epsilon, \epsilon)$ introduces a simplification into the proof of the theorem, and a less degree of restriction on the function $\rho(t)$ is requisite. This simplification was suggested* by Neder, who gave the theorem in a form very similar to the above formulation. He employed the function $\dfrac{\sin nt}{\tan\frac{1}{2}t}$ instead of $\dfrac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}$ (see § 323).

It is seen from the theorem of § 422, and an examination of the foregoing proof, that the theorem may be extended to express the necessary and sufficient condition that the series should converge uniformly in a prescribed interval of x. This condition is that

$$\frac{1}{2\pi}\int_{-2\epsilon}^{2\epsilon} F(x+t)\,\rho(t)\,\frac{d^2}{dt^2}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

should converge uniformly to a limit $s(x)$ for all values of x in the prescribed interval. This extension was also given by Neder.

425. From the above theorems the following consequences at once follow:

The convergence of a series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, for which $a_n = o(1)$, $b_n = o(1)$, at a point x, depends only on the nature of the series as represented by the Riemann sum in an arbitrarily small neighbourhood $(x - 2\epsilon, x + 2\epsilon)$ of the point x, where $0 < \epsilon < \frac{1}{2}\pi$.

The uniform convergence of the series in an interval (a, b) depends only on the behaviour of the series, as represented by the Riemann sum in an interval $(a - 2\epsilon, b + 2\epsilon)$, where ϵ is arbitrary, subject to $0 < \epsilon < \frac{1}{2}\pi$. In case the given series is the Fourier's series corresponding to a function $f(x)$, summable in the interval $(-\pi, \pi)$, it has been shewn in § 360, that $\int_{-\pi}^{x}\left\{\int_{-\pi}^{x} f(x)\,dx\right\}dx$ differs from the sum of the series $\frac{1}{2}A_0x^2 - \Sigma\dfrac{A_n(x)}{n^2}$ by a linear function, and thus $F(x) - \int_{-\pi}^{x}\left\{\int_{-\pi}^{x} f(x)\,dx\right\}dx$ is a linear function. From this it is seen that the theorems of § 341 follow from the above.

* *Math. Annalen*, vol. LXXXIV (1921), p. 119.

426. For a Fourier's series corresponding to $f(x)$ the sum (R) is almost everywhere $f(x)$. For, since $\Sigma \frac{a_n \sin nx - b_n \cos nx}{n}$ converges uniformly to a continuous function $\int_{-\pi}^{x} f(x)\,dx + C'$, we have $F'(x) = \int_{a}^{x} f(x)\,dx + C'$, at every point, and since $\int_{-\pi}^{x} f(x)\,dx$ has a differential coefficient equal to $f(x)$ almost everywhere, it is seen that $F''(x) = f(x)$ almost everywhere.

This result also holds good* when the series is a Fourier's (D) series. For, in that case $\Sigma \frac{a_n \sin nx - b_n \cos nx}{n}$ converges to a continuous function, and $\int_{-\pi}^{x} f(x)\,dx$ has a differential coefficient $f(x)$, almost everywhere (see I, § 470).

INVESTIGATIONS SUBSEQUENT TO THOSE OF RIEMANN

427. The important discovery of the fundamental distinction between series which converge uniformly, and those which converge non-uniformly in a prescribed interval, remained for a long time without influence upon the development of the theory of series in general, and in particular of trigonometrical series. It was shewn by Weierstrass that the legitimacy of term by term integration of a convergent series follows from the uniform convergence of the series; by previous writers no such restriction upon the universal validity of the process had been recognized. It was first pointed out by Heine† that a full recognition of the consequences of the theory of uniformity of convergence made it necessary to undertake a re-examination of the foundations of the theory of trigonometrical series. The investigations of Dirichlet and others had established that a function which satisfies certain conditions can be represented by means of a trigonometrical series in which the coefficients have the form given by Fourier; unless however it be assumed that a series so obtained converges uniformly, it cannot be immediately proved that it is the only trigonometrical series by which the function can be represented. The customary proof that a function is capable only of a single representation by means of a trigonometrical series was based upon the assumption that, if a convergent series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

converge to zero for all values of x in the interval $(-\pi, \pi)$, it is legitimate to multiply the series by $\cos nx$ or $\sin nx$, and then to integrate term by term, between the limits $-\pi, \pi$; thus shewing that $a_n = 0$, $b_n = 0$, for every value of n. If however it is not known that the series converges

* See Priwaloff, *Rendiconti di Palermo*, vol. XLI (1916), p. 203.

† *Crelle's Journal*, vol. LXXI (1870), p. 353; see also *Kugelfunctionen*, vol. I, p. 55.

uniformly, or at all events boundedly, the process of term by term integration is not necessarily legitimate, and thus the proof is invalid. In fact it is conceivable that a non-uniformly convergent series might exist whose sum is zero for every value of the variable. It thus appeared that, when a Fourier's series exists which represents a function $f(x)$, it cannot be immediately inferred that no other trigonometrical series exists which represents the same function.

A Fourier's series that represents a function $f(x)$ which has discontinuities is certainly non-uniformly convergent in the neighbourhood of such continuities, and in default of proof to the contrary, it may also be non-uniformly convergent in the neighbourhood of points at which $f(x)$ is continuous. Thus, for example, if $f(x)$ is continuous in its whole domain, and is representable by a Fourier's series, it cannot be assumed that the series is uniformly convergent (see § 324). The value of the representation of a function $f(x)$ by a series $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ would be seriously impaired, if it were not known that the series was, at all events in general, uniformly convergent. For it could not be assumed that, if $\psi(x)$ denotes a continuous function, the integral $\int_a^b f(x)\,\psi(x)\,dx$ would be represented by the series

$$\tfrac{1}{2}a_0 \int_a^b \psi(x)\,dx + \sum_{n=1} \int_a^b (a_n \cos nx + b_n \sin nx)\,\psi(x)\,dx;$$

the employment of Fourier's series in physical and other investigations would consequently be much restricted.

These considerations gave rise to a series of investigations with the view of establishing the uniqueness of the representation of a function by means of a trigonometrical series, and of investigating whether the coefficients in the series are necessarily expressible in the Fourier form. The two main questions which arise in this connection are (1), whether a trigonometrical series can exist, with coefficients not all zero, which represents the number zero? and (2), under what conditions is a trigonometrical series which represents a function the Fourier's series corresponding to that function? Heine* proved that the Fourier's series which represents a bounded function that satisfies the conditions known as Dirichlet's, viz. that it has only a finite number of discontinuities and is in general monotone, is uniformly convergent in the portions of the interval $(-\pi, \pi)$ which remain when arbitrarily small neighbourhoods of the points of discontinuity are removed from the interval. This property of the series, of being in general uniformly convergent, suffices to remove, in the case of a most important class of functions, the restriction which has been above mentioned relating to those applications of Fourier's series which involve a term by term integration.

* *Crelle's Journal*, vol. LXXI (1870), p. 353.

It having thus been shewn that a function satisfying Dirichlet's conditions is representable by a series which converges in general uniformly, Heine proved that, if a function is representable at all by a series which converges in general uniformly, there can exist only one such series. This is equivalent to the theorem that, if a series converges in general uniformly in the interval $(-\pi, \pi)$, and represents zero, then all the coefficients vanish, and the sum of the series is therefore zero for all values of the variable. Heine proved further that this theorem holds even when, for a finite number of values of the variable, the series is not known to converge, or when it is at least not assumed that its sum is zero for such values of the variable. The possibility remained, however, that when a function is thus uniquely represented by means of a series which is in general uniformly convergent, other series not possessing this property of uniform convergence may exist, which also represent the same function. It should be remarked that uniform convergence is at the present time of less relative importance than would appear from these investigations; for bounded convergence is now known to suffice for many purposes for which uniform convergence was formerly employed.

It was next proved by G. Cantor* that, if the expression

$$a_n \cos nx + b_n \sin nx$$

be such that, for every value of x in a given interval (α, β), the limit $\lim\limits_{n \to \infty} (a_n \cos nx + b_n \sin nx)$ is zero, then a_n, b_n converge to zero, as n is indefinitely increased, and hence that the series

$$\tfrac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$$

can only converge for all values of x in (α, β) if a_n, b_n have the limit zero, as n is increased indefinitely. This theorem is independent of any assumption that the convergence is uniform. Cantor† then deduced that, if a trigonometrical series $\tfrac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ converges to zero, for every value of x with the exception of a finite number of values, for which it is unknown whether the series converges, all the coefficients a_n, b_n must vanish. Kronecker† shewed that this theorem can be proved without assuming the previous one. These proofs depend upon the use of Schwarz's theorem that, if $F(x)$ denotes a function which is such that

$$\lim_{\epsilon = 0} \frac{F(x+\epsilon) - 2F(x) + F(x-\epsilon)}{\epsilon^2} = 0,$$

then $F(x)$ must be a linear function of x.

The next step‡ was made by G. Cantor, in extending the proof of the uniqueness of the representation of a function by means of a trigonometrical series to the case in which the function may have an indefinitely great

* *Crelle's Journal*, vol. LXXII (1870), p. 130, also in a simplified form in *Math. Annalen*, vol. IV (1871), p. 139.

† *Crelle's Journal*, vol. LXXIII (1871), p. 294.

‡ *Math. Annalen*, vol. V (1872), p. 123.

number of points of discontinuity, these points forming a set of the first species. Starting with Weierstrass' theorem, that an infinite set of points possesses at least one limiting point, Cantor developed the theory of the successive derivatives of a set of points, and proved that, if a limited function has discontinuities which form a set, one of whose derivatives contains only a finite number of points, then, if the function is representable by a trigonometric series at all, there can be only one such series. In this connection the theory of sets of points was first considered, and thus the whole development of this subject, and of the more abstract theory of transfinite numbers, arose historically from the requirements of the theory of trigonometrical series. Proofs were given by Dini* and Ascoli† that, for restricted classes of functions, a series which represents such functions must be a Fourier's series.

An important advance in the theory was made by Du Bois-Reymond‡, who proved that a series

$$\tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx),$$

which is such that a_n, b_n have the limit zero, as n is indefinitely increased, has $f(x)$ for its sum-function, the coefficients must always have the form

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

whenever these expressions exist as R-integrals. The function $f(x)$ is not necessarily everywhere single-valued, and the theorem is extended to cases in which $f(x)$ may have infinite discontinuities at a finite set of points. This theorem includes the theorem as to the uniqueness of the representation of bounded functions, integrable (R).

The most general formulations of the theorems as to the uniqueness of the representation of a function by a trigonometrical series are due to Harnack, Hölder, de la Vallée Poussin, W. H. Young, and others; an account of their results will be given later. Important extensions of Du Bois-Reymond's results were given§ by M. Riesz.

THE LIMITS OF THE COEFFICIENTS IN A TRIGONOMETRICAL SERIES

428. The following theorem, due to Harnack‖, yields a sufficient condition that the coefficients in a trigonometrical series converge to zero:

If, in a given interval (α, β), *the series* $\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *be such that, for each number* $\delta\ (> 0)$, *an interval in* (α, β) *exists such that, at*

* *Sopra la serie di Fourier.* Pisa, 1872, p. 247.

† *Annali di Matematica* (2), vol. VI (1875), p. 252, also *Math. Annalen*, vol. VI (1873), p. 231.

‡ *Abhandlungen der bayerischen Akademie*, vol. XII (1875), p. 119.

§ *Math. Annalen*, vol. LXXI (1912), p. 54.

‖ *Bulletin des sciences math.* (2), vol. VI (1882), also *Math. Annalen*, vol. XIX (1882), p. 250.

each point of it, the difference $\bar{f}(x) - \underline{f}(x)$, *of the upper and lower sum-functions of the series, is* $< \delta$, *then* $a_n = o(1)$, $b_n = o(1)$.

In particular, if the points at which $\bar{f}(x) - \underline{f}(x) \geqq \delta$ form, for each value of δ, a non-dense set, the condition of the theorem holds good.

Harnack's theorem is a generalization of the theorem of Cantor* that:

If the series is convergent at every point of an interval (α, β), *then* $a_n = o(1)$, $b_n = o(1)$.

It follows from Harnack's theorem that, if the trigonometrical series converge at all points of a set which is everywhere dense in (α, β), and be such that a_n, b_n do not converge to zero, then, for some value of δ, the set of points at which $\bar{f}(x) - \underline{f}(x) \geqq \delta$ must be everywhere dense.

No assumption is made as to the form of the coefficients a_n, b_n.

That, in the case of a Fourier's series, a_n and b_n converge to zero has been established in § 334.

In order to prove Harnack's theorem, we observe that, for each point x at which $\bar{f}(x) - \underline{f}(x) < \delta$, there is a value m, of n, such that

$$| a_n \cos nx + b_n \sin nx | < 3\delta, \text{ for } n \geqq m;$$

we suppose an interval to exist, at each point of which this condition is satisfied. If x be any fixed point within this interval, a neighbourhood $(x - \eta, x + \eta)$ of x can be so determined that

$$| a_n \cos n(x \pm \eta) + b_n \sin n(x \pm \eta) | < 3\delta, \text{ for } n \geqq m_\eta;$$

the value of m_η will depend in general upon η. We deduce at once that

$$| (a_n \cos nx + b_n \sin nx) \cos n\eta | < 6\delta, \quad | (a_n \sin nx - b_n \cos nx) \sin n\eta | < 6\delta;$$

on multiplication by $\cos nx \sin n\eta$, $\sin nx \cos n\eta$, and addition of the two expressions in the inequalities, we have $| a_n \sin 2n\eta | < 6\delta$, for $n \geqq m_\eta$; and similarly it is seen that $| b_n \sin 2n\eta | < 6\delta$, for $n \geqq m_\eta$. These inequalities hold for all small enough values of η, the value of m_η depending on η.

Let $6\delta = \delta'$, $2\eta = \alpha$, then, for each value of α in a certain interval (a, b), a value of n can be determined, such that

$$| a_n \sin n\alpha |, \quad | a_{n+1} \sin \overline{n+1}\,\alpha |, \ldots | a_{n+s} \sin \overline{n+s}\,\alpha |, \ldots$$

are all $< \delta'$.

Let us suppose that, if possible, a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ exists, all of whose terms are numerically $\geqq \delta''$, where $\delta'' > \delta'$. It will then be proved that there exists a certain value of α, in (a, b), such that the sequence $a_{n_1} \sin n_1\alpha$, $a_{n_2} \sin n_2\alpha$, $a_{n_3} \sin n_3\alpha$, ... contains one infinite set of terms each of which is numerically $\geqq \delta'$. This being contrary to the hypothesis that, for each value of α, in (a, b), $| a_n \sin n\alpha | < \delta'$, for all sufficiently great values of n, leads to a contradiction; and thus it is impossible that such a sequence as $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ can exist.

* *Crelle's Journal*, vol. LXXII (1870), p. 130, also in a simplified form in *Math. Annalen*, vol. IV (1871), p. 139.

To establish this, it will be shewn that the sequence $a_{n_1}, a_{n_2}, \ldots$ contains a sequence $a_{n_1'}, a_{n_2'}, a_{n_3'}, \ldots$ such that, for a certain value $\bar{a}$, of α, in (a, b) the numbers $n_1'\bar{a}, n_2'\bar{a}, n_3'\bar{a}, \ldots$ all differ from an odd multiple of $\frac{1}{2}\pi$ by less than an arbitrarily chosen positive number ρ.

If $\frac{1}{2}\pi y_1 - \rho < n\alpha < \frac{1}{2}\pi y_1 + \rho$, then $\frac{\frac{1}{2}\pi y_1 - \rho}{n} < \alpha < \frac{\frac{1}{2}\pi y_1 + \rho}{n}$. Now let it be assumed that $a < \frac{\frac{1}{2}\pi y_1 - \rho}{n}$, $b > \frac{\frac{1}{2}\pi y_1 + \rho}{n}$, which is equivalent to the assumption that $\frac{2}{\pi}(na + \rho) < y_1 < \frac{2}{\pi}(nb - \rho)$. There exists a value of y_1 which is an odd integer, satisfying this condition, provided

$$\frac{2}{\pi}\{n(b-a) - 2\rho\} > 2,$$

that is if $n \geqq \frac{\pi + 2\rho}{b - a}$.

Taking for n_1' the least of the numbers $n_1, n_2, n_3, \ldots$ which is $\geqq \frac{\pi + 2\rho}{b - a}$, a corresponding odd integer y_1 can be determined, and we take α to lie within the interval (a', b'), where

$$a' = (\tfrac{1}{2}\pi y_1 - \rho)/n_1', \quad b' = (\tfrac{1}{2}\pi y_1 + \rho)/n_1';$$

this interval (a', b') lies within (a, b), and is of length $2\rho/n_1'$.

Next, an odd integer y_2 can be so determined that

$$\frac{2}{\pi}(n_2'a' + \rho) < y_2 < \frac{2}{\pi}(n_2'b' - \rho),$$

provided $n_2' \geqq \frac{\pi + 2\rho}{b' - a'} \geqq \frac{\pi + 2\rho}{2\rho} n_1'$. The number n_2' can be chosen from the sequence $n_1, n_2, \ldots$ so as to satisfy this condition, if α lies in the interval (a'', b''), where $a'' = (\frac{1}{2}\pi y_2 - \rho)/n_2'$, $b'' = (\frac{1}{2}\pi y_2 + \rho)/n_2'$; and thus (a'', b'') is within (a', b'), and is of length $2\rho/n_2'$.

Proceeding in this manner, a sequence $n_1', n_2', \ldots$ of numbers all belonging to the sequence $n_1, n_2, \ldots$ is determined, such that if $\bar{\alpha}$ be the point which lies within all the intervals $(a, b), (a', b'), (a'', b''), \ldots$, the numbers $n_1'\bar{\alpha}, n_2'\bar{\alpha}, n_3'\bar{\alpha}, \ldots$ all differ by less than ρ from odd multiples of $\frac{1}{2}\pi$. Since ρ can be chosen arbitrarily, $n_1', n_2', \ldots$ can be so determined that $|a_{n_1'} \sin n_1'\bar{\alpha}|, |a_{n_2'} \sin n_2'\bar{\alpha}|, \ldots$ are all $\geqq \delta'$, and this is contrary to the hypothesis that $|a_n \sin n\bar{\alpha}|$ is, for all sufficiently large values of n, $< \delta'$.

Therefore no sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ exists, all of whose terms are numerically $\geqq \delta''$; and if δ'' be first chosen, δ' may be chosen afterwards. Therefore, from and after some value of n, $|a_n|$ must be $< \delta''$; and since this holds for every value of δ'', it follows that $\lim\limits_{n \sim \infty} a_n = 0$. In a similar manner it is seen that $\lim\limits_{n \sim \infty} b_n = 0$.

429. It follows from Harnack's theorem that, if the trigonometrical series is non-convergent only at points of a set of the first category, then $a_n = o(1)$, $b_n = o(1)$. For, let $\{W_n(x)\}$, $\{w_n(x)\}$ be the monotone sequences associated with $f(x)$ (see § 112), the first of which is non-increasing and converges to $\bar{f}(x)$, and the second of which is non-diminishing and converges to $\underline{f}(x)$. Since $W_n(x)$ is an l-function, and $w_n(x)$ is a u-function, $W_n(x) - w_n(x)$ is an l-function, which converges to $\bar{f}(x) - \underline{f}(x)$; and the sequence $\{W_n(x) - w_n(x)\}$ is non-increasing. A point x, at which

$$\bar{f}(x) - \underline{f}(x) > \delta$$

belongs to the set of points at which $W_n(x) - w_n(x) > \delta$, and that set is (see § 191) an open set. Every point for which $W_{n+1}(x) - w_{n+1}(x) > \delta$ belongs to the set of points for which $W_n(x) - w_n(x) > \delta$. Thus the set of points at which $\bar{f}(x) - \underline{f}(x) > \delta$ is contained in the inner limiting set of a sequence of open sets, each of which contains the next, that is, it is an ordinary inner limiting set. It follows (I, § 100) that, if the points at which $\bar{f}(x) - \underline{f}(x) > \delta$, are everywhere-dense in any interval, they form in that interval a set of the second category; and therefore the set at which $\bar{f}(x) - \underline{f}(x) \geqq \delta$ is of the second category. By hypothesis this is not the case; and therefore the set of points at which $\bar{f}(x) - \underline{f}(x) > \delta$ is non-dense in any interval, and this for each value of δ; therefore the set for which $\bar{f}(x) - \underline{f}(x) \geqq \delta$ is non-dense in every interval. Consequently, any interval contains another interval in which $\bar{f}(x) - \underline{f}(x) < \delta$; and therefore, by Harnack's theorem, $a_n = o(1)$, $b_n = o(1)$.

It has thus been established* that:

If the trigonometrical series $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ *converges everywhere in an interval* (α, β) *except at points belonging to a set of the first category, then* $a_n = o(1)$, $b_n = o(1)$.

430. The following general theorem will be established:

If, in any interval (α, β), *it is known that* $a_n \cos nx + b_n \sin nx$ *converges to zero at every point of a set* G, *of positive measure, as* $n \sim \infty$, *then* a_n *and* b_n *converge to zero, and thus* $a_n \cos nx + b_n \sin nx$ *converges everywhere to zero.*

Writing $a_n \cos nx + b_n \sin nx$ in the form $k_n \sin n(x - \gamma_n)$, where $k_n = (a_n^2 + b_n^2)^{\frac{1}{2}}$; if k_n does not converge to zero, there must be a sequence $n_1, n_2, \ldots$ of values of n, and a positive number ϵ, such that $k_{n_1}, k_{n_2}, \ldots$ are all greater than ϵ. If $|\sin n(x - \gamma_n)| \leqq \sin \eta$, where $0 < \eta < \frac{\pi}{2}$, $n(x - \gamma_n)$ must lie between $r\pi \pm \eta$, where r is an integer (positive, negative, or zero); or x must lie between $\frac{r\pi}{n} + \gamma_n \pm \frac{\eta}{n}$, the length of each of which intervals is

* See W. H. Young, *Messenger of Math.* vol. XXXVIII (1909), p. 44.

$\frac{2\eta}{n}$. The number of values of r such that x may lie in a given interval cannot exceed a fixed multiple of n, independent of η, and therefore the measure of the set of values of x in (α, β) such that $|\sin n(x-\gamma_n)| \leqq \sin\eta$ cannot exceed a fixed multiple of η, say $s\eta$, which is arbitrarily small, since η is arbitrarily chosen. It follows that, in a set of points of measure $\geqq \beta-\alpha-s\eta$, the condition $|\sin n(x-\gamma_n)| > \sin\eta$ is satisfied. Considering now the sequence $n_1, n_2, \ldots$ of values of n, there is a set of points E_{n_r} of measure $\geqq \beta-\alpha-s\eta$, for which $k_{n_r}|\sin n_r(x-\gamma_{n_r})| > \epsilon\sin\eta$. There must exist a set E, of points, each of which belongs to E_{n_r} for an indefinitely great set of values of r, and this set has measure $\geqq \beta-\alpha-s\eta$ which is $> \beta-\alpha-m(G)$, if η be chosen small enough. At any point of this set, $k_n\sin n(x-\gamma_n)$ cannot converge to 0, because it is numerically $>\epsilon\sin\eta$ for an infinite set of values of n. This is contrary to the hypothesis in the theorem.

It follows from the theorem that, if a_n, b_n do not converge to zero, $a_n\cos nx + b_n\sin nx$ can only converge to zero at points of a set of measure zero. It is hence seen that a series $\frac{1}{2}a_0 + \Sigma(a_n\cos nx + b_n\sin nx)$ can be convergent only at points of a set of measure zero, in case a_n, b_n do not converge to zero.

431. The more general theorem has been established* by Steinhaus that:

Almost everywhere in the interval $(-\pi, \pi)$,

$$\overline{\lim_{n\sim\infty}}\,|a_n\cos nx + b_n\sin nx| = \overline{\lim_{n\sim\infty}}\,(a_n^2+b_n^2)^{\frac{1}{2}}.$$

This includes the preceding theorem; for if $a_n\cos nx + b_n\sin nx$ converges to zero at points in a set of measure greater then zero, it follows that $\overline{\lim}\,(a_n^2+b_n^2)^{\frac{1}{2}} = 0$, or $a_n = o(1)$, $b_n = o(1)$. In case, for a given trigonometrical series, a_n and b_n do not converge to zero, it follows from Steinhaus' theorem that, almost everywhere the series is non-convergent. This has been proved directly† by Lebesgue, as follows:

When k_n does not converge to zero, as $n\sim\infty$, a sequence of integers $\{n_p\}$ can be determined such that k_{n_p} is, for every value of p, greater than some positive number η. If ϵ is an arbitrarily chosen positive number, $<\eta$, $|k_n\sin n_p(x-\gamma_{n_p})| > \epsilon$, except for points x, of the interval $(-\pi, \pi)$, which form a set of measure $\leq 4\sin^{-1}\frac{\epsilon}{\eta}$. It then follows that the measure of the set of points of convergence is $\leq 4\sin^{-1}\frac{\epsilon}{\eta}$. Since ϵ is arbitrarily small, it follows that the set of points of convergence has measure zero.

* *Wiadomósci Matomatycyne*, vol. XXIV (1920), p. 197. A proof has also been given by Rajchman, *Fundamenta Math.* vol. III (1922), p. 301.

† *Leçons sur les séries trigonométriques* (1906), p. 110.

432. *In* a series* $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$, *the upper and lower sums of the series cannot be finite in any interval, except in the points of a set of measure zero, unless* $a_n = o(1)$ *and* $b_n = o(1)$.

The series may be written in the form

$$\frac{1}{2}a_0 + \Sigma k_n \cos n(x - \gamma_n),$$

where $$k_n = (a_n^2 + b_n^2)^{\frac{1}{2}}.$$

Since $$k_n \cos n(x - \gamma_n) = s_n(x) - s_{n-1}(x),$$

$$\overline{\lim_{n\sim\infty}} \mid k_n \cos n(x - \gamma_n) \mid \leq \overline{\lim} \mid s_n(x) \mid + \overline{\lim} \mid s_{n-1}(x) \mid \leq 2\,\overline{\lim} \mid s_n(x) \mid,$$

hence, if $s_n(x)$ has finite upper and lower limits, $\mid k_n \cos n(x - \gamma_n) \mid$ must be bounded for all values of n, when x is fixed.

It will be shewn that, unless k_n is bounded for all values of n,

$$\mid k_n \cos n(x - \gamma_n) \mid,$$

for a fixed value of x, cannot be bounded, except for fixed values of x belonging to a set of measure zero. If k_n is not bounded, a monotone increasing sequence of values of k_n can be extracted.

Let E_n be the set of points for which $k_n \mid \cos n(x - \gamma_n) \mid > k_n^{\frac{1}{2}}$, or $\mid \cos n(x - \gamma_n) \mid > \dfrac{1}{k_n^{\frac{1}{2}}}$. As n and k_n increase indefinitely, $m(E_n)$ tends to the measure of the interval. If x belongs to an infinite number of the sets $E_1, E_2, \ldots E_n, \ldots$, $k_n \cos n(x - \gamma_n)$ cannot be bounded. But the set of all such points x has measure equal to that of the whole interval, and therefore only at a point x of a set of measure zero can $k_n \cos n(x - \gamma_n)$ be bounded for all the values of n.

It follows from the theorem that† *if, in any interval, the upper and lower sum-functions are everywhere finite,* a_n *and* b_n *are bounded.*

PROPERTIES OF THE GENERALIZED SECOND DERIVATIVE OF A FUNCTION

433. If a continuous function $\phi(x)$ has a maximum in the interval (a, b), at such a point x, $\phi(x + h) - \phi(x)$, $\phi(x - h) - \phi(x)$ are both ≤ 0, for all sufficiently small values of h; and therefore $\overline{\mathscr{D}}^2\phi(x) \leq 0$, at a maximum (see I, § 256).

If it be known that, at a maximum, there exists at least one symmetrical derivative, it is clear, since $D^+\phi(x) \leq 0$, $D^-\phi(x) \geq 0$, that

$$D^+\phi(x) = D^-\phi(x) = 0,$$

the single symmetrical derivative having the value zero.

The following theorem is of importance in the theory:

If a continuous function has its upper generalized second derivative

* See de la Valleé Poussin, *Bulletin de l'acad. roy. de Belgique* (1913), p. 10.

† See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX, p. 427.

positive (>0) *at every point of the open interval* (a, b), *then, at every point of the open interval,*

$$F(x) < F(a) + \frac{x-a}{b-a}\{F(b) - F(a)\}.$$

For if the function

$$\phi(x) \equiv F(x) - F(a) - \frac{x-a}{b-a}\{F(b) - F(a)\}$$

has positive values, or is zero, within the interval, there must be such a value that is a maximum of $\phi(x)$, and at such a point $\overline{\mathscr{D}}^2\phi(x)$, and consequently $\underline{\mathscr{D}}^2 F(x)$, must be $\leqq 0$, which is contrary to the condition which $\underline{\mathscr{D}}^2 F(x)$ satisfies.

A similar statement is that if $\mathscr{D}^2 F(x) < 0$, in the open interval (a, b), then

$$F(x) > F(a) + \frac{x-a}{b-a}\{F(b) - F(a)\}.$$

434. *If* $F(x)$ *be continuous in the closed interval of definition* (a, b), *and have, at each point of the open interval* (a, b), *at least one symmetrical derivative, then, if there are values of* x *in the interval at which*

$$F(x) > F(a) + \frac{x-a}{b-a}\{F(b) - F(a)\},$$

there exists a set of points E *at which the upper and lower generalized second derivatives are both negative* (<0), *and such that* E *contains a perfect set. A similar statement holds when there are points at which*

$$F(x) < F(a) + \frac{x-a}{b-a}\{F(b) - F(a)\};$$

in that case the set E *consists of points at which* $\overline{\mathscr{D}}^2 F(x)$, $\underline{\mathscr{D}}^2 F(x)$ *are both positive* (>0).

This theorem and the preceding one are due to de la Vallée Poussin*. Let $\phi_K(x) \equiv F(x) - F(a) - K(x-a)$, where $K = \dfrac{F(b) - F(a)}{b-a}$. Since $\phi_K(x)$ vanishes at a and b, and has positive values, there exists a point or a closed set of points at which $\phi_K(x)$ has an absolute maximum; let x_K be the upper extreme of all such points, then we have

$$\overline{\mathscr{D}}^2 F(x_K) = \overline{\mathscr{D}}^2\phi_K(x_K) \leqq 0,$$

and also $\phi_K(x)$ has at the point x_K a symmetrical derivative of value zero. This point x_K therefore belongs to the set E_1, of all points at which $\overline{\mathscr{D}}^2 F(x) \leqq 0$. If k have a value $> K$, and such that $k - K$ is sufficiently small, it is easily seen that the function $\phi_k(x) = F(x) - F(a) - k(x-a)$ has positive values. Let ξ_k be the greatest value of x at which $\phi_k(x) = 0$, thus $k = \dfrac{F(\xi_k) - F(a)}{\xi_k - a}$.

* *Bulletin de l'académie royale de Belgique* (1912), pp. 701–707

Considering the function $\phi_k(x)$ in the interval (a, ξ), as before, there exists a point x_k at which $\phi_k(x)$ has an absolute maximum, and which is the greatest value of x at which this is the case. The point x_k belongs to E_1, and $\phi_k(x)$ has a single symmetrical derivative of value zero; at this point $F(x)$ has a single symmetrical derivative of value k. To each value of k in some interval (K, K_1), where $K_1 > K$, there corresponds a point x_k which belongs to E_1, and it is impossible that x_k have equal values for two such values of k, since the single symmetrical derivative of $F(x)$ has different values. To the points of the interval (K, K_1) there correspond points x_k belonging to a set having the cardinal number of the continuum.

If $k_1 < k_2$, then $x_{k_1} > x_{k_2}$; for, if possible let us suppose that $x_{k_2} > x_{k_1}$. Since $\phi_{k_1}(x_{k_1}) > \phi_{k_1}(x_{k_2})$, we have

$$\phi_{k_1}(x_{k_1}) - (k_2 - k_1)\, x_{k_1} > \phi_{k_1}(x_{k_2}) - (k_2 - k_1)\, x_{k_1} > \phi_{k_1}(x_{k_2}) - (k_2 - k_1)\, x_{k_2},$$

or $\phi_{k_2}(x_{k_1}) > \phi_{k_2}(x_{k_2})$, which is impossible, since x_{k_2} gives an absolute maximum of $\phi_{k_2}(x)$.

It follows that, if k have the values of an increasing sequence contained in the interval (K, K_1) which converges to $\bar{k}$, the corresponding points x_k form a diminishing sequence which converges to $x_{\bar{k}}$, a point of E_1. A similar remark applies to a diminishing sequence of values of k, and it thus follows that E_1 contains a closed set which, since it is unenumerable, contains a perfect set.

Applying the result which has been obtained to the function

$$\Phi(x) \equiv F(x) + \epsilon (x - a)^2$$

which, for all sufficiently small values of ϵ, must at certain points be greater than $\Phi(a) + \dfrac{\Phi(b) - \Phi(a)}{b - a}(x - a)$, it follows that the set of points at which $\overline{\mathscr{D}}^2 F(x) + 2\epsilon < 0$ contains a perfect set. Since ϵ is arbitrarily small, it follows that the set of points at which $\overline{\mathscr{D}}^2 F(x) < 0$ contains a perfect set.

In a similar manner it can be shewn that the set of points at which $\underline{\mathscr{D}}^2 F(x) > 0$ must contain a perfect set, in case there are values of x at which $F(x) - F(a) - \dfrac{F(b) - F(a)}{b - a}(x - a)$ is negative.

435. The following generalization of Schwarz's theorem given in I, § 272, may be deduced at once from de la Vallée Poussin's theorem. This is a more complete generalization than that given in I, § 273:

If a function $F(x)$, continuous in the closed interval (a, b), be known to have a generalized second differential coefficient of value zero, except at points of a set G which contains no perfect set, and if further it have everywhere in the open interval (a, b) at least one symmetrical (first) derivative, then the function $F(x)$ must be linear in the whole interval.

It will be observed that the condition $\mathscr{D}^2 F(x) = 0$, at a point x, includes the condition that, at such a point, there is a symmetrical first derivative; in fact all the derivatives are symmetrical. For if

$$\lim_{h\sim 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = 0,$$

it follows that

$$\lim_{h\sim 0} \left\{\frac{F(x+h) - F(x)}{h} - \frac{F(x-h) - F(x)}{-h}\right\} = 0$$

and thus all the derivatives on one side must correspond to equal derivatives on the other side, so that

$$D^+F(x) = D^-F(x), \text{ and } D_+F(x) = D_-F(x).$$

The theorem follows from the fact that

$$F(x) - F(a) - \frac{F(b) - F(a)}{b - a}(x - a)$$

can have neither positive nor negative values in the interval, because the two perfect sets in which $\overline{\mathscr{D}}^2 F(x) < 0$, and $\underline{\mathscr{D}}^2 F(x) > 0$, respectively, are both non-existent.

436. *If a function $F(x)$ has, in an interval to which x is interior, a continuous differential coefficient $F'(x)$, its upper and lower generalized second derivatives cannot exceed the greatest of the four derivatives of $F'(x)$, and cannot be less than the smallest of them.*

By the theorem of I, § 264 it is seen that the limits of

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2}$$

both lie in the interval formed by the limits of

$$\frac{F'(x+h) - F'(x-h)}{2h},$$

or of
$$\frac{1}{2}\left\{\frac{F'(x+h) - F'(x)}{h} + \frac{F'(x-h) - F'(x)}{-h}\right\},$$

and therefore in the interval defined by

$$\tfrac{1}{2}\{D_+F'(x) + D_-F'(x)\} \text{ and } \tfrac{1}{2}\{D^+F'(x) + D^-F'(x)\}$$

or in the interval contained by the greater of the two numbers $D_+F'(x)$, $D_-F'(x)$ and the greater of the two numbers $D^+F'(x)$, $D^-F'(x)$.

437. The following theorem, due to Hölder*, will prove useful in the later theory:

If $F(x)$ be continuous in an interval (a, b) to which the interval $(x_1 - \alpha, x_1 + \alpha)$ is interior, and if in $(x_1 - \alpha, x_1 + \alpha)$ the generalized upper

* *Math. Annalen*, vol. XXIV (1884), p. 183. The theorem has also been established otherwise by Lebesgue, for the case in which $\mathscr{D}^2 F(x)$ is definite at each point; see *Annales sc. de l'école normale sup.* (3) (1903), vol. XX, p. 458.

and lower second differential coefficients $\overline{\mathscr{D}}^2F(x)$, $\underline{\mathscr{D}}^2F(x)$ are both bounded, being both, at every point, in the interval (L, U), then

$$\frac{F(x_1+\alpha)+F(x_1-\alpha)-2F(x_1)}{\alpha^2}$$

is in the interval (L, U).

Let

$$\phi(x) \equiv F(x) - F(x_1-\alpha) - \frac{x-x_1+\alpha}{2\alpha}\{F(x_1+\alpha)-F(x_1-\alpha)\}$$
$$+ \tfrac{1}{2}C(x-x_1+\alpha)(x_1+\alpha-x),$$

where C is a constant. It is seen that

$$\phi(x_1) = \tfrac{1}{2}\alpha^2\left\{C - \frac{F(x_1+\alpha)+F(x_1-\alpha)-2F(x_1)}{\alpha^2}\right\},$$

and thus $\phi(x_1)$ is $\gtrless 0$, according as $C \gtrless \dfrac{F(x_1+\alpha)+F(x_1-\alpha)-2F(x_1)}{\alpha^2}$.

Let C be so chosen that $\phi(x_1) > 0$. Since $\phi(x)$ is continuous in the interval $(x_1-\alpha, x_1+\alpha)$, and vanishes at the points $x_1-\alpha$, $x_1+\alpha$, there must be in the interval at least one point z at which $\phi(x)$ has a maximum, and is positive.

Since

$$\frac{\phi(z+h)+\phi(z-h)-2\phi(z)}{h^2} = \frac{F(z+h)+F(z-h)-2F(z)}{h^2} - C,$$

and since, for all sufficiently small values of h,

$$\frac{\phi(z+h)+\phi(z-h)-2\phi(z)}{h^2} \leqq 0,$$

it follows that $\underline{\mathscr{D}}^2F(z) \leqq \overline{\mathscr{D}}^2F(z) \leqq C$. Since $L \leqq \underline{\mathscr{D}}^2F(z)$, $U \geqq \overline{\mathscr{D}}^2F(z)$, it follows that $L \leqq C$; and this holds for every value of C that is consistent with the condition $\phi(x_1) > 0$. It has thus been shewn that

$$L \leqq \frac{F(x_1+\alpha)+F(x_1-\alpha)-2F(x_1)}{\alpha^2}.$$

In a similar manner, by choosing C so that $\phi(x_1)$ is negative, and considering a minimum of $\phi(x)$, it can be shewn that

$$U \geqq \frac{F(x_1+\alpha)+F(x_1-\alpha)-2F(x_1)}{\alpha^2}.$$

The following theorem follows at once from the above theorem:

If $F(x)$ be continuous in an interval (a, b), and if, in an interval (a_1, b_1) interior to (a, b), the upper and lower generalized differential coefficients $\overline{\mathscr{D}}^2F(x)$, $\underline{\mathscr{D}}^2F(x)$ are in an interval (L, U), then, for all points x in (a_1, b_1),

$$\frac{F(x+h)+F(x-h)-2F(x)}{h^2}$$

lies in the interval (L, U), provided h be such that $x+h$, $x-h$ are in the interval (a, b).

438. *If $f(x)$ be any function that is summable in the interval (a, b), and which is therefore finite almost everywhere in that interval, and if η be a prescribed positive number, a continuous function $\phi_1(x)$ can be constructed, such that it exceeds $\int_a^x f(x)\,dx$ by less than η, whatever value x may have in the interval (a, b), and such that, at every point at which $f(x)$ is finite, its four derivatives all exceed $f(x)$.*

Similarly, a continuous function $\phi_2(x)$ can be constructed which is everywhere less than $\int_a^x f(x)\,dx$ by less than η, and of which the four derivatives are all less than $f(x)$, at each point at which $f(x)$ is finite.

The functions $\phi_1(x)$, $\phi_2(x)$ have been denominated by de la Vallée Poussin, to whom this result is due, as *majorante* and *minorante* respectively, relatively to the function $f(x)$. In § 262 they have been termed major and minor functions associated with $f(x)$.

First, let $f(x)$ be everywhere $\geqq 0$ in the interval (a, b). Consider the numbers $0, \epsilon, 2\epsilon, \dots n\epsilon, \dots$; let e_n be the set of points x, at which

$$n\epsilon \leqq f(x) < (n+1)\epsilon;$$

then

$$\sum_{n=0}^{\infty} n\epsilon m(e_n) \leqq \int_a^b f(x)\,dx < \sum_{n=0}^{\infty} (n+1)\epsilon m(e_n) < \int_a^b f(x)\,dx + \epsilon(b-a).$$

Let all the points of e_n be enclosed within intervals of a set $\{\delta_r^{(n)}\}$ which do not overlap, and are such that

$$m(e_n) < \sum_{r=1}^{\infty} m(\delta_r^{(n)}) < m(e_n) + \epsilon_n,$$

where the numbers $\{\epsilon_n\}$ are so chosen that the series $\sum_{n=1}^{\infty} (n+1)\epsilon_n$ converges to a value less than unity.

If $S_r^{(n)}(x)$ denote the sum of all the intervals and portions of intervals of the set $\{\delta_r^{(n)}\}$ that lie in the interval (a, x), let $\phi_1(x) = \sum_{n=0}^{\infty} (n+1)\epsilon \,.\, S_r^{(n)}(x)$; it will be shewn that $\phi_1(x)$ satisfies the prescribed conditions in relation to the non-negative function $f(x)$.

We see at once that

$$\int_a^b f(x)\,dx < \phi_1(b) < \int_a^b f(x)\,dx + \epsilon(b-a) + \epsilon$$

and *a fortiori* that

$$\int_a^x f(x)\,dx < \phi_1(x) < \int_a^x f(x) + \epsilon(b-a) + \epsilon;$$

the number ϵ can be so chosen that $\epsilon(b-a) + \epsilon = \eta$.

Let x be a point of e_n, and therefore interior to an interval $\delta^{(n)}$, of the set $\{\delta_r^{(n)}\}$. We have then

$$\phi_1(x+h) - \phi_1(x) = \sum_{n=0}^{\infty} (n+1)\epsilon [S^{(n)}(x+h) - S^{(n)}(x)];$$

and all the terms of the series are positive when $h > 0$, and do not diminish as h increases. It follows that, if h be so small that $x + h$ is in the interval $\delta^{(n)}$, then $\phi_1(x+h) - \phi_1(x) \geqq (n+1)\epsilon h$, for all positive values of h that are sufficiently small. It is clear that, if h be negative, the inequality $\geqq$ must be replaced by $\leqq$. Hence, if $|h|$ be sufficiently small,

$$\frac{\phi_1(x+h) - \phi_1(x)}{h} \geqq (n+1)\epsilon,$$

and therefore the four derivatives of $\phi_1(x)$ are all $\geqq (n+1)\epsilon$, and consequently $> f(x)$.

Next, let $f(x)$ be unrestricted as regards its sign. If N be a positive number, let $f_N(x) = f(x)$, when $f(x) > -N$, and let $f_N(x) = -N$, when $f(x) \leqq -N$. If ζ be a positive number $< \eta$, a function $\psi(x)$ can be so determined as to satisfy the conditions that

$$\int_a^x [f_N(x) + N]\,dx < \psi(x) < \eta - \zeta + \int_a^x [f_N(x) + N]\,dx,$$

and that all the derivatives of $\psi(x)$ are greater than $f_N(x) + N$, at any point at which $f_N(x)$ is finite. The number N can be so chosen that

$$\int_a^x f_N(x)\,dx - \int_a^x f(x)\,dx$$

is $< \zeta$. Let $\phi_1(x)$ be defined by $\phi_1(x) = \psi(x) - N(x-a)$, then

$$\int_a^x f(x)\,dx < \int_a^x f_N(x)\,dx < \phi_1(x) < \eta - \zeta + \int_a^x f_N(x)\,dx < \eta + \int_a^x f(x)\,dx,$$

and all the derivatives of $\phi_1(x)$ are greater than $f_N(x)$, or than $f(x)$, at any point at which $f(x)$ is finite. Hence the required function $\phi_1(x)$ has been constructed.

In order to construct the function $\phi_2(x)$, we observe that, if $\psi(x)$ be a major function relatively to $-f(x)$, the function $-\psi(x)$ is the required minor function relatively to $f(x)$.

It is easily seen that a monotone non-increasing sequence of major functions, associated with $f(x)$, can be constructed, and similarly a monotone non-diminishing sequence of minor functions. For, if $\phi_1^{(1)}(x)$, $\phi_1^{(2)}(x)$ correspond to the values of η_1, η_2, where $\eta_1 > \eta_2$, in any interval in which $\phi_1^{(2)}(x)$ is $\geqq \phi_1^{(1)}(x)$, $\phi_1^{(2)}(x)$ can be replaced by $\phi_1^{(1)}(x)$, and then $\phi_1^{(2)}(x) \leqq \phi_1^{(1)}(x)$ everywhere. A sequence $\{\phi_1^{(n)}(x)\}$ thus formed must converge to $\int_a^x f(x)\,dx$.

439. *If $F(x)$ be continuous in the interval (a, b), and there exists a finite summable function $f(x)$ such that $\overline{\mathscr{D}}^2 F(x) \geqq f(x) \geqq \underline{\mathscr{D}}^2 F(x)$ at every point of (a, b), then $F(x) - \int_a^x dx \int_a^x f(x)\,dx$ is a linear function of x in (a, b).*

The theorem also holds if the summable function $f(x)$ has infinite values in a set of points E which contains no perfect set, provided that, at all points of E, the function $F(x)$ has at least one symmetrical derivative.

Let $\phi_1(x)$, $\phi_2(x)$ be the major and the minor functions associated with $f(x)$, constructed in accordance with § 438.

The two functions $\quad \psi_1(x) \equiv F(x) - \int_a^x \phi_2(x),$

$$\psi_2(x) = F(x) - \int_a^x \phi_1(x),$$

and the function $\quad \psi(x) = F(x) - \int_a^x dx \int_a^x f(x)\,dx$

are such that $\psi_1(x) > \psi(x) > \psi_2(x)$, and that $\psi_1(x) - \psi_2(x) < 2(b-a)\eta$. The three functions have all the same value at the point a.

We have $\quad \overline{\mathscr{D}}^2 \psi_1(x) > \overline{\mathscr{D}}^2 F(x) - f(x) > 0$

and $\quad \underline{\mathscr{D}}^2 \psi_2(x) < \underline{\mathscr{D}}^2 F(x) - f(x) < 0$

at all points at which $f(x)$ has a finite value.

In virtue of the theorem of § 434 it follows that

$$\psi_1(x) < F(a) + \frac{x-a}{b-a}\{\psi_1(b) - \psi_1(a)\}$$

and $\quad \psi_2(x) > F(a) + \frac{x-a}{b-a}\{\psi_2(b) - \psi_2(a)\},$

when $x > a$. It now follows that $\psi(x)$ is between these two linear functions which differ from one another by less than $2\eta(x-a)$. Taking a monotone non-diminishing sequence of major functions, and a monotone non-increasing sequence of minor functions, constructed as in § 438, corresponding to the values of η in a sequence $\{\eta_n\}$ which converges to zero, we see that $\psi(x)$ lies, for every value of n, between the values of two linear functions $A_n x + B_n$, $A_n' x + B_n'$, where A_n, A_n' have the same limit A, as $n \sim \infty$, and B_n, B_n' have the same limit B; it then follows that $\psi(x)$ is the linear function $Ax + B$.

It follows from the above theorem that, in case $F(x)$ satisfies the conditions of the theorem, it has everywhere a continuous differential coefficient $\int_a^x f(x)\,dx + p$, where $px + q$ is the linear function. Moreover, it will almost everywhere have the second differential coefficient $f(x)$.

THE CONVERGENCE OF A TRIGONOMETRICAL SERIES AT A POINT

440. If the trigonometrical series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$ converge at a point x, and if the sum-function of the series have definite limits $f(x+0)$, $f(x-0)$, on the right, and on the left, at the point x, it does not follow that the series necessarily converges at points in a neighbourhood of the point x at which the series converges. From the existence of $f(x+0), f(x-0)$, it follows however that, corresponding to an arbitrarily chosen positive number δ, a neighbourhood of the point x can be determined, such that $\overline{f}(x) - \underline{f}(x) < \delta$, for all points x in that neighbourhood. From Harnack's theorem, given in § 428, it now follows that $a_n = o(1)$, $b_n = o(1)$.

If $F(x)$ denotes $\frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty} \frac{1}{n^2}(a_n \cos nx + b_n \sin nx)$, it has been shewn in § 421 that

$$\lim_{\epsilon \sim 0} \frac{F(x+\epsilon) - 2F(x) + F(x-\epsilon)}{\epsilon^2} = f(x)$$

at the given point x of convergence of the series.

We now have

$$2f(x) = \lim_{\epsilon \sim 0} \left\{ 4\frac{F(x+2\epsilon) - 2F(x) + F(x-2\epsilon)}{4\epsilon^2} - 2\frac{F(x+\epsilon) - 2F(x) + F(x-\epsilon)}{\epsilon^2} \right\}$$

$$= \lim_{\epsilon \sim 0} \left\{ \frac{F(x+2\epsilon) - 2F(x+\epsilon) + F(x)}{\epsilon^2} + \frac{F(x) - 2F(x-\epsilon) + F(x-2\epsilon)}{\epsilon^2} \right\}.$$

In accordance with the theorem of § 437,

$$\frac{F(x+2\epsilon) - 2F(x+\epsilon) + F(x)}{\epsilon^2}$$

lies between the extreme values of

$$\lim_{\alpha \sim 0} \frac{F(z+\alpha) - 2F(z) + F(z-\alpha)}{\alpha^2},$$

for $x \leqq z \leqq x + 2\epsilon$. It has been shewn in § 437 that, for each value of z, this limit lies between values which depend on the upper and lower values of $f(z)$. It follows that, for an assigned positive number δ, the positive number ϵ can be so determined that

$$f(x+0) - \delta < \lim_{\alpha \sim 0} \frac{F(z+\alpha) - 2F(z) + F(z-\alpha)}{\alpha^2} < f(x+0) + \delta,$$

for every value of z such that $x \leqq z \leqq x + 2\epsilon$.

We thus see that $\lim_{\epsilon \sim 0} \frac{F(x+2\epsilon) - 2F(x+\epsilon) + F(x)}{\epsilon^2} = f(x+0)$

Similarly, it can be shewn that

$$\lim_{\epsilon \sim 0} \frac{F(x-2\epsilon) - 2F(x-\epsilon) + F(x)}{\epsilon^2} = f(x-0).$$

It has now been proved that $f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$.

The following theorem has been established:

If a trigonometrical series converge at a point, then the value to which it converges is the mean of the limits of the sum-function, on the right, and on the left, at the point, provided those limits exist as definite numbers.

This theorem holds for every trigonometrical series, whether it be a Fourier's series, or not.

THE UNIQUENESS OF A TRIGONOMETRICAL SERIES WHICH REPRESENTS A FUNCTION

441. In order to establish the uniqueness of a trigonometrical series which, in an assigned sense (not necessarily in the sense that the series converges to the value of the function almost everywhere), represents a given function, it is sufficient to establish, first that the series is a Fourier's series, and secondly, that there cannot exist two distinct Fourier's series, both of which have the given relation with the given function. The latter is equivalent to proving that, if two such Fourier's series exist, the Fourier's series which is the difference of the two must have all its coefficients zero. In case the mode of representation of the function is postulated to be such that the series converges almost everywhere to the values of a single-valued function which is taken to be summable in $(-\pi, \pi)$, then it is clear that there cannot be two Fourier's series both of which represent the function. It has been shewn in § 361 that there cannot be two non-equivalent summable functions which have one and the same Fourier's series.

442. In the case which will be given first, the uniqueness of a trigonometrical series can be established without shewing that it is a Fourier's series.

Let it be assumed that the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges to zero at every point of the interval $(-\pi, \pi)$ with the exception of an enumerable set of points E, at which it is not assumed that the series converges. Since an enumerable set is of the first category, it follows from the theorem of § 429 that a_n and b_n converge to zero. Accordingly, Riemann's function $\frac{1}{4}a_0x^2 - \Sigma \dfrac{a_n \cos nx + b_n \sin nx}{n^2}$ exists as a continuous function, and $\mathscr{D}^2F(x) = 0$, for every point at which the series converges to zero; also by Riemann's second theorem, $F(x)$ has symmetrical derivatives at every point. It follows, in accordance with the extension of Schwarz's theorem given in § 435, that $F(x)$ is linear in any interval $(-m\pi, m\pi)$; thus $F(x) = ax + b$, in any interval $(-m\pi, m\pi)$. It is thus

seen that, in an interval $(-m\pi, m\pi)$, $\frac{1}{2}a_0x^2 - (ax+b)$ is represented by the periodic series $-\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}$, and therefore $\frac{1}{2}a_0x^2 - (ax+b)$ must be a periodic function, which can only be the case if $a_0 = 0$, $a = 0$.

Since the series converges uniformly to $-b$, we can multiply by $\cos nx$, or by $\sin nx$, and integrate term by term between the limits $-\pi, \pi$; it is thus seen that $a_n = 0$, $b_n = 0$, and therefore all the coefficients of the given series vanish identically. By considering the difference of two series, the following theorem can be established:

No two distinct trigonometrical series can exist which converge to the same value for all points of the interval $(-\pi, \pi)$, with the exception of an enumerable set of points at which the series are not known to converge to the same sum, or to converge at all.

This theorem, which is due to W. H. Young, is an extension of the older theorem of Cantor, in which the exceptional set of points is restricted to be a reducible set. It is also an extension of the still earlier theorem of Cantor* that a trigonometrical series is the unique representation of a function which has an indefinitely great number of points of discontinuity which form a set of the first species.

443. *If the Riemann function $F(x)$, corresponding to a trigonometrical series*

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx),$$

such that a_n, b_n converge to zero, as $n \sim \infty$, is of the form

$$F(x) = \int_{-\pi}^{x} dx \int_{-\pi}^{x} f(x)\,dx + px + q,$$

where p, q are constants, and $f(x)$ is a function summable in $(-\pi, \pi)$, then the series is the Fourier's series corresponding to $f(x)$.

We have $\qquad F'(\pi) = \int_{-\pi}^{\pi} f(x)\,dx + p, \quad F'(-\pi) = p;$

and thus $\qquad \int_{-\pi}^{\pi} f(x)\,dx = F'(\pi) - F'(-\pi) = a_0\pi,$

and hence $a_0 = \alpha_0$, where $\frac{1}{2}\alpha_0 + \sum_{n=1} (\alpha_n \cos nx + \beta_n \sin nx)$ denotes the Fourier's series corresponding to $f(x)$.

By the theorem of § 360 the series $\sum_{n=1}^{\infty} \frac{\alpha_n \sin nx - \beta_n \cos nx}{n}$ is, when a constant is added to it, the Fourier's series which converges uniformly to $\int_{-\pi}^{x} f(x)\,dx - \frac{1}{2}a_0 x$. Thus $-\frac{\beta_n}{n}, \frac{\alpha_n}{n}$ are the Fourier's constants corresponding to the function $F'(x) - \frac{1}{2}a_0x - p$, or to $F'(x) - \frac{1}{2}a_0x$.

* See *Crelle's Journal*, vol. LXXII (1870), p. 130, and *Math. Annalen*, vol. V (1872), p. 123.

Now
$$\frac{a_n}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \{\tfrac{1}{4}a_0 x^2 - F(x)\} \cos nx\, dx$$
$$= \frac{1}{n\pi}\int_{-\pi}^{\pi} \{F'(x) - \tfrac{1}{2}a_0 x\} \sin nx\, dx = \frac{\alpha_n}{n^2},$$
and similarly we see that $\frac{b_n}{n^2} = \frac{\beta_n}{n^2}$. Therefore $\alpha_n = a_n$, $\beta_n = b_n$; and it has been shewn that $\alpha_0 = a_0$, which establishes the theorem.

This theorem, taken in conjunction with the theorem of § 439, yields the following result:

If the series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, for which a_n, b_n converge to zero, be such that its upper and lower sum-functions are summable in $(-\pi, \pi)$, and such that both of them are finite at every point, with the possible exception of points belonging to a set E which contains no perfect component, then the given series is the Fourier's series corresponding to either the upper or the lower sum-function of the series, and consequently to either $\mathscr{D}^2F(x)$ or $\mathscr{D}^2f(x)$.

This theorem is theoretically more general than that of § 442, as the exceptional set of points is not necessarily enumerable, but may be an unenumerable set which contains no perfect set, if such a set exists.

From the theorem of § 436, the upper and lower generalized second differential coefficients of $F(x)$ are both finite except at the points of E. If $f(x)$ denote either the upper or the lower sum-function of the series, then $F(x) - \int_{-\pi}^{x} dx \int_{-\pi}^{x} f(x)\, dx$ is linear in the interval (a, b), and consequently the given series is the Fourier's series corresponding to $f(x)$. It follows that, when the conditions of the theorem are satisfied, the upper and lower sum-functions must be equal almost everywhere, and therefore the Fourier's series is convergent almost everywhere.

A particular case of the above theorem is that*:

If the series $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ has its upper and lower sum-functions bounded, it is a Fourier's series.

It is unnecessary in this theorem to include in the statement the condition that a_n, b_n converge to zero, as $n \sim \infty$, because it can be shewn that this is necessarily the case if the upper and lower sum-functions are bounded in the interval (see § 432). This was the first theorem obtained which referred to the upper and lower sum-functions of the series.

A particular case of the general theorem is the following theorem of Lebesgue†:

There is only a single trigonometrical series which converges everywhere to a given bounded function; viz. the Fourier's series corresponding to the

* See W. H. Young, *Proc. Lond. Math. Soc.* (2), vol. IX, p. 427.

† See *Leçons sur les séries trigonométriques* (1906), p. 122, also *Annales sc. de l'école normale* (3), vol. XX (1903), p. 467.

function. There may be a reducible set of points at which the series is not known to converge.

There exist series which converge everywhere, and of which the sum-function is unbounded, which are not Fourier's series. For example, the series $\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$ is everywhere convergent, but it is not a Fourier's series, since the series $\sum_{n=2} \frac{\cos nx}{n \log n}$ is not convergent when $x = 0$.

The above general theorem* may be replaced by the following:

Every trigonometrical series for which the Riemann sum everywhere exists either as a single finite number, or as indeterminate between finite upper and lower boundaries, and is such that the function $\phi(x)$ which is at each point equal to the numerically smaller of the upper and lower Riemann functions is ummable, is the Fourier's series corresponding to $\phi(x)$.

It will be observed that it is unnecessary to assume *a priori* that a_n and b_n converge to zero. But when there is an exceptional set of points at which the limits of indetermination are not finite, we have the following statement:

The above theorem holds for a trigonometrical series such that a_n, b_n converge to zero, as $n \sim \infty$, even when there is a set E, of points at which the function $\phi(x)$ is not finite, provided E contains no perfect component, and provided the function $\phi(x)$ is summable.

As regards the existence of sets of points which are unenumerable and which do not contain a perfect set, it has been shewn† by Alexandroff that such a set cannot be measurable (B); that in fact every unenumerable set that is measurable (B) contains a perfect set.

444. If $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ be the Fourier's series corresponding to a given summable function $f(x)$, we have (see § 360)

$$\int_{-\pi}^{x} f(x)\, dx = C + \tfrac{1}{2}a_0 x + \Sigma \frac{a_n \sin nx - b_n \cos nx}{n},$$

and since the function $\int_{-\pi}^{x} f(x)\, dx$ is of bounded variation, we have

$$\int_{-\pi}^{x} dx \int_{-\pi}^{x} f(x)\, dx = C' + Cx + \tfrac{1}{4}a_0 x^2 - \Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}.$$

and thus the function $F(x)$ differs from $\int_{-\pi}^{x} dx \int_{-\pi}^{x} f(x)\, dx$ by a linear function. At every point x we have $F'(x) = \int_{-\pi}^{x} f(x)\, dx$, and almost everywhere $F''(x) = f(x)$.

* See de la Vallée Poussin, *Bulletin de l'acad. roy. de Belgique*, 1912, p. 717.

† *Comptes Rendus*, vol. CLXII (1916), p. 323; see also Hausdorff, *Math. Annalen*, vol. LXXVII (1916), p. 436.

Therefore the Riemann sum of the Fourier's series exists everywhere, and is almost everywhere equal to $f(x)$.

If another trigonometrical series exists besides the Fourier's series, and is also summable by Riemann's procedure almost everywhere, having $F(x)$ almost everywhere for the Riemann sum, it would appear that the difference of the two series would have a Riemann sum almost everywhere, and it would be equal to zero. If such a series, with coefficients not all zero, exists, its Riemann sum must be infinite at points of some set which contains a perfect set. To define such a series, let H be a non-dense perfect set of measure zero, in $(-\pi, \pi)$. Let the intervals contiguous to H be placed in correspondence with the rational numbers of the interval $(0, 1)$ taken in ascending order. Let $\phi(x)$ have the value, at any point within one of the contiguous intervals, of the rational number to which the interval corresponds; and at any point not interior to such an interval, let $\phi(x)$ be defined so as to be continuous. Then $\phi'(x)$ exists at all points of $C(H)$, and has the value zero. The continuous function $\phi(x)$ is monotone, and thus it is representable as the sum of a uniformly convergent Fourier's series

$$\phi(x) = \tfrac{1}{2}a_0 + \sum_1^\infty \frac{a_n \sin nx - b_n \cos nx}{n}.$$

The function $$F(x) = -\sum \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

has $\phi(x) - \frac{1}{2}a_0$ for its differential coefficient, everywhere, and it has a second differential coefficient equal to zero almost everywhere. Thus the series $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ is summable by Riemann's procedure almost everywhere, and has zero for the value of that sum. It has been shewn by de la Vallée Poussin, to whom this construction is due, that the Cesàro sum of the series is almost everywhere zero, like the Riemann sum.

445. It has been shewn in § 443 that:

If the trigonometrical series $\frac{1}{2}a_0 + \sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ *converges almost everywhere to the values of a function* $f(x)$ *which is summable in* $(-\pi, \pi)$, *and the set of points at which it does not converge, or oscillate between finite limits, contains no perfect component, then the series is a Fourier's series, and it is the unique trigonometrical series which satisfies the prescribed conditions.*

It is unnecessary to assume in the statement of the theorem that $a_n = o(1)$, $b_n = o(1)$, because these relations follow from the convergence almost everywhere, of the series (see § 430).

It follows that, if a trigonometrical series converges to zero everywhere except at the points of a set E which contains no perfect component, then all the coefficients of the series vanish. It has been proved* by Rajchman

* See *Prace Mathematyczno-fizyczne*, vol. XXX (1919), p. 30.

that the same result holds if, instead of ordinary convergence, the Poisson sum of the series be taken.

An example has been given by* Menchoff of a trigonometrical series which converges to zero at all points except those of a certain perfect set of measure zero, such that the coefficients do not vanish.

It has however been shewn† by Rajchman that there exist closed sets of measure zero, unenumerable, and therefore containing perfect sets, of a certain type first‡ considered by Hardy and Littlewood, such that the coefficients all vanish if the trigonometrical series converges everywhere to zero except at the points of a closed set of this special type. Thus the condition of the above theorem can be replaced by a less stringent condition. Moreover this result also holds when, instead of ordinary convergence, the Poisson sum is zero except in the exceptional set of points, so that if $\frac{1}{2}a_0 + \lim_{h\sim 1} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, h^n = 0$, everywhere except in a perfect set of the special type, then $a_n = 0$, for $n = 0, 1, 2, 3, \dots$, and $b_n = 0$, for $n = 1, 2, 3, \dots$.

The following theorem has been established§ by Rajchman:

If the trigonometrical series $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos 2n\pi x + b_n \sin 2n\pi x)$, *such that* $a_n = o(1)$, $b_n = o(1)$, *converges everywhere to zero, except possibly at the points of a closed set of type* (H); *or more generally if, except at the points of that closed set,*

$$\tfrac{1}{2}a_0 + \lim_{r\sim 1} \sum_{n=1}^{\infty} (a_n \cos 2n\pi x + b_n \sin 2n\pi x)\, r^n = 0,$$

then $a_0 = 0$, $a_n = b_n = 0$.

It has been stated|| by Rajchman that the fact of the existence of perfect sets which have the above property, in relation to the uniqueness of a trigonometrical series, had been demonstrated, but not published, by Mlle Nina Bary, in 1921, in the Mathematical Seminary of the University of Moscow. Rajchman's independent result relating to the sets of type (H) has been later generalized¶ by Mlle Bary, who shewed that a set of points $M(H_1, H_2, \dots)$, where $\{H_n\}$ is a sequence of sets, all of type (H), has the same property, in relation to the uniqueness of a trigonometrical series, as a single set of type (H).

The closed sets introduced by Hardy and Littlewood, and considered further by Steinhaus, and explicitly defined by Rajchman, who terms them sets of type (H), can be defined as follows:

* *Comptes Rendus*, vol. CLXIII (1916), p. 433.

† *Fundamenta Math.* vol. III (1922), p. 287.

‡ *Acta Math.* vol. XXXVII (1914), p. 155.

§ *Fundamenta Math.* vol. III (1922), p. 287.

|| *Ibid.* vol. IV (1923), p. 367.

¶ *Comptes Rendus*, vol. CLXXVII (1923), p. 1195. Rajchman has given another proof of his result, dependent on the theory of formal multiplication, *Comptes Rendus*, vol. CLXXVII (1923), p. 493. Some further results are given by Zygmund, in the same volume, p. 576. See also Zygmund, *Math. Zeitschr.* vol. XXIV (1925), p. 40.

Let G be a closed set contained in the interval $(0, 1)$. In case either 0 or 1 is a point of G, it will be assumed that both of these points are points of G. Let a point $P(x)$, whose polar coordinates are $r = 1$, $\theta = 2\pi x$, be made to correspond to each point x, of G. To the closed set G, in the interval $(0, 1)$, there will correspond a closed set $\bar{G}$, of points on the circumference of the circle with radius unity. If k be a positive integer, and $P(kx)$ be the point whose polar coordinates are $r = 1$, $\theta = 2\pi kx$, it is clear that $P(kx) = P(kx - Ekx)$, where Ekx denotes the greatest integer $\leqq kx$. Let $\bar{G}_k$ denote the set of points $P(kx)$, where x has all the values in G. In order that a point $P(y)$, where $0 \leqq y < 1$, may not belong to $\bar{G}_k$, it is necessary and sufficient that the k numbers

$$\frac{y}{k},\ \frac{y}{k}+\frac{1}{k},\ \frac{y}{k}+\frac{2}{k}, \dots, \frac{y}{k}+\frac{k-1}{k}$$

do not belong to G.

We take the set G_k to be the set in $(0, 1)$ which corresponds to $\bar{G}_k$, so that, if y is a point of G_k, $r = 1$, $\theta = 2\pi y$ is a point of $\bar{G}_k$. Let $2\pi d_k$ be the length of the greatest arc of the circle which does not contain in its interior any point of $\bar{G}_k$; and thus the length of the greatest arc contiguous to $\bar{G}_k$. We have, for each value of k, $0 \leqq d_k \leqq 1$, and thus $\overline{\lim\limits_{k\sim\infty}}\, d_k > 0$, unless $\lim\limits_{k\sim\infty} d_k$ exists, and has the value zero.

The closed set G will be said to be of type (H), provided $\overline{\lim\limits_{k\sim\infty}}\, d_k > 0$.

In the case of Cantor's perfect set (I, § 118), we have $d_{3^h} = \frac{1}{3}$, for $h = 1, 2, 3, \dots$; therefore this set is of type (H).

446. With a view to the extension of the theorems of §§ 443, 445, relating to a trigonometrical series $\frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$, it will be sufficient to leave out the coefficient a_0 and to consider the series $\sum\limits_{n=1} A_n$, where A_n denotes $a_n \cos nx + b_n \sin nx$.

Let $F_0(x)$ denote $-\sum\limits_{n=1} \frac{A_n}{n^2}$, $F_2(x)$ the series $\sum\limits_{n=1} \frac{A_n}{n^4}$, and generally let $F_{2r}(x)$ denote $(-1)^{r+1} \Sigma \frac{A_n}{n^{2r+2}}$. It will be assumed that either a_n, b_n are bounded, or more generally that $\frac{a_n}{n^{1-p}}$, $\frac{b_n}{n^{1-p}}$, where $(0 < p \leqq 1)$, are bounded. In either case the series $\Sigma \frac{A_n}{n^2}$ is uniformly convergent.

Let
$$K_0(h) = F_0(x+h) + F_0(x-h) - 2F_0(x),$$
$$K_2(h) = F_2(x+h) + F_2(x-h) - 2F_2(x) - h^2 F_0(x),$$

etc.,

and generally

$$K_{2r}(h) = F_{2r}(x+h) + F_{2r}(x-h) - 2F_{2r}(x) - h^2 F_{2r-2}(x)$$
$$- \frac{2h^4}{4!} F_{2r-4}(x) - \frac{2h^6}{6!} F_{2r-6}(x) - \dots - \frac{2h^{2r}}{(2r)!} F_0(x).$$

These functions are formed by the rule

$$K_2(h) = \int_0^h dh \int_0^h K_0(h)\, dh, \quad K_{2r}(h) = \int_0^h dh \int_0^h K_{2r-2}(h)\, dh.$$

Denoting by $G_r(h)$ the expression $K_{2r}(h) \Big/ \frac{2h^{2r+2}}{(2r+2)!}$, it will be shewn, by employing the theorem of § 158, that the $\overline{\lim\limits_{h\sim 0}}\, G_r(h)$ and $\underline{\lim}\, G_r(h)$ are given by $\frac{1}{2}(\overline{C}_r + \underline{C}_r) \pm \frac{1}{2}\lambda(\overline{C}_r - \underline{C}_r)$, where $\overline{C}_r$ and $\underline{C}_r$ are the upper and lower sums (C, r) of the series ΣA_n; where these Cesàro sums are assumed to be finite, and λ is a fixed number.

We have $G_r(h) = \sum\limits_{n=1}^{\infty} A_n \phi(nh)$, where $\phi(nh)$ is given by

$$\phi(nh) = \frac{(-1)^r(2r+2)!}{(nh)^{2r+2}} \left[(1 - \cos nh) - \frac{n^2h^2}{2!} + \frac{n^4h^4}{4!} - \dots + (-1)^{r-1}\frac{(2h)^{2r}}{(2r)!}\right].$$

Writing t for nh, we have

$$\phi(t) = \frac{(-1)^r(2r+2)!}{t^{2r+2}}\left(1 - \cos t - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots + (-1)^{r-1}\frac{t^{2r}}{(2r)!}\right),$$

from which it follows that $\lim\limits_{t\sim 0} \phi(t) = 1$, and that $t^2\phi(t)$ is bounded for all positive values of t.

In order to apply the theorem of § 158, we shall shew that $\phi^{(r+1)}(t)\, t^{r+3}$ is bounded for all values of t. Since

$$\frac{\phi(t)}{(-1)^r(2r+2)!} = (1 - \cos t)\, t^{-2r-2} + \frac{(-1)^{r-1}}{(2r)!} t^{-2} + \dots - \frac{1}{2!} t^{-2r}.$$

$\dfrac{\phi^{(r+1)}(t)}{(-1)^r(2r+2)!}$ consists of the term in t^{-r-3}, a number of terms of higher negative powers, and of terms containing one of the factors $\cos t$ or $\sin t$ and as the other factor a power of t which is $-2r-2, -2r-3, \dots, -3r-3$. It follows that $t^{r+3}\phi^{(r+1)}(t)$ consists of a constant term and of terms containing negative powers of t; hence $t^{r+3}\phi^{(r+1)}(t)$ is bounded for all values of $t > c > 0$. Applying the theorem of § 158, taking $k = 2$, it follows that the upper and lower limits of $G_r(h)$, as $h \sim 0$, lie in an interval

$$\tfrac{1}{2}(\overline{C}^{(r)} + \underline{C}^{(r)}) \pm \tfrac{1}{2}\lambda(\overline{C}^{(r)} - \underline{C}^{(r)}).$$

The following theorem has now been established:

If $F_{2r}(x)$ denote the series $(-1)^{(r+1)}\sum_{n=1}^{\infty}\frac{A_n}{n^{2r+2}}$, and the series $F_0(x) = -\sum_{n=1}^{\infty}\frac{A_n}{n^2}$ be assumed to be convergent, and if

$$K_{2r}(h) \equiv F_{2r}(x+h) + F_{2r}(x-h) - 2F_{2r}(x) - h^2F_{2r-2}(x) - \frac{2h^4}{4!}F_{2r-4}(x) - \dots - \frac{2h^{2r}}{(2r)!}F_0(x)$$

is the function defined by $K_{2r}(h) = \int_0^h dh \int_0^h K_{2r-2}(h)\,dh$, and $G_r(h)$ denote

$$K_{2r}(h) \Big/ \frac{2h^{2r+2}}{(2r+2)!},$$

then the upper and lower limits of $G_r(h)$, as $h \sim 0$, are given by

$$\tfrac{1}{2}(\overline{C}_r + \underline{C}_r) \pm \tfrac{1}{2}\lambda(\overline{C}_r - \underline{C}_r),$$

when λ is a fixed number, and $\overline{C}_r$, $\underline{C}_r$ are the upper and lower Cesàro sums (C, r) of the series $\sum_{n=1} A_n$, assumed to be finite.

447. By repeated application of the theorem given in I, § 264, it is seen that the upper and lower limits of $G_r(h)$, as $h \sim 0$, lie in the interval bounded by the upper and lower limits of $\frac{F_0(x+h) + F_0(x-h) - 2F_0(x)}{h^2}$, where $F_0(x)$ is identical with Riemann's function $F(x)$.

When $a_n = O(n^{1-p})$, $b_n = O(n^{1-p})$, where $0 < p \leq 1$, the function $F(x)$ is continuous, and any limit of $G_r(h)$, as $h \sim 0$, is also a limit of

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2}.$$

Let it now be assumed that, at every point of the interval $(-\pi, \pi)$, $\overline{C}^{(r)}$, $\underline{C}^{(r)}$ are both finite, and that they are summable in the interval; it then follows that a function exists which is summable in $(-\pi, \pi)$, finite at every point, and is at each point x in the interval formed by $\overline{\mathscr{D}}^2F(x)$, $\underline{\mathscr{D}}^2F(x)$. Applying the theorem of § 443, it now follows that the series ΣA_n is a Fourier's series.

The following theorem has now been established:

If $\frac{a_n}{n^k}$, $\frac{b_n}{n^k}$ are bounded, where k is some number such that $0 \leq k < 1$, and if the series $\Sigma(a_n \cos nx + b_n \sin nx)$ is such that its upper and lower Cesàro sums of integral order r are finite at each point of $(-\pi, \pi)$, and summable in that interval, then the series is a Fourier's series.

In the particular case $r = 1$, if it be assumed that the upper and lower Cesàro sums of order k, where k is such that $0 < k < 1$, are everywhere finite, it follows that $\frac{a_n}{n^k}$, $\frac{b_n}{n^k}$ are bounded and that the Cesàro sum

of order 1 is bounded. We have then the following theorem which has been given by W. H. Young*:

If the upper and lower Cesàro sums of order k, where $0 \leq k < 1$, of the series $\Sigma (a_n \cos nx + b_n \sin nx)$ are everywhere finite, and define summable functions in the interval $(-\pi, \pi)$, the series is a Fourier's series.

If the more stringent assumption be made that a_n, b_n converge to zero, as $n \sim \infty$, the function $F(x)$ has at each point symmetrical derivatives; and we obtain the following theorem:

If a_n, b_n converge to 0, as $n \sim \infty$, and if the series $\Sigma (a_n \cos nx + b_n \sin nx)$ is such that the upper and lower Cesàro sums, of integral order r, are summable in $(-\pi, \pi)$, and finite at every point which does not belong to a set E which contains no perfect component, then the series is a Fourier's series.

This theorem was also given, for the case $r = 1$, by W. H. Young (*loc. cit.*). The mode of proof given above is a modification of this proof. A more complicated proof of the general theorem has been given† by A. Rajchman.

448. The following theorem relating to Cesàro summation of any order k, positive but not necessarily integral, may be deduced from de la Vallée Poussin's theorem (§ 432):

If the series $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ has, in any interval, its upper and lower Cesàro sums (C, k), where $k > 0$, finite at each point which does not belong to a set of measure zero, then $\frac{a_n}{n^k}, \frac{b_n}{n^k}$ are bounded.

It follows from the condition of the theorem that the upper and lower sums of the series $\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^k}$ are both finite at almost all points of the interval, and therefore $\frac{a_n}{n^k}, \frac{b_n}{n^k}$ are bounded.

The following theorem may be deduced from that of § 430:

If the series $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ is summable (C, k), where $k > 0$, at all points of a set H of positive measure, then $\frac{a_n}{n^k}, \frac{b_n}{n^k}$ converge to zero as $n \sim \infty$.

For it is known that at any point at which the series is summable (C, k), $\frac{a_n \cos nx + b_n \sin nx}{n^k}$ converges to zero. This being the case at points of a set H, of positive measure, the result follows from § 430.

A particular case of this theorem was given‡ by M. Riesz, that if the series $\frac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx)$ is summable $(C, 1)$ in an interval, then $\frac{a_n}{n}, \frac{b_n}{n}$ converge to zero.

* *Proc. Roy. Soc.* (A), vol. LXXXIX (1914), p. 150.

† *Monatshefte fur Math. u. Physik*, vol. XXVI (1915), p. 263.

‡ *Math. Annalen*, vol. LXXI (1912), p. 58.

449. The general theorem of § 443 includes the condition that the upper and lower sum-functions of the given series are summable in $(-\pi, \pi)$ whether they be bounded or not. It is however convenient to possess tests that a given trigonometrical series is a Fourier's series that do not involve this condition, but instead depend upon whether these upper and lower functions are bounded in the whole or in a part of the interval. It has been shewn in § 360 that, if the integrated series $\Sigma \frac{a_n \sin nx - b_n \cos nx}{n}$ converges to an integral, then the given series is a Fourier's series; but in practice it is difficult in any particular case to ascertain whether this condition is satisfied or not.

It is accordingly convenient to possess tests in which a lesser knowledge of the properties of this integrated series is involved. With a view to remedying as far as possible these practical defects of the theorems of § 442, the following theorem will be established:

If a trigonometrical series has its upper and lower sum-functions bounded, except in the neighbourhood of points belonging to a closed enumerable set E, and if the integrated series $\sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$ converges boundedly (or in particular uniformly) in the whole interval $(-\pi, \pi)$ to a continuous function, then the trigonometrical series is either a Fourier's series or a Fourier HL-series.

In accordance with the theorem of § 432, since $\Sigma (a_n \cos nx + b_n \sin nx)$ is bounded in an interval, a_n and b_n are bounded, and thus the series $-\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}$ converges uniformly in $(-\pi, \pi)$ to a continuous function $F(x)$. Denoting by $\Phi(x)$ the sum-function of the series

$$\sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n},$$

since this series by hypothesis converges boundedly, term by term integration may be applied to it, and thus $\int_{-\pi}^{x} \Phi(x)\, dx = F(x) - F(-\pi)$. It follows that $F'(x)$ exists everywhere in $(-\pi, \pi)$, and is continuous, being equal to $\Phi(x)$.

Consider an interval (a, b) interior to an interval contiguous to E: then the upper and lower sum-functions of the series $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ are bounded in (a, b). It follows from the theorems of §§ 421, 437 that, if h be sufficiently small, $\frac{F(x+h) + F(x-h) - 2F(x)}{h^2}$ is bounded as a function of (x, h), for all such values of h, and all values of x in (a, b).

Denoting $\int_{-\pi}^{x} F(x)\,dx$ by $F_1(x)$, and employing the second theorem given in § 221, it is seen that the upper and lower limits, as $h \sim 0$, of

$$\frac{F_1(x+h)+F_1(x-h)-2F_1(x)}{h^2}$$

are L-integrals in (a, b), since they are both upper and lower semi-integrals. Since $F_1(x)$ possesses everywhere in (a, b) a second differential coefficient $F'(x)$, both these upper and lower limits coincide with $F'(x)$, which is therefore an integral in (a, b) and consequently has, almost everywhere in (a, b), a differential coefficient $F''(x)$.

If (α, β) be the interval, contiguous to E, in which (a, b) is contained, we have $\int_a^b F''(x)\,dx = F'(b) - F'(a)$; $F''(x)$ existing at almost all points of (a, b) (see I, § 406, last theorem). Since $F'(x)$ is continuous, we have, as a, b converge to α, β respectively.

$$\lim_{b\sim\beta,\, a\sim\alpha} \int_a^{\beta} F''(x)\,dx = F'(\beta) - F'(\alpha),$$

and therefore $\int_{\alpha}^{\beta} F''(x)\,dx$ exists, either as an L-integral, or else as an HL-integral. Since (α, β) is any one of the intervals contiguous to E, all the abutting intervals may be amalgamated, and we see that, if (α_1, β_1) be any interval contiguous to the first derivative E', of E, the integral of $F''(x)$ exists in any interval interior to (α_1, β_1), and is equal to the difference of the values of $F'(x)$ at the ends of the interval. Proceeding to the limit as before, we see that $\int_{\alpha_1}^{\beta_1} F''(x)\,dx$ exists, as either an L-integral or as an HL-integral, and its value is $F'(\beta_1) - F'(\alpha_1)$. Proceeding in this manner, it is seen that $\int_{\alpha_n}^{\beta_n} F''(x)\,dx$ exists in any interval (α_n, β_n) contiguous to $E^{(n)}$, the nth derivative of E. If $(\alpha_\omega, \beta_\omega)$ is contiguous to $E^{(\omega)}$, the first transfinite derivative of E (if it exists), the integral of $F''(x)$ exists in any interval interior to $(\alpha_\omega, \beta_\omega)$, and consequently as before in $(\alpha_\omega, \beta_\omega)$, and is equal to $F'(\beta_\omega) - F'(\alpha_\omega)$. Proceeding in this manner, the set E will be exhausted, since E is enumerable, before some transfinite derivative is reached. Hence, in the interval $(-\pi, \pi)$, $F''(x)$ exists almost everywhere, and $\int_a^x F''(x)\,dx$ exists as an L-integral, or else as an HL-integral. Thus the function $\Phi(x)$ exists as an L-integral, or else as an HL-integral, and it is the sum-function of the series $\Sigma \dfrac{a_n \sin nx - b_n \cos nx}{n}$. It follows, in accordance with §§ 360, 364, that the given trigonometrical series is a Fourier's series or a Fourier's HL-series, according as $F'(x)$ is an L-integral or an HL-integral.

EXAMPLES

(1) Consider the series $\sum_{n=1}^{\infty}\frac{1}{n^k}\sin nx$, $\sum_{n=1}^{\infty}\frac{1}{n^k}\cos nx$, where $0<k\leqq 1$. Since the series $\sum_{n=1}^{\infty}\sin nx$, $\sum_{n=1}^{\infty}\cos nx$ oscillate boundedly, except in the second case in the neighbourhood of the point $x=0$, by the theorem of § 24, the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^k}$, $\sum_{n=1}^{\infty}\frac{\cos nx}{n^k}$ are convergent boundedly, except that the second does not converge at $x=0$. Since the integrated series $\sum_{n=1}^{\infty}\frac{\cos nx}{n^{k+1}}$, $\sum_{n=1}^{\infty}\frac{\sin nx}{n^{k+1}}$ converge uniformly in $(-\pi, \pi)$, it follows by the above theorem that both the series considered are either Fourier's series or else Fourier's HL-series. If $k>\frac{1}{2}$, it follows from the Riesz-Fischer theorem that both series are Fourier's series.

(2) Consider the series $\Sigma\frac{\sin nx}{\log n}$. The integrated series $\Sigma\frac{\cos nx}{n\log n}$ is divergent at the point $x=0$, and thus the series cannot (see §§ 360, 364) be a Fourier's series or a Fourier's (HL) series. On the other hand the series $\Sigma\frac{\sin nx}{(\log n)^{1+k}}$, where $k>0$, is such that the integrated series $\Sigma\frac{\cos nx}{n(\log n)^{1+k}}$ converges uniformly in the interval $(-\pi, \pi)$. Hence the series is either a Fourier's series or a Fourier's HL-series.

450. In the theorem of § 449 the condition that $\Sigma\frac{a_n\sin nx-b_n\cos nx}{n}$ converges boundedly to a continuous sum-function may be relaxed; simple convergence to a continuous sum-function $\Phi(x)$ being sufficient. For, by Fatou's theorem (§ 413), applied to the summable function $F(x)$, it is seen that $\Phi(x)$ is equal to $F'(x)$ wherever $F'(x)$ exists, which is the case almost everywhere in (a, b); for $\lim_{h\to 1}\sum_{n=1}^{\infty}\left(\frac{a_n\sin nx-b_n\cos nx}{n}\right)h^n=\Phi(x)$, since a_n, b_n are bounded (see § 133). If $F_2(x)$ denote $\int_{-\pi}^{x}F_1(x)\,dx$, it is seen as before that the upper and lower limits, as $h\sim 0$, of

$$\frac{F_2(x+h)+F_2(x-h)-2F_2(x)}{h^2}$$

are integrals in (a, b). Since $F_2(x)$ possesses everywhere a second differential coefficient $F(x)$, these upper and lower limits coincide with $F(x)$, and therefore $F(x)$ is an integral, and consequently

$$F(x)-F(-\pi)=\int_{-\pi}^{x}F'(x)\,dx=\int_{-\pi}^{x}\Phi(x)\,dx.$$

Since $\Phi(x)$ is continuous, it follows that $F'(x)$ exists everywhere, and is equal to $\Phi(x)$. From this point onwards, the procedure is as before. Consequently the following theorem is established:

If a trigonometrical series be such that the upper and lower sum-functions are bounded, except in the neighbourhood of points belonging to a closed enumerable set, and if the integrated series $\sum_{n=1}^{\infty}\frac{a_n\sin nx-b_n\cos nx}{n}$ *converges to a continuous function in the whole interval* $(-\pi, \pi)$, *then the trigonometrical series is either a Fourier's series or a Fourier's HL-series.*

This theorem was given* by W. H. Young, whose proof, however, would appear to require some addition to make it complete.

RESTRICTED FOURIER'S SERIES

451. The properties of a certain kind of trigonometrical series of the form $\frac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$, which are in general not Fourier's series, have been investigated† by W. H. Young; to series of this class he has given the name *restricted Fourier's series*. We shall in the first instance give an account of a specially interesting sub-class of restricted Fourier's series called *ordinary restricted Fourier's series*, which may be characterized as follows:

A trigonometrical series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ *is said to be an ordinary restricted Fourier's series, or ORF-series, if it satisfies the conditions* (1), $a_n = o(1)$, $b_n = o(1)$, *and* (2). *the integrated series*

$$\sum_{n=1} \frac{1}{n} (a_n \sin nx - b_n \cos nx)$$

converges in an open interval (α, β) *contained in* $(-\pi, \pi)$, *or in each of a set of such open intervals, to a function* $F(x)$ *which is an L-integral. The function* $\frac{dF(x)}{dx}$, *defined almost everywhere in* (α, β), *is then said to be the function associated with the ORF-series in the open interval* $\alpha < x < \beta$.

It should be observed that, from condition (1), the convergence of the series $\Sigma \left(\frac{a_n}{n}\right)^2$, $\Sigma \left(\frac{b_n}{n}\right)^2$ follows, and then, employing the Riesz-Fischer theorem (§ 379), it follows that the integrated series is a Fourier's series; thus the function $F(x)$ is the function corresponding to a Fourier's series, although it is an L-integral only within the interval or intervals of restriction. The interest of these *ORF*-series arises from the fact, which will be established, that, in an interval of restriction, they possess many of the cardinal properties of Fourier's series; they may accordingly, within such an interval, be employed in Analysis in like manner as a Fourier's series.

The following theorem, which is the analogue of Riemann's property of Fourier's series, will be established:

The upper and lower functions of an ORF-series at a point x, interior to an interval of restriction, depend only on the character of the associated function in an arbitrarily small neighbourhood of the point x.

* *Proc. Lond. Math. Soc.* ser. (2), vol. IX, p. 430. It seems here to be assumed without sufficient justification that the sum of the integrated series is equal to $F'(x)$. The continuity of the sum of the integrated series is by itself not sufficient to justify this.

† See *Proc. Lond. Math. Soc.* (2), vol. XVII (1918), pp. 195–236; also *ibid.* pp. 353–366; and *Proc. Roy. Soc.* (A), vol. XCIII (1917), pp. 276–292. See also *Bulletin de la soc. mat. de France*, vol. LII (1924), p. 585.

Let $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ be the *ORF*-series, and let $f(x)$ be the function associated with it in the open interval (α, β); where $a_n = o(1)$, $b_n = o(1)$. Let $F(x)$ be the function corresponding to the Fourier's series $\sum \frac{a_n \sin nx - b_n \cos nx}{n}$; then we have $F(x) - F(\alpha) = \int_\alpha^x f(x)\,dx$, for $\alpha < x < \beta$ (see I, § 406). The nth partial sum of the *ORF*-series is given by

$$s_n(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^\pi \{F(x+t) + F(x-t)\} \sin(n+\tfrac{1}{2})t \operatorname{cosec} \tfrac{1}{2}t\,dt,$$

at an interior point of the interval (α, β). If $(x - \epsilon, x + \epsilon)$ be an interval interior to (α, β), the integral in the expression for $s_n(x)$ may be divided into two parts, the first taken from 0 to ϵ, and the second from ϵ to π. The value of the first of these integrals depends only on the properties of $f(x)$ in the interval $(x - \epsilon, x + \epsilon)$. In order to prove the theorem, it is sufficient to shew that

$$\frac{d}{dx}\int_\epsilon^\pi [F(x+t) + F(x-t)] \sin(n+\tfrac{1}{2})t \operatorname{cosec} \tfrac{1}{2}t\,dt$$

converges to zero, as $n \sim \infty$. The expression may be written in the form

$$\frac{d}{dx}\int_\epsilon^\pi [F(x+t) + F(x-t)] \sin nt \cot \tfrac{1}{2}t\,dt + k_n,$$

where $$k_n = k_{-n} = \frac{d}{dx}\int_\epsilon^\pi [F(x+t) + F(x-t)] \cos nt\,dt.$$

We have

$$k_n = \frac{d}{dx}\int_0^\pi [F(x+t) + F(x-t)] \cos nt\,dt - \frac{d}{dx}\int_0^\epsilon [F(x+t) + F(x-t)] \cos nt\,dt;$$

the first expression on the right-hand side is equal to

$$\pi\frac{d}{dx}\left(\frac{a_n \sin nx - b_n \cos nx}{n}\right),$$

or to $$\pi(a_n \cos nx + b_n \sin nx),$$

which converges to zero, as $n \sim \infty$. In the second expression the differentiation can be carried out under the sign of integration, since

$$F(x+t) + F(x-t),\ \cos nt$$

are both L-integrals in the interval $(0, \epsilon)$ (see § 249, Ex. (6)); it is therefore equal to $-\int_0^\epsilon [f(x+t) + f(x-t)] \cos nt\,dt$, which converges to zero, as $n \sim \infty$, since $f(x+t) + f(x-t)$ is summable in the interval $(0, \epsilon)$. It has accordingly been proved that $k_n = o(1)$.

In order to deal with $\frac{d}{dx}\int_\epsilon^\pi [F(x+t) + F(x-t)] \sin nt \cot \frac{1}{2}t\,dt$, let $\phi(t)$ be defined as an odd function of t in the interval $(-\pi, \pi)$, such

that $\phi(0) = 0$, $\phi(t) = \cot \frac{1}{2}t$ in the interval (ϵ, π), and let it be continuous at $t = \epsilon$, and such that $\phi'(t)$, $\phi''(t)$ exist and are bounded in the interval $(0, \epsilon)$. The function $\phi(t)$ is the integral of an integral, and its Fourier's series $\sum\limits_{p=1} c_p \sin pt$ converges uniformly. The differentiated series also converges uniformly to $\phi'(t)$, and $c_p = 0\left(\frac{1}{p^3}\right)$; thus $\sum\limits_{p=1} |c_p|$ converges to a number C. If it were necessary, $\phi(t)$ could be so defined as to have any number of its derivatives bounded.

We now consider

$$\frac{d}{dx}\int_\epsilon^\pi \{F(x+t) + F(x-t)\}\,\phi(t)\sin nt\,dt;$$

by integration by parts we have

$$\begin{aligned}\int_\epsilon^\pi \{F(x+t) + F(x-t)\}\,\phi(t)\sin nt\,dt \\ = -\{G(x+\epsilon) - G(x-\epsilon)\}\,\phi(\epsilon)\sin n\epsilon \\ - n\int_\epsilon^\pi \{G(x+t) - G(x-t)\}\,\phi(t)\cos nt\,dt \\ - \int_\epsilon^\pi \{G(x+t) - G(x-t)\}\,\phi'(t)\sin nt\,dt;\end{aligned}$$

where $G(x) = \int_{-\pi}^{x} F(x)\,dx$. The function $F(x)$ is the differential coefficient of $G(x)$ almost everywhere, and at every point of $(x-\epsilon, x+\epsilon)$; we have thus, remembering that $G(x+t) - G(x-t)$, $\phi(t)\cos nt$, $\phi'(t)\sin nt$ are integrals in (ϵ, π) (see § 249, Ex. (6)), for

$$\frac{d}{dx}\int_\epsilon^\pi \{F(x+t) + F(x-t)\}\,\phi(t)\sin nt\,dt,$$

the expression

$$\begin{aligned}-\{F(x+\epsilon) - F(x-\epsilon)\}\,\phi(\epsilon)\sin n\epsilon \\ - n\int_\epsilon^\pi \{F(x+t) - F(x-t)\}\,\phi(t)\cos nt\,dt \\ - \int_\epsilon^\pi \{F(x+t) - F(x-t)\}\,\phi'(t)\sin nt\,dt.\end{aligned}$$

Since the Fourier's series for $\phi(t)$, $\phi'(t)$ both converge uniformly, we may substitute them in the integrals, and integrate term by term. We can then transform back the coefficients of each factor c_p, where $\sin pt$ takes the place of $\phi(t)$, and we thus obtain the expression

$$\sum_{p=1}^{\infty} c_p \frac{d}{dx}\int_\epsilon^\pi \{F(x+t) + F(x-t)\}\sin pt \sin nt\,dt;$$

hence we have

$$\frac{d}{dx}\int_\epsilon^\pi \{F(x+t) + F(x-t)\}\,\phi(t)\sin nt\,dt = \tfrac{1}{2}\sum_{p=1}^{\infty} c_p\,(k_{p-n} - k_{p+n}).$$

If η be an arbitrarily chosen positive number, since $k_n = o\,(1)$, we have $|\,k_n\,| < \eta$, for $|\,n\,| \geqq m$. It follows that $\left|\sum\limits_{p=1}^{\infty} c_p k_{p+n}\right| < \eta C$, provided $n \geqq m$. To estimate $\sum\limits_{p=1}^{\infty} c_p k_{p-n}$, we have

$$\left|\sum_{p=1}^{\infty} c_p k_{p-n}\right| \leq \left|\sum_{p=n+m}^{\infty} c_p k_{p-n}\right| + \left|\sum_{p=1}^{n-m} c_p k_{p-n}\right|$$
$$+ \left|\sum_{p=n-m+1}^{n} c_p k_{p-n}\right| + \left|\sum_{p=n+1}^{n+m-1} c_p k_{p-n}\right|.$$

The first expression on the right-hand side is less than $\eta \sum\limits_{p=n+m}^{\infty} |\,c_p\,|$, or than $C\eta$; the second is less than $\eta \sum\limits_{p=1}^{p=n-m} |\,c_p\,|$, or than $C\eta$; the third is less than $M \sum\limits_{p=n-m+1}^{n} |\,c_p\,|$, where M is the maximum of the numbers $|\,k_0\,|$, $|\,k_1\,|, \ldots, |\,k_{m-1}\,|$, and this converges to zero, as $n \sim \infty$, when m is kept fixed; the fourth expression is less than $M \sum\limits_{p=n+1}^{n+m-1} |\,c_p\,|$, or than $M \sum\limits_{p=n+1}^{\infty} |\,c_p\,|$, which converges to zero, as $n \sim \infty$. It has now been shewn that

$$\left|\,\overline{\lim_{n\sim\infty}} \sum_{p=1}^{\infty} c_p\,(k_{p-n} - k_{p+n})\right| \leq 2C\eta;$$

and since η is arbitrary, $\lim\limits_{n\sim\infty} \sum\limits_{p=1}^{\infty} c_p\,(k_{p-n} - k_{p+n}) = 0$. It has now been shewn that $\lim\limits_{n\sim\infty} \dfrac{d}{dx}\displaystyle\int_{\epsilon}^{\pi} [F\,(x+t) + F\,(x-t)]\,\phi\,(t) \sin ntdt = 0$, and the theorem has thus been established.

It can be shewn that, if x is confined to a closed interval interior to (α, β), the convergence of

$$\frac{d}{dx}\int_{\epsilon}^{\pi} [F\,(x+t) + F\,(x-t)] \sin\,(n+\tfrac{1}{2})\,t \operatorname{cosec} \tfrac{1}{2}tdt$$

to zero is uniform in the specified interval of x.

Since

$$\frac{d}{dx}\int_{0}^{\pi} [F\,(x+t) + F\,(x-t)] \cos ntdt = \pi\,(a_n \cos nx + b_n' \sin nx),$$

the expression on the left-hand side is numerically less than $\pi\,(|\,a_n\,| + |\,b_n\,|)$, and it therefore converges to zero, as $n \sim \infty$, uniformly with respect to x.

It is also known (see § 334) that $\displaystyle\int_{0}^{\epsilon} [f\,(x+t) + f\,(x-t)] \cos ntdt$ converges to zero uniformly for all values of x in a closed interval interior to (α, β). Hence k_n converges to zero uniformly for all such values of x.

Since $| k_n(x) | \leqq \pi (| a_n | + | b_n |) + \int_{x-\epsilon}^{x+\epsilon} | f(t) | dt$, and ϵ is so chosen that $(x - \epsilon, x + \epsilon)$ is in a closed interval (A, B) interior to (α, β), we see that, if $M = \pi \sum_{n=0}^{n=m} (| a_n | + | b_n |) + m \int_A^B | f(t) | dt$, none of the numbers $k_0(x), k_1(x), \ldots, k_{m-1}(x)$ can numerically exceed M, for any of the values of x. Thus the uniform convergence is established.

452. It appears from the theorem proved in § 251 that, at any point x in the open interval (α, β) of restriction, the *ORF*-series is convergent if the Fourier's series of the function which has the value of $f(x)$ in the interval $(x - \epsilon, x + \epsilon)$, and has elsewhere the value zero, is convergent at x. Moreover, if (α_1, β_1) is any closed interval contained in (α, β), the *ORF*-series converges uniformly in (α_1, β_1) if the Fourier's series of the summable function which has the value $f(x)$ when x is in (α_1, β_1), and which elsewhere has the value of some summable function, is uniformly convergent. The general result may be stated as follows:

Sufficient conditions for the convergence of an ORF-series in the open interval (α, β) of restriction, at a point x in (α, β), and also sufficient conditions for the uniform convergence of the series in a closed interval (α_1, β_1) contained in (α, β), are identical with the corresponding sufficient conditions for the Fourier's series corresponding to the summable function which agrees with $f(x)$ in a neighbourhood of the point x, or in an interval contained in (α, β), and which has elsewhere the value zero, or the value of some summable function.

It is thus seen that, in any closed interval contained in an interval of restriction, an *ORF*-series may be employed like a Fourier's series; for example, it may, subject to the same conditions as in the case of a Fourier's series, be substituted in the integrand of any integral whose limits are within the interval of restriction, the integration being then carried out term by term.

453. The trigonometrical series $\Sigma (a_n \cos nx + b_n \sin nx)$, corresponding to a function $f(x)$ which has a Denjoy integral, or in particular a Harnack-Lebesgue integral, in the interval $(-\pi, \pi)$, is in general not such that the conditions $a_n = o(1)$, $b_n = o(1)$ are satisfied. But in any such case when they are satisfied, the series $\Sigma \frac{a_n \sin nx - b_n \cos nx}{n}$ is a Fourier's series which corresponds to a function $F(x)$ which is an indefinite D-integral, but not an L-integral, and is consequently a continuous function. But in each open interval that is contiguous to the set H, of points of non-summability of $f(x)$, $F(x)$ is an L-integral, to the values of which the Fourier's series converges; thus the Fourier's D-series is an *ORF*-series, to which the results of § 452 are applicable.

In case the conditions $a_n = o(1)$, $b_n = o(1)$ are not satisfied, there may be values of x for which $\lim_{n\sim\infty}(a_n \cos nx + b_n \sin nx) = 0$. If ξ be a point within one of the intervals contiguous to the set H of points of non-summability of the function $f(x)$, and for which $\lim_{n\sim\infty}(a_n \cos n\xi + b_n \sin n\xi) = 0$, the result of § 452 is applicable to the neighbourhood of the point ξ, because in the proof in § 451, no use has been made of the conditions $a_n = o(1)$, $b_n = o(1)$, the conditions having alone been employed that the Fourier's series $\Sigma \dfrac{a_n \sin nx - b_n \cos nx}{n}$ exists, and that

$$\lim_{n\sim\infty}(a_n \cos nx + b_n \sin nx) = 0,$$

at the particular point x. The Fourier's D-series may be convergent at a point x at which this limit has the value zero, but the set of points of convergence cannot have a measure greater than zero, unless $a_n = o(1)$, $b_n = o(1)$, because, as has been shewn in § 430, if $a_n \cos nx + b_n \sin nx$ converges to zero at all points of a set of positive measure, it then follows that $a_n = o(1)$, $b_n = o(1)$.

We have accordingly obtained the following properties of a Fourier's D-series:

A Fourier's D-series, or in particular a Fourier's HL-series,

$$\sum_{n=1}(a_n \cos nx + b_n \sin nx),$$

corresponding to a function $f(x)$, and for which $a_n = o(1)$, $b_n = o(1)$, behaves, in any closed interval interior to an interval contiguous to the set H, of points of non-summability of $f(x)$, in exactly the same manner, as regards convergence, uniform convergence, or oscillation, as the Fourier's series corresponding to the summable function which, in the closed interval, has the same values as $f(x)$, and outside that interval has the value zero. The series may be employed in that closed interval in the same manner, and subject to the same conditions, as the Fourier's series.

In case the conditions $a_n = o(1)$, $b_n = o(1)$ are not satisfied, the points of convergence of the Fourier's D-series, in the intervals contiguous to H, form at most a set of points of measure zero. At any point interior to an interval contiguous to H, at which $a_n \cos nx + b_n \sin nx$ converges to zero, as $n \sim \infty$, the series behaves, as regards convergence or oscillation, in the same manner as the Fourier's series corresponding to the summable function which has in a neighbourhood $(x - \delta, x + \delta)$ of the point x the same values as $f(x)$, and has everywhere else the value zero.

454. The more general class of restricted Fourier's series may be defined as follows:

The series obtained by differentiating p times the Fourier's series corresponding to a summable function $F(x)$ is said to be a restricted Fourier's

series of the pth class, and to be restricted to one or more open intervals (α, β), *if, in each such open interval,* $F(x)$ *is a pth indefinite L-integral. The pth differential coefficient of* $F(x)$, *which exists almost everywhere in* (α, β), *is said to be the function associated with the restricted Fourier's series.*

It is easily seen that an *ORF*-series belongs to the first class of restricted Fourier's series.

The theorem in § 341 that the convergence, or the nature of the upper and lower sum-functions, of a Fourier's series at a particular point depends only on the values of the function to which the series corresponds in an arbitrarily small neighbourhood of the point has been extended to the case of the pth derived series, when summation (C, p) takes the place of ordinary summation. W. H. Young has in fact established* the following theorem:

The upper and lower sums (C, p) *of the pth derived series of a Fourier's series corresponding to* $f(x)$ *at a particular point depend only on the values of* $f(x)$ *in an arbitrarily small neighbourhood of the point.*

It has also been shewn† by W. H. Young that:

The derived series of a Fourier's series corresponding to a function of bounded variation in $(-\pi, \pi)$ *converges* (C, k), *where* $k > 0$, *almost everywhere to the differential coefficient of the function.*

By employing both of the last theorems, the following theorem can be obtained:

If $f(x)$ *be, in a certain interval* (a, b), *of bounded variation, the first derived series of the Fourier's series of* $f(x)$ *converges* $(C, 1)$, *almost everywhere in* (a, b), *to the differential coefficient of the function* $f(x)$.

For the convergence $(C, 1)$ at an interior point of (a, b), of the derived series, does not depend upon the nature of $f(x)$ outside (a, b), and is therefore the same as if $f(x)$ were of bounded variation in $(-\pi, \pi)$. In that case the series converges $(C, 1)$ almost everywhere in (a, b), to the value $f'(x)$.

CONVERGENCE AND SUMMABILITY OF THE SERIES ALLIED WITH A FOURIER'S SERIES

455. The series allied with a Fourier's series

$$\tfrac{1}{2}a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin x), \text{ corresponding to the function } f(x).$$

has been defined in § 400 to be the series $\sum_{n=1} (a_n \sin nx - b_n \cos nx)$. We proceed to consider the question of the convergence of the allied series at a particular point.

* *Proc. Lond. Math. Soc.* (2), vol. XVII (1915), pp. 212–217.

† *Ibid.* (2), vol. XIII (1913), pp. 21–23.

We have

$$\sum_1^n (a_n \sin nx - b_n \cos nx) = \frac{1}{\pi}\sum_1^n \int_{-\pi}^{\pi} f(x') \sin n(x - x')\, dx'$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x') \frac{\sin\frac{n+1}{2}(x-x')\sin\frac{n}{2}(x-x')}{\sin\frac{1}{2}(x-x')}\, dx';$$

and thus the partial sum on the left-hand side can be expressed in the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\left[\cot\frac{x-x'}{2} - \cos n(x-x')\cot\frac{x-x'}{2} + \sin n(x-x')\right] dx'.$$

Since $\int_{-\pi}^{\pi} f(x') \sin n(x-x')\, dx'$ converges to zero, as $n \sim \infty$, uniformly for all values of x, the limits of the partial sum depend upon those of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\cot\frac{x-x'}{2}\{1 - \cos n(x-x')\}\, dx',$$

or, writing $x' = x + t$, upon those of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\tfrac{1}{2}t\,(1 - \cos nt)\, dt,$$

or of
$$\frac{1}{2\pi}\int_0^{\pi}\{f(x-t) - f(x+t)\}\cot\tfrac{1}{2}t\,(1-\cos nt)\, dt.$$

It follows that the allied series converges at a point x to the value

$$\frac{1}{2\pi}\int_0^{\pi}\{f(x-t) - f(x+t)\}\cot\tfrac{1}{2}t\, dt,$$

provided this expression has a definite meaning, and provided further the condition

$$\lim_{n\sim\infty}\int_0^{\pi}\{f(x-t) - f(x+t)\}\cot\tfrac{1}{2}t\cos nt\, dt = 0$$

is satisfied.

Since $\cot\frac{1}{2}t - \frac{2}{t}$ is bounded and measurable in the interval $(0, \pi)$, it follows from the theorem of Riemann-Lebesgue that

$$\lim_{n\sim\infty}\int_0^{\pi}\{f(x-t) - f(x+t)\}\left(\cot\tfrac{1}{2}t - \frac{2}{t}\right)\cos nt\, dt = 0,$$

hence the second condition is equivalent to

$$\lim_{n\sim\infty}\int_0^{\pi}\frac{f(x-t) - f(x+t)}{t}\cos nt\, dt = 0.$$

In case $\frac{f(x-t) - f(x+t)}{t}$ is summable in an interval which contains the point $t = 0$, both the conditions of convergence of the series at the point x are satisfied, since

$$\int_0^{\pi}\{f(x-t) - f(x+t)\}\left(\cot\tfrac{1}{2}t - \frac{2}{t}\right) dt$$

exists as an L-integral.

In general however $\dfrac{f(x-t)-f(x+t)}{t}$ is not summable in the neighbourhood of the point $t=0$, but

$$\frac{1}{2\pi}\int_0^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt$$

may exist as a non-absolutely convergent integral, denoting

$$\frac{1}{2\pi}\lim_{\epsilon\sim 0}\int_\epsilon^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt.$$

The result has now been established that:

The allied series converges at a point x to the value

$$\frac{1}{2\pi}\int_0^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt,$$

provided this expression has a definite meaning, and provided further that

$$\lim_{n\sim\infty}\int_0^\pi \frac{f(x-t)-f(x+t)}{t}\cos nt\,dt=0,$$

the integrals in these two conditions being in general non-absolutely convergent at $t=0$.

This result was obtained* by Pringsheim, who was the first to investigate the convergence of the series in a rigorous manner.

It is easily seen that, $f(x)$ being a periodic function,

$$\frac{1}{2\pi}\int_0^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt$$

is equivalent to $$\frac{1}{\pi}\int_0^\infty \frac{f(x-t)-f(x+t)}{t}\,dt.$$

456. The case in which $f(x)$ is a function of bounded variation in the interval $(-\pi,\pi)$ was investigated† by W. H. Young. For the convergence at a point x, of continuity of the function, we can however consider the more general case in which the function is only of bounded variation in some neighbourhood $(x-\delta, x+\delta)$, of the point x.

We have then to consider the expression

$$\frac{1}{2\pi}\int_0^\delta [f(x-t)-f(x+t)]\cot\tfrac{1}{2}t\,(1-\cos nt)\,dt,$$

since, when the limits of the integral are (δ,π), the expression converges, as $n\sim\infty$, to the definite value

$$\frac{1}{2\pi}\int_\delta^\pi [f(x-t)-f(x+t)]\cot\tfrac{1}{2}t\,dt.$$

* *Sitzungsber. d. Münch. Akad.* (1900), p. 87.

† *Ibid.* (1911), p. 361.

We have, when n is so large that $\delta > \pi/n$,

$$\left|\int_0^{\frac{\pi}{n}} \psi(t)\cot\tfrac{1}{2}t\,(1-\cos nt)\,dt\right| < \int_0^{\frac{\pi}{n}} |\psi(t)|\,\frac{2}{t}\,|(1-\cos nt)|\,dt$$

$$< 4\int_0^{\frac{\pi}{n}} |\psi(t)|\,\frac{\sin\frac{1}{2}nt}{t}\,dt$$

$$< 4\epsilon_n\int_0^{\pi}\frac{\sin\frac{1}{2}\theta}{\theta}\,d\theta,$$

where $\psi(t)$ denotes $f(x-t)-f(x+t)$, and ϵ_n is the maximum of $|\psi(t)|$ in the interval $(0, \pi/n)$. If n be sufficiently large, we have

$$\left|\int_0^{\frac{\pi}{n}} \psi(t)\cot\tfrac{1}{2}t\,(1-\cos nt)\,dt\right| < \eta,$$

where η is an arbitrarily chosen positive number.

The function $\psi(t)$ may be expressed as $P(t)-Q(t)$, where $P(t)$, $Q(t)$ are monotone and non-increasing in the interval $(0, \delta)$; we have then

$$\int_{\frac{\pi}{n}}^{\delta} P(t)\cot\frac{t}{2}\cos nt\,dt = \cot\frac{\pi}{2n}\,P\left(\frac{\pi}{n}\right)\int_{\frac{\pi}{n}}^{\delta'}\cos nt\,dt,$$

where δ' is in the interval $\left(\frac{\pi}{n}, \delta\right)$; hence the integral on the left-hand side is numerically less than $\frac{2}{n}\cot\frac{\pi}{2n}\,P\left(\frac{\pi}{n}\right)$; a similar result holds for the function Q. Since $P\left(\frac{\pi}{n}\right)$, $Q\left(\frac{\pi}{n}\right)$ both converge to zero, as $n\sim\infty$, we see that

$$\left|\frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta}\{f(x-t)-f(x+t)\}\cot\frac{t}{2}\cos nt\,dt\right| < \eta,$$

if n be sufficiently large. It now follows that

$$\frac{1}{2\pi}\int_0^{\delta}\{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,(1-\cos nt)\,dt$$

differs from
$$\frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta}\{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt$$

by less than 2η, if n be sufficiently large. If now

$$\int_0^{\delta}\{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt$$

exists, we have

$$\lim_{n\sim\infty}\frac{1}{2\pi}\int_0^{\delta}\{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,(1-\cos nt)\,dt$$
$$=\frac{1}{2\pi}\int_0^{\delta}\{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt.$$

The following theorem has now been established:

If, at a point of continuity x, of the function $f(x)$, the function be of bounded variation in some neighbourhood of x, the allied series converges at x to the value

$$\frac{1}{2\pi}\lim_{\epsilon\sim 0}\int_\epsilon^\pi \{f(x-\delta)-f(x+\delta)\}\cot\tfrac{1}{2}t\,dt,$$

provided this limit has a definite value.

The following analogue of de la Vallée Poussin's condition (§ 346) for the convergence of Fourier's series has been obtained* by W. H. Young:

If $\frac{1}{u}\int_0^u \{f(x-t)-f(x+t)\}\,dt$ is of bounded variation, as a function of u, the allied series converges to

$$\lim_{n\sim\infty}\frac{1}{\pi}\int_{\frac{\pi}{n}}^\infty \frac{f(x-t)-f(x+t)}{t}\,dt,$$

provided this limit has a definite value.

457. As regards the summability of the allied series, the first step was taken by W. H. Young, who proved that the series is summable $(C, 1)$ if the integral

$$\frac{1}{2\pi}\int_0^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt,$$

which will be denoted by I, exists, at least as a non-absolutely convergent integral, provided also

$$\int_0^u |f(x-t)-f(x+t)|\,dt = o(u).$$

The latter condition is certainly satisfied at a point x at which $|f(x)-C|$ is the differential coefficient of its indefinite integral, whatever value C may have. The set of points at which this condition is satisfied contains almost every point of the interval $(-\pi, \pi)$ (see I, § 432), and may be termed the L-set. In order to shew that the allied series is summable $(C, 1)$ almost everywhere, it is accordingly necessary to shew that the integral

$$\int_0^\pi \{f(x-t)-f(x+t)\}\cot\tfrac{1}{2}t\,dt$$

exists almost everywhere.

It was proved† by Plessner that, if

$$V(h,x) = \sum_{n=1}^\infty (a_n\sin nx - b_n\cos nx)\,h^n,$$

* *Loc. cit.* p. 368. See also *Proc. Lond. Math. Soc.* (2), vol. x (1911), pp. 266, 271, where various theorems relating to the allied series are given.

† *Zur Theorie der konjugierten trigonometrischen Reihen*, Mittheilungen des Math. Seminars der Univ. Giessen, No. x, 1923

where $|h| < 1$, then

$$\lim_{h \sim 1}\left[V(h, x) - \frac{1}{2\pi}\int_\alpha^\pi \{f(x-t) - f(x+t)\}\cot\frac{t}{2}\,dt\right] = 0,$$

where $\alpha = \sin^{-1}(1 - h)$, provided

$$\int_0^u |f(x-t) - f(x+t)|\,dt = o(u);$$

it follows that in the L-set, the integral I exists at a point at which the Poisson sum exists. The particular case of this theorem in which x is a point of continuity of $f(x)$ had been obtained earlier* by Fatou. It was further proved by Plessner that the Poisson sum exists almost everywhere, and it then follows that I exists almost everywhere. Taken in conjunction with the theorem of Young, referred to above, it follows that the allied series converges $(C, 1)$ almost everywhere.

The summability (C, k), for $k > 0$, has been considered in a memoir† of Hardy and Littlewood, which contains an account of the development of the theory of the summability of the series. It is there shewn that, in the L-set, the allied series is either summable (C, k) for every value of k (> 0), or not summable (C, k) for any value of k, nor by Poisson's method. This result combined with that of Plessner leads to the extension of the property of Fourier's series to the allied series;

The allied series is summable (C, k), for $k > 0$, almost everywhere.

An extension is also given in this memoir of the theorem stated in § 374, relating to the conditions that, at a particular point, the series should be summable (C, k) for some value of k.

458. In case $f(x)$ is a function whose square is summable in $(-\pi, \pi)$, it follows from the Riesz-Fischer theorem that the allied series is the Fourier's series of a function of which the square is summable.

The following theorem was given‡ by Lusin:

If $\{f(x)\}^2$ is summable in $(0, 1)$, the integral $\int_0^1 \frac{f(x-t) - f(x+t)}{t}\,dt$ has a definite value almost everywhere, and represents a function $\phi(x)$ such that $\{\phi(x)\}^2$ is summable in the interval.

A proof of this theorem has been given§ by Besikovitch, who shewed that

$$\int_0^1 \{\phi(x)\}^2\,dx \leq 2\pi^2 \int_0^1 \{f(x)\}^2\,dx.$$

* *Acta Math.* vol. XXX (1906), p. 360.

† *Proc. Lond. Math. Soc.* (2), vol. XXIV (1925), p. 211.

‡ *Comptes Rendus*, vol. CLVI (1913), p. 1655.

§ *Fundamenta Math.* vol. IV (1923), p. 172.

The more general case of a function $f(x)$ such that $|f(x)|^p$ is summable in $(-\pi, \pi)$, for some value of p (> 1) has been studied by* M. Riesz (see § 397) who has given an indication of his proof that:

If $|f(x)|^p$ *be summable in* $(-\pi, \pi)$, *where* p *has some value* > 1, *then the series allied to the Fourier's series of* $f(x)$ *is the Fourier's series of a function* $\phi(x)$ *such that* $|\phi(x)|^p$ *is summable in* $(-\pi, \pi)$.

Detailed proofs of this theorem, with a development of its consequences, will appear in forthcoming memoirs by M. Riesz, and by Titchmarsh.

DOUBLE FOURIER'S SERIES

459. The theory of double Fourier's series has been investigated by Ascoli†, Picard‡, Cerni§, Krause‖, Hardy¶, Vergerio**, W. H. Young††, C. N. Moore‡‡, Küstermann§§, and Titchmarsh‖‖. A detailed account of many of these investigations, with some further developments, has been given by Geiringer¶¶. Those respects in which multiple Fourier's series differ from single Fourier's series are sufficiently represented by the case of the double series; for simplicity of statement, only the case of Fourier's double series will accordingly be dealt with here.

If $f(x^{(1)}, x^{(2)})$ be a function of $(x^{(1)}, x^{(2)})$, periodic with respect to $x^{(1)}$ and with respect to $x^{(2)}$, in each case with period 2π, and summable in the rectangle $(-\pi, -\pi; \pi, \pi)$; let us consider the series

$$\sum_{n^{(1)}=0,\, n^{(2)}=0}^{\infty,\,\infty} (a_{n^{(1)}, n^{(2)}} \cos n^{(1)}x^{(1)} \cos n^{(2)}x^{(2)} + b_{n^{(1)}, n^{(2)}} \cos n^{(1)}x^{(1)} \sin n^{(2)}x^{(2)}$$
$$+ c_{n^{(1)}, n^{(2)}} \sin n^{(1)}x^{(1)} \cos n^{(2)}x^{(2)} + d_{n^{(1)}, n^{(2)}} \sin n^{(1)}x^{(1)} \sin n^{(2)}x^{(2)}),$$

where $\quad a_{n^{(1)}, n^{(2)}} = \dfrac{1}{\pi^2} \displaystyle\int_{(\Delta)} f(x^{(1)}, x^{(2)}) \cos n^{(1)}x^{(1)} \cos n^{(2)}x^{(2)}\, d(x^{(1)}, x^{(2)})$

for $\quad n^{(1)} > 0, \quad n^{(2)} > 0,$

and

$$a_{n^{(1)}, 0} = \frac{1}{2\pi^2} \int_{(\Delta)} f(x^{(1)}, x^{(2)}) \cos n^{(1)}x^{(1)}\, d(x^{(1)}, x^{(2)}),$$

$$a_{0, n^{(2)}} = \frac{1}{2\pi^2} \int_{(\Delta)} f(x^{(1)}, x^{(2)}) \cos n^{(2)}x^{(2)}\, d(x^{(1)}, x^{(2)}),$$

$$a_{0,0} = \frac{1}{4\pi^2} \int_{(\Delta)} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)}),$$

* *Proc. Lond. Math. Soc.* (2), vol. XXII, *Records*, Jan. 17, 1924, p. iv, also *Comptes Rendus*, vol. CLXXVIII (1924), p. 1464.

† *Rendiconti Lomb.* (2), vol. XX, p. 543.

‡ *Traité d'Analyse*, 2nd ed. (1901), p. 294.

§ *Rend. Lombard.* vol. XXXIV (1901), p. 921.

‖ *Leipz. Ber.* (1903), pp. 164, 239.

¶ *Quarterly Journ.* vol. XXXVII (1906), p. 53.

** *Gior. di Batalgini*, vol. XLIX (1911), p. 181.

†† *Proc. Lond. Math. Soc.* (2), vol. XI (1912), p. 133.

‡‡ *Trans. Amer. Math. Soc.* vol. XIV (1913), p. 73; *Bull. Amer. Math. Soc.* vols. XVII, XVIII, *Comptes Rendus*, vol. CLV (1912), p. 126; *Math. Annalen*, vol. LXXIV (1913), p. 555.

§§ "Inaugural dissertation," *Ueber Fourier'sche Doppelreichen und das Poisson'sche Doppel integral*, Munich, 1913.

‖‖ *Proc. Roy. Soc.* vol. CVI (1924), p. 299.

¶¶ *Monatshefte f. Math. u. Physik*, vol. XXIX (1918), p. 65.

with similar expressions for $b_{n^{(1)}, n^{(2)}}$, $c_{n^{(1)}, n^{(2)}}$, $d_{n^{(1)}, n^{(2)}}$; where Δ denotes the fundamental rectangle $(-\pi, -\pi; \pi, \pi)$. This series is said to be the double Fourier's series corresponding to the summable function $f(x^{(1)}, x^{(2)})$. It is clear (see § 237) that each of the integrals which express the coefficients may be replaced by the corresponding repeated integral. We may denote the correspondence of the series with the function by

$$f(x^{(1)}, x^{(2)}) \sim \sum_{n^{(1)}=0,\ n^{(2)}=0}^{\infty,\ \infty} (a_{n^{(1)}, n^{(2)}} \cos n^{(1)} x^{(1)} \cos n^{(2)} x^{(2)} + b_{n^{(1)}, n^{(2)}} \cos n^{(1)} x^{(1)} \sin n^{(2)} x^{(2)} + c_{n^{(1)}, n^{(2)}} \sin n^{(1)} x^{(1)} \cos n^{(2)} x^{(2)} + d_{n^{(1)}, n^{(2)}} \sin n^{(1)} x^{(1)} \sin n^{(2)} x^{(2)}).$$

It is clear that the formal expression of the series may be obtained by expressing $f(x^{(1)}, x^{(2)})$ as a single Fourier's series of cosines and sines of multiples of $x^{(2)}$, and then each coefficient in that series as a Fourier's series of cosines and sines of multiples of $x^{(1)}$.

If we denote by $K_{n^{(1)}, n^{(2)}}$ the sum of the four terms corresponding to $n^{(1)}, n^{(2)}$, and by $s_{n^{(1)}, n^{(2)}}$ the sum $\sum_{0,0}^{n^{(1)}, n^{(2)}} K_{n^{(1)}, n^{(2)}}$, we have

$$s_{n^{(1)}, n^{(2)}} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} d\xi^{(1)} [\tfrac{1}{2} + \cos(\xi^{(1)} - x^{(1)}) + \dots + \cos n^{(1)} (\xi^{(1)} - x^{(1)})]$$
$$\int_{-\pi}^{\pi} f(\xi^{(1)}, \xi^{(2)}) [\tfrac{1}{2} + \cos(\xi^{(2)} - x^{(2)}) + \dots + \cos n^{(2)} (\xi^{(2)} - x^{(2)})]\, d\xi^{(2)}$$
$$= \frac{1}{4\pi^2} \int_{(-\pi, -\pi)}^{(\pi, \pi)} f(\xi^{(1)}, \xi^{(2)}) \frac{\sin (2n^{(1)} + 1) \dfrac{\xi^{(1)} - x^{(1)}}{2}}{\sin \dfrac{\xi^{(1)} - x^{(1)}}{2}} \frac{\sin (2n^{(2)} + 1) \dfrac{\xi^{(2)} - x^{(2)}}{2}}{\sin \dfrac{\xi^{(2)} - x^{(2)}}{2}}\, d(\xi^{(1)}, \xi^{(2)})$$
$$= \frac{1}{\pi^2} \int_{-\frac{1}{2}(\pi + x^{(1)})}^{\frac{1}{2}(\pi - x^{(1)})} \int_{-\frac{1}{2}(\pi + x^{(2)})}^{\frac{1}{2}(\pi - x^{(2)})} f(x^{(1)} + 2t^{(1)}, x^{(2)} + 2t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{\sin t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{\sin t^{(2)}}\, dt^{(1)}\, dt^{(2)}$$
$$= \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{\sin t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{\sin t^{(2)}}\, dt^{(1)}\, dt^{(2)};$$

where $m^{(1)} = 2n^{(1)} + 1$, $m^{(2)} = 2n^{(2)} + 1$, and $F(t^{(1)}, t^{(2)})$ denotes

$$f(x^{(1)} + 2t^{(1)}, x^{(2)} + 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} + 2t^{(2)}) + f(x^{(1)} + 2t^{(1)}, x^{(2)} - 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} - 2t^{(2)}),$$

the function $f(x^{(1)}, x^{(2)})$ being taken to be periodic with respect to $x^{(1)}$ and to $x^{(2)}$.

The investigation of the properties of the double Fourier's series, as regards convergence, divergence, or oscillation, at a point $(x^{(1)}, x^{(2)})$ depends upon a discussion of the nature of the double limit

$$\lim_{m^{(1)} \sim \infty,\ m^{(2)} \sim \infty} \frac{1}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)} \sin m^{(2)} t^{(2)}}{\sin t^{(1)} \sin t^{(2)}}\, dt^{(1)}\, dt^{(2)}.$$

It can easily be shewn that, for this double limit,

$$\lim_{m^{(1)}\sim\infty,\, m^{(2)}\sim\infty} \frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)}t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)}t^{(2)}}{t^{(2)}}\, dt^{(1)}\, dt^{(2)}$$

may be substituted. For $\dfrac{1}{t^{(1)}t^{(2)}} - \dfrac{1}{\sin t^{(1)} \sin t^{(2)}}$ is bounded and summable over the cell $(0, 0; \frac{1}{2}\pi, \frac{1}{2}\pi)$, and when multiplied by the summable function $F(t^{(1)}, t^{(2)})$ the product is summable. Since $\sin m^{(1)}t^{(1)} \sin m^{(2)}t^{(2)}$ is bounded, and since the integral of it taken over any cell contained in $(0, 0; \frac{1}{2}\pi, \frac{1}{2}\pi)$ converges to zero, as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$, it follows from the general convergence theorem of § 279 that

$$\lim_{m^{(1)}\sim\infty,\, m^{(2)}\sim\infty} \frac{1}{\pi^2}\int_{(0,\,0)}^{(\frac{1}{2}\pi\ \frac{1}{2}\pi)} F(t^{(1)}, t^{(2)}) \sin m^{(1)}t^{(1)} \sin m^{(2)}t^{(2)} \left(\frac{1}{t^{(1)}t^{(2)}} - \frac{1}{\sin t^{(1)} \sin t^{(2)}}\right) d(t^{(1)}, t^{(2)}) = 0;$$

from which the result follows.

460. If the general convergence Theorem I, of § 279, be applied to any of the integrals $\int_{(\Delta)} f(x^{(1)}x^{(2)}) \frac{\cos}{\sin} n^{(1)}x^{(1)} \frac{\cos}{\sin} n^{(2)}x^{(2)}\, d(x^{(1)}, x^{(2)})$ the double limit as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$, will have the value zero, provided the conditions (1) and (2) of the theorem are satisfied by

$$\frac{\cos}{\sin} n^{(1)}x^{(1)} \frac{\cos}{\sin} n^{(2)}x^{(2)} = \Phi(x^{(1)}, x^{(2)}).$$

Since $|\Phi| \leq 1$, the condition (1) is satisfied; also the integral of Φ over any rectangle contained in Δ is numerically less than $\dfrac{4}{n^{(1)}n^{(2)}}$, which converges to zero, as $n^{(1)}$, $n^{(2)}$ become indefinitely great; thus the condition (2) is satisfied.

It follows that *the double limits of the four Fourier's coefficients* a_{mn}, b_{mn}, c_{mn}, d_{mn} *as* $m \sim \infty$, $n \sim \infty$ *are all zero.*

If we denote $\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)}) \frac{\cos}{\sin} n^{(2)}x^{(2)} dx^{(2)}$ by $\phi(x^{(1)}, n^{(2)})$, this function is equivalent, for each value of $n^{(2)}$, to a function which is summable in the interval $(-\pi, \pi)$ with respect to $x^{(1)}$, and $\int_{-\pi}^{\pi} \phi(x^{(1)}, n^{(2)}) \frac{\sin}{\cos} n^{(1)}x^{(1)} dx^{(1)}$ converges to zero as $n^{(1)} \sim \infty$, the number $n^{(2)}$ remaining fixed. Consequently it has been shewn that:

The coefficients $a_{n^{(1)}, n^{(2)}}$, $b_{n^{(1)}, n^{(2)}}$, $c_{n^{(1)}, n^{(2)}}$, $d_{n^{(1)}, n^{(2)}}$, *in a double Fourier's series, converge to zero, as one of the numbers* $n^{(1)}$, $n^{(2)}$ *diverges to* ∞, *the other number remaining fixed.*

461. It has been shewn that, at any point $(x^{(1)}, x^{(2)})$, which we may take to be interior to the cell $(-\pi, -\pi; \pi, \pi)$, the behaviour of the double

Fourier's series, as regards convergence, divergence, or oscillation, depends upon the limit as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$, of the integral of

$$f(\xi^{(1)}, \xi^{(2)}) \frac{\sin (2n^{(1)}+1)\frac{\xi^{(1)}-x^{(1)}}{2} \sin (2n^{(2)}+1)\frac{\xi^{(2)}-x^{(2)}}{2}}{\sin \frac{\xi^{(1)}-x^{(1)}}{2} \sin \frac{\xi^{(2)}-x^{(2)}}{2}}$$

over the cell $(-\pi, -\pi; \pi, \pi)$.

Let us now consider the function $\Phi(\xi^{(1)} - x^{(1)}, \xi^{(2)} - x^{(2)}; n^{(1)}, n^{(2)})$ which has the value

$$\frac{\sin (2n^{(1)}+1)\frac{\xi^{(1)}-x^{(1)}}{2} \sin (2n^{(2)}+1)\frac{\xi^{(2)}-x^{(2)}}{2}}{\sin \frac{\xi^{(1)}-x^{(1)}}{2} \sin \frac{\xi^{(2)}-x^{(2)}}{2}}$$

when both $|\xi^{(1)} - x^{(1)}| \geqq \mu$, $|\xi^{(2)} - x^{(2)}| \geqq \mu$, where μ is a positive number such that $x^{(1)}$ and $x^{(2)}$ are both in the interval $(-\pi + \mu, \pi - \mu)$. Let the function ϕ have the value zero, when either of these conditions is not satisfied, or when neither of them is satisfied. We have $|\Phi| < \operatorname{cosec}^2 \frac{1}{2}\mu$, and thus the condition (1) of Theorem I, in § 279, is satisfied by Φ. Again the integral of Φ over any cell contained in $(-\pi, -\pi; \pi, \pi)$ is the integral over a cell, for no point of which $|\xi^{(1)} - x^{(1)}| < \mu$, or $|\xi^{(2)} - x^{(2)}| < \mu$; or it is the sum of the integrals over at most four cells, all of which satisfy this condition. If we consider the integral of Φ over one such cell, its value is

$$\int_\alpha^\beta \sin (2n^{(1)}+1)\frac{\theta^{(1)}}{2} \operatorname{cosec} \frac{\theta^{(1)}}{2}\, d\theta^{(1)} \,.\int_{\alpha'}^{\beta'} \sin (2n^{(2)}+1)\frac{\theta^{(2)}}{2} \operatorname{cosec} \frac{\theta^{(2)}}{2}\, d\theta^{(2)},$$

where $\operatorname{cosec} \frac{\theta^{(1)}}{2}$ is numerically $\geqq \operatorname{cosec} \frac{\mu}{2}$, and is monotone; the same condition holding for $\operatorname{cosec} \frac{\theta^{(2)}}{2}$. Applying to these integrals the second mean value theorem, we see that they are less than fixed multiples of $(2n^{(1)}+1)^{-1}$, $(2n^{(2)}+1)^{-1}$ respectively. Therefore the integral converges to zero, as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$; and accordingly the condition (2) of Theorem I, of § 279, is satisfied.

It follows that the behaviour of $s_{n^{(1)}, n^{(2)}}$ at the point $(x^{(1)}, x^{(2)})$ depends only on that of the integral

$$\int f(\xi^{(1)}, \xi^{(2)}) \sin (2n^{(1)}+1)\frac{\xi^{(1)}-x^{(1)}}{2} \operatorname{cosec} \frac{\xi^{(1)}-x^{(1)}}{2} \sin (2n^{(2)}+1)\frac{\xi^{(2)}-x^{(2)}}{2} \operatorname{cosec} \frac{\xi^{(2)}-x^{(2)}}{2}\, d(\xi^{(1)}, \xi^{(2)})$$

taken over the cross-neighbourhood (see § 291) of the point $(x^{(1)}, x^{(2)})$, defined as the set of points for which either

$$|\xi^{(1)} - x^{(1)}| \leqq \mu, \text{ or } |\xi^{(2)} - x^{(2)}| \leqq \mu,$$

or where both these conditions are satisfied; the point $(x^{(1)}, x^{(2)})$ being at a distance $> \mu$ from the boundary of the cell. Moreover, the convergence to zero of

$$\int_{(-\pi,-\pi)}^{(\pi,\pi)} f(\xi^{(1)}, \xi^{(2)})\, \Phi(\xi^{(1)} - x^{(1)}, \xi^{(2)} - x^{(2)}, n^{(1)}, n^{(2)})\, d(\xi^{(1)}, \xi^{(2)})$$

as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$, is uniform for all points $(\xi^{(1)}, \xi^{(2)})$ in a closed set that is interior to Δ, when μ is taken sufficiently small.

A main point in which double, or multiple, Fourier's series differ from single Fourier's series depends upon the fact that the behaviour of the former, as regards convergence, divergence, or oscillation, at a point, does not, as in the latter case, depend only upon the nature of the function in a neighbourhood of the point, but upon its nature in a cross-neighbourhood of the point.

FUNCTIONS OF BOUNDED VARIATION

462. Nearly all the writers on the subject of double Fourier's series have considered the convergence of the double series corresponding to functions which satisfy the condition of being of bounded variation in accordance with the definition of functions of bounded variation given by Hardy and Krause (see I, § 254). The more general definition given by Arzelà (I, § 253) will be employed here in an extended form. As a preliminary, some remarks as to the scope of this definition are requisite. In the cell $(a^{(1)}, a^{(2)}; b^{(1)}, b^{(2)})$, the definition of a function of bounded variation, given by Arzelà, depends upon the consideration of the family of monotone curves joining the two corners $(a^{(1)}, a^{(2)})$ and $(b^{(1)}, b^{(2)})$ of the cell. It was shewn in I, § 253, that the necessary and sufficient condition that a function should be of bounded variation in the cell is that the function should be expressible as the difference of two bounded monotone functions, these two functions being either both non-diminishing with respect to both $x^{(1)}$ and $x^{(2)}$, or else non-increasing with respect to both those variables.

Arzelà's definition may, however be extended to apply to the case in which the monotone curves employed in it are curves joining the other pair of opposite corners of the cell, viz. $(b^{(1)}, a^{(2)})$ and $(a^{(1)}, b^{(2)})$. It thus appears that there exists a second species of functions of bounded variation, such that a function of this species is expressible as the difference of two functions, each of which is monotone non-diminishing with respect to $x^{(1)}$, and monotone non-increasing with respect to $x^{(2)}$; or else in each case the reverse.

Both species of functions will be regarded as included in the definition of functions of bounded variation. It is clear that if $f(x^{(1)}, x^{(2)})$ is of bounded variation, of the first species, and if $(x^{(1)}, x^{(2)\prime})$ be the optical image of the point $(x^{(1)}, x^{(2)})$ in the straight line through the centre of the cell

parallel to the $x^{(1)}$-axis, and if $\bar{f}(x^{(1)}, x^{(2)})$ has the value, at $(x^{(1)}, x^{(2)})$, of the function f at the point $x^{(1)}, x^{(2)\prime}$, that is $f(x^{(1)}, x^{(2)\prime})$, then the function $\bar{f}(x^{(1)}, x^{(2)})$ is of bounded variation, of the second species.

A function which is expressible as the difference of two monotone functions, which are both monotone in the same sense, is said to be *monotonoid*. When the function is the difference of two functions, each of which is monotone non-increasing with respect to one variable $x^{(1)}$, and monotone non-diminishing with respect to the other $x^{(2)}$, then the function may be said to be *quasi-monotonoid*.

Thus a function of bounded variation is either monotonoid or else quasi-monotonoid.

It was shewn in I, § 307, that, for a monotone function $\phi(x^{(1)}, x^{(2)})$, defined in the cell $(a^{(1)}, a^{(2)}; b^{(1)}, b^{(2)})$, the functional limit $\phi(x^{(1)}+0, x^{(2)}+0)$ which represents the double limit of $\phi(x^{(1)}+h^{(1)}, x^{(2)}+h^{(2)})$, as $h^{(1)}$ and $h^{(2)}$ converge to zero from positive values, has a definite value. Also the double limit $\phi(x^{(1)}-0, x^{(2)}-0)$ has a definite value. But the double limits $\phi(x^{(1)}+0, x^{(2)}-0)$, $\phi(x^{(1)}-0, x^{(2)}+0)$ do not necessarily exist as definite numbers (see the correction to I, § 307, at the end of the present volume). But the limit

$$\lim_{\zeta\sim 0}\ \lim_{h^{(1)}\sim 0,\ h^{(2)}\sim 0} \phi(x^{(1)}+h^{(1)}, x^{(2)}-\zeta+h^{(2)}), \text{ or } \lim_{\zeta\sim 0}\phi(x^{(1)}+0, x^{(2)}-\zeta+0),$$

where $\zeta > 0$, necessarily exists. For $\phi(x^{(1)}+0, x^{(2)}-\zeta+0)$ exists, and is clearly monotone with respect to ζ. Similarly $\lim\limits_{\zeta\sim 0}\phi(x^{(1)}+\zeta-0, x^{(2)}-0)$ exists. Whenever $\phi(x^{(1)}+0, x^{(2)}-0)$ exists, we have

$$\begin{aligned}\phi(x^{(1)}+0, x^{(2)}-0) &= \lim_{\zeta\sim 0}\phi(x^{(1)}+0, x^{(2)}-\zeta+0)\\ &= \lim_{\zeta\sim 0}\phi(x^{(1)}+\zeta-0, x^{(2)}-0).\end{aligned}$$

For, when $\phi(x^{(1)}+0, x^{(2)}-0)$ exists, we have

$$|\phi(x^{(1)}+h^{(1)}, x^{(2)}-h^{(2)})-\phi(x^{(1)}+0, x^{(2)}-0)| < \epsilon,$$

provided $h^{(1)}$, $h^{(2)}$ are both less than some number η; if ζ be sufficiently small, it is then clear that $\phi(x^{(1)}+0, x^{(2)}-\zeta+0)$ differs from

$$\phi(x^{(1)}+0, x^{(2)}-0)$$

by not more than ϵ, hence, since ϵ is arbitrary, we have

$$\phi(x^{(1)}+0, x^{(2)}-0) = \lim_{\zeta\sim 0}\phi(x^{(1)}+0, x^{(2)}-\zeta+0).$$

Similarly, $\lim\limits_{\zeta\sim 0}\phi(x^{(1)}-\zeta+0, x^{(2)}+0)$ and $\lim\limits_{\zeta\sim 0}\phi(x^{(1)}-0, x^{(2)}+\zeta-0)$ both exist, and in case $\phi(x^{(1)}-0, x^{(2)}+0)$ exists, all the three have the same value.

For example, let $\phi(x^{(1)}, x^{(2)}) = (x^{(1)}+1)(x^{(2)}+1)$, for $x^{(2)} < -x^{(1)}$, and

$$\phi(x^{(1)}, x^{(2)}) = (x^{(1)}+2)(x^{(2)}+2)$$

for $x^{(2)} \geqq -x^{(1)}$. In the cell $(-1, -1; 1, 1)$, $\phi(x^{(1)}, x^{(2)})$ is monotone, since the four factors are all positive. At the point $(0, 0)$,

$$\lim_{h^{(1)}\sim 0,\, h^{(2)}\sim 0} \phi(h^{(1)}, -h^{(2)}) \text{ and } \lim_{h^{(1)}\sim 0,\, h^{(2)}\sim 0} \phi(-h^{(1)}, h^{(2)})$$

do not exist. But $\lim\limits_{\zeta\sim 0} \phi(-\zeta + 0, +0)$ and the other similar limits exist.

It is clear that these remarks apply not only to a monotone function but also to one which is monotonoid, since the latter is the difference of two monotone functions.

It has been shewn* by R. C. Young that, in the case of a quasi-monotone function of any one of the four classes specified in I, § 255, all four double limits $\phi(x^{(1)} \pm 0, x^{(2)} \pm 0)$ exist as definite numbers (see the correction referred to above).

463. As regards a function which is monotone with respect to the variables in opposite senses, the following theorem may be established†:

It is a sufficient condition that a function $\phi(x^{(1)}, x^{(2)})$ *which is monotone with respect to* $x^{(1)}$ *and to* $x^{(2)}$, *but in opposite senses, should be monotonoid is that either* (1), *one of the four partial derivatives of* $\phi(x^{(1)}, x^{(2)})$ *with respect to* $x^{(1)}$ *should be bounded in the cell, or* (2), *that one of the four partial derivatives of* $\phi(x^{(1)}, x^{(2)})$ *with respect to* $x^{(2)}$ *should be bounded in the cell.*

We need only consider the case in which $\phi(x^{(1)}, x^{(2)})$ is monotone non-diminishing with respect to $x^{(1)}$, and monotone non-increasing with respect to $x^{(2)}$. Let it be assumed that one of the four partial derivatives, say $D^{+}_{x^{(1)}} \phi(x^{(1)}, x^{(2)})$, which is necessarily $\geqq 0$, is bounded in the cell Δ, and let A be its upper boundary.

In a straight line parallel to the $x^{(1)}$-axis, all four derivatives of ϕ with respect to $x^{(1)}$ have one and the same upper boundary; therefore A is the upper boundary in Δ, of any one of these four derivatives.

The incrementary ratio $\dfrac{\phi(x^{(1)} + h^{(1)}, x^{(2)}) - \phi(x^{(1)}, x^{(2)})}{h^{(1)}}$ has A for its upper boundary, when all pairs of points $(x^{(1)} + h^{(1)}, x^{(2)})$, $(x^{(1)}, x^{(2)})$, in Δ, are taken into account (see I, § 280). The function

$$Ax^{(1)} - \phi(x^{(1)}, x^{(2)}) \equiv \psi(x^{(1)}, x^{(2)})$$

is such that $\psi(x^{(1)} + h^{(1)}, x^{(2)}) - \psi(x^{(1)}, x^{(2)}) \geqq 0$, for $h^{(1)} > 0$; therefore $\psi(x^{(1)}, x^{(2)})$ is monotone non-diminishing with respect to $x^{(1)}$; and it is so

* *L'enseignement Math.* vol. XXIV (1925), p. 79.

† It is asserted by Geiringer (*loc. cit.* p. 109) that Küstermann had proved (*loc. cit.* p. 28) that in all cases in which $\phi(x^{(1)}, x^{(2)})$ is monotone non-diminishing with respect to $x^{(1)}$, and monotone non-increasing with respect to $x^{(2)}$, or the reverse, $\phi(x^{(1)}, x^{(2)})$ is monotonoid. This assertion is however not correct. Küstermann proved only that it holds good when ϕ has everywhere a partial differential coefficient with respect to one of the variables, which is bounded in the cell, the existence of a finite second partial differential coefficient being also assumed. This condition is much more stringent than that which is given above.

also with respect to $x^{(2)}$; and therefore it is a monotone function. Thus $\phi(x^{(1)}, x^{(2)})$ is expressible as the difference of the two monotone non-diminishing functions $Ax^{(1)}$, $Ax^{(1)} - \phi(x^{(1)}, x^{(2)})$. The other cases may be treated in a similar manner.

When one of the conditions in the theorem is satisfied, the four double limits of $\phi(x^{(1)} + h^{(1)}, x^{(2)} + h^{(2)})$, as $h^{(1)}, h^{(2)}$ converge to zero, from values that are either both positive, both negative, or one positive and the other negative, all exist.

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464. We proceed to consider the value of the double limit of that part of the integral with respect to $(t^{(1)}, t^{(2)})$ representing $s_{n^{(1)}, n^{(2)}}(x^{(1)}, x^{(2)})$, which is taken over a cross-neighbourhood of the point $(x^{(1)}, x^{(2)})$. Using the notation of § 459, this integral is

$$\int F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)}),$$

where $F(t^{(1)}, t^{(2)})$ denotes

$$f(x^{(1)} + 2t^{(1)}, x^{(2)} + 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} + 2t^{(2)}) \\ + f(x^{(1)} + 2t^{(1)}, x^{(2)} - 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} - 2t^{(2)}),$$

the integration being taken over the three cells $(0, 0; \epsilon, \epsilon)$, $(0, \epsilon; \epsilon, c)$, $(\epsilon, 0; c, \epsilon)$, where $\frac{1}{2}\pi \geqq c > \epsilon$.

(1) Let us consider

$$\int_{(0,0)}^{(\epsilon,\epsilon)} \psi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)}),$$

where $\psi(t^{(1)}, t^{(2)})$ is taken to be monotone, non-increasing, and bounded, in the cell $(0, 0; \epsilon, \epsilon)$. The integral may be written in the form

$$\psi(\epsilon - 0, \epsilon - 0) \int_{(0,0)}^{(\epsilon,\epsilon)} \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)}) \\ + \int_{(0,0)}^{(\epsilon,\epsilon)} \phi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)}),$$

where $\phi(t^{(1)}, t^{(2)})$ denotes the monotone non-increasing function

$$\psi(t^{(1)}, t^{(2)}) - \psi(\epsilon - 0, \epsilon - 0).$$

The first part of the expression is equal to

$$\psi(\epsilon - 0, \epsilon - 0) \int_0^{m^{(1)}\epsilon} \frac{\sin\theta}{\theta} d\theta \,.\int_0^{m^{(2)}\epsilon} \frac{\sin\theta}{\theta} d\theta.$$

As $m^{(1)}, m^{(2)}$ increase indefinitely, the integrals both converge to $\frac{1}{2}\pi$; also $|\psi(\epsilon - 0, \epsilon - 0) - \psi(+0, +0)|$ is arbitrarily small, if ϵ be sufficiently small. Hence the expression differs from $\frac{1}{4}\pi^2\psi(+0, +0)$ by less than the arbitrarily chosen positive number ζ, if $\epsilon \leqq \epsilon_0$, and $m^{(1)}, m^{(2)}$ are both $\geqq$ an integer dependent on ζ and ϵ_0.

The integral

$$\int_{(0,0)}^{(\epsilon,\epsilon)} \phi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$$

can be divided into parts taken over the cells

$$\left(\frac{\iota\pi}{m^{(1)}}, \frac{\iota'\pi}{m^{(2)}}; \frac{\overline{\iota+1}\pi}{m^{(1)}}, \frac{\overline{\iota'+1}\pi}{m^{(2)}}\right),$$

where ι has the values $0, 1, 2, \ldots s^{(1)}$, and ι' has the values $0, 1, 2, 3, \ldots s^{(2)}$; the integers $s^{(1)}$, $s^{(2)}$ being such that $\frac{s^{(1)}\pi}{m^{(1)}} < \epsilon \leq \frac{\overline{s^{(1)}+1}\pi}{m^{(1)}}$, and

$$\frac{s^{(2)}\pi}{m^{(2)}} < \epsilon \leq \frac{\overline{s^{(2)}+1}\pi}{m^{(2)}}.$$

It is convenient to take $\phi(t^{(1)}, t^{(2)}) = 0$, when $t^{(1)} > \epsilon$ or $t^{(2)} > \epsilon$; the integration can then be taken over the whole of the cells for which $\iota = s^{(1)}$ or $\iota' = s^{(2)}$.

The integral can be expressed in the form $\sum_{\iota=0}^{\iota=s^{(1)}} \sum_{\iota'=0}^{\iota'=s^{(2)}} (-1)^{\iota+\iota'} u(\iota, \iota')$, where

$$u(\iota, \iota') = \int_{(0,0)}^{\left(\frac{\pi}{m^{(1)}}, \frac{\pi}{m^{(2)}}\right)} \phi\left(t^{(1)} + \frac{\iota\pi}{m^{(1)}}, t^{(2)} + \frac{\iota'\pi}{m^{(2)}}\right) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)} + \frac{\iota\pi}{m^{(1)}}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)} + \frac{\iota'\pi}{m^{(2)}}} d(t^{(1)}, t^{(2)}).$$

It is seen that $u(\iota, \iota')$ is positive, and monotone decreasing with respect to ι and ι'.

If $U_\iota = u(\iota, \iota) - u(\iota, \iota+1) + u(\iota, \iota+2) - \ldots + (-1)^{\iota+s^{(2)}} u(\iota, s^{(2)})$

and

$V_\iota = -u(\iota+1, \iota) + u(\iota+2, \iota) - u(\iota+3, \iota) + \ldots + (-1)^{\iota+s^{(1)}} u(s^{(1)}, \iota)$,

it is easily seen that the integral can be expressed as

$$(U_0 + V_0) + (U_1 + V_1) + \ldots + (U_{s^{(1)}} + V_{s^{(2)}}).$$

We have
$$u(\iota, \iota) > U_\iota > u(\iota, \iota) - u(\iota, \iota+1)$$
$$-u(\iota+1, \iota) + u(\iota+2, \iota) > V_\iota > -u(\iota+1, \iota);$$

and hence we have

$$\Sigma u(\iota, \iota) - \Sigma\{u(\iota+1, \iota) - u(\iota+2, \iota)\} > \Sigma(U_\iota + V_\iota)$$
$$> \Sigma\{u(\iota, \iota) - u(\iota, \iota+1)\} - \Sigma u(\iota+1, \iota)\},$$

and therefore $\sum_{\iota=0} u(\iota, \iota) > \sum_{\iota=0} (U_\iota + V_\iota) > -\sum_{\iota=0} u(\iota, \iota)$;

and thus $\left|\sum_{\iota=0} (U_\iota + V_\iota)\right| < \sum_{\iota=0} u(\iota, \iota)$.

The numerical value of the integral to be estimated is accordingly less than

$$u(0, 0) + u(1, 1) + \ldots + u(s^{(1)}, s^{(1)}).$$

Now

$$u(0,0) = \int_{(0,0)}^{\left(\frac{\pi}{m^{(1)}}, \frac{\pi}{m^{(2)}}\right)} \phi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$$

$$< \phi(+0, +0) \left\{\int_0^\pi \frac{\sin\theta}{\theta} d\theta\right\}^2,$$

which is less than $\pi^2 \{\psi(+0, +0) - \psi(\epsilon - 0, \epsilon - 0)\}$, and this is $< \zeta$, provided ϵ is chosen to be not greater than some value ϵ_1.

The sum $\sum\limits_{\iota=1} u(\iota, \iota)$ is less than $\phi(+0, +0) \sum\limits_{\iota=1} \frac{1}{\iota^2}$, or is less than a fixed multiple of $\{\psi(+0, +0) - \psi(\epsilon - 0, \epsilon - 0)\}$, and this is $< \zeta$, provided ϵ is less than some value ϵ_2.

It has now been shewn that the integral under consideration differs from $\frac{\pi^2}{4} \psi(+0, +0)$ by less than 3ζ, provided ϵ is not greater than the smallest of the numbers $\epsilon_0, \epsilon_1, \epsilon_2$, for all values of $m^{(1)}, m^{(2)}$ which are not less than an integer dependent on ϵ.

(2) We proceed to consider the integral

$$\int_{(0,\epsilon)}^{(\epsilon, c)} \psi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)}).$$

The integral may be divided into parts taken over the cells

$$\left(\frac{\iota\pi}{m^{(1)}}, \frac{\iota'\pi}{m^{(2)}}; \frac{\overline{\iota+1}\pi}{m^{(1)}}, \frac{\overline{\iota'+1}\pi}{m^{(2)}}\right),$$

where $\iota = 0, 1, 2, \ldots s^{(1)}$, $s^{(1)}$ being determined by the condition

$$\frac{s^{(1)}\pi}{m^{(1)}} < \epsilon \leqq \frac{(s^{(1)}+1)\pi}{m^1};$$

and we may assume that $\psi(t^{(1)}, t^{(2)}) = 0$, when $t^{(1)} > \epsilon$, so that the integration may be taken over the whole of the cells for which $\iota = s^{(1)}$. The integer ι' has the values $p, p+1, p+2, \ldots s^{(2)}$, where p is such that

$$\frac{p\pi}{m^{(2)}} \frac{\pi}{m^{(2)}} < \epsilon \leqq \frac{\overline{p+1}\pi}{m^{(2)}},$$

and $s^{(2)}$ is such that $\frac{s^{(2)}\pi}{m^{(2)}} < c \leqq \frac{(s^{(2)}+1)\pi}{n^{(2)}}$; and we may assume that $\psi(t^{(1)}, t^{(2)}) = 0$, when $t^{(2)} > c$.

The part of the integral taken over the cells for which $\epsilon \leqq t^{(2)} \leqq \frac{\overline{p+1}\pi}{m^{(2)}}$ is

$$\int_\epsilon^{\frac{\overline{p+1}\pi}{m^{(2)}}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} dt^{(2)} \int_0^{\frac{\pi}{m^{(1)}}} \left\{\psi(t^{(1)}, t^{(2)}) - \psi\left(t^{(1)} + \frac{\pi}{m^{(1)}}, t^{(2)}\right)\right.$$

$$\left. + \psi\left(t^{(1)} + \frac{2\pi}{m^{(1)}}, t^{(2)}\right) \ldots\right\} \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} dt^{(1)},$$

and this is numerically less than

$$\psi(+0,+0)\int_0^{\pi}\frac{\sin\theta}{\theta}\,d\theta\left|\int_{m^{(2)}\pi}^{\overline{p+1}\pi}\frac{\sin\theta}{\theta}\,d\theta\right|.$$

As $m^{(2)}$ is indefinitely increased, p is so also, hence the expression converges to zero, as $m^{(2)}\sim\infty$. It is accordingly numerically $<\zeta$, provided $m^{(2)}$ is sufficiently large. The remaining part of the integral may be denoted by $\sum_{\iota=0}^{\iota=s^{(1)}}\sum_{\iota'=p+1}^{\iota'=s^{(2)}}(-1)^{\iota+\iota'}u(\iota,\iota')$, where $u(\iota,\iota')$ denotes

$$\int_{(0,0)}^{\left(\frac{\pi}{m^{(1)}},\frac{\pi}{m^{(2)}}\right)}\psi\left(t^{(1)}+\frac{\iota\pi}{m^{(1)}},t^{(2)}+\frac{\iota'\pi}{m^{(2)}}\right)\frac{\sin m^{(1)}t^{(1)}}{t^{(1)}+\frac{\iota\pi}{m^{(1)}}}\cdot\frac{\sin m^{(2)}t^{(2)}}{t^{(2)}+\frac{\iota'\pi}{m^{(2)}}}\,d(t^{(1)}.t^{(2)}).$$

We see that $u(\iota,\iota')$ is positive and monotone decreasing with respect to ι and ι'.

Let

$$U_\iota=u(\iota,p+\iota+1)-u(\iota,p+\iota+2)+u(\iota,p+\iota+3)-\dots,$$
$$V_\iota=-u(\iota+1,p+\iota+1)+u(\iota+2,p+\iota+1)-u(\iota+3,p+\iota+1)+\dots.$$

As in case (1) it then appears that the integral is equal to $\sum_{\iota=0}^{\iota=s^{(1)}}(U_\iota+V_\iota)$, and that this is less in absolute value than

$$u(0,p+1)+u(1,p+2)+\dots+u(s^{(1)},p+1+s^{(1)}).$$

This is less than

$$\int_{(0,0)}^{\left(\frac{\pi}{m^{(1)}},\frac{\pi}{m^{(2)}}\right)}\psi(+0,+0)\frac{\sin m^{(1)}t^{(1)}}{t^{(1)}}\cdot\frac{\sin m^{(2)}t^{(2)}}{t^{(2)}+\frac{\overline{p+1}\pi}{m^{(2)}}}\,d(t^{(1)},t^{(2)})$$
$$+\psi(+0,+0)\sum_{\iota=1}\frac{1}{\iota(p+\iota+1)}.$$

The first integral is less than $\psi(+0,+0)\int_0^{\pi}\frac{\sin\theta}{\theta}\,d\theta\int_0^{\pi}\frac{d\theta}{\theta+(p+1)\pi}$, or than a fixed multiple of $\log\frac{p+2}{p+1}$, which converges to zero, as $m^{(2)}\sim\infty$. Thus the integral is numerically less than ζ, if $m^{(2)}$ is sufficiently great.

The series $\sum_{\iota=1}\frac{1}{\iota(p+\iota+1)}$ is $<\sum_{\iota=1}^{\iota=\lambda}\frac{1}{\iota(p+\iota+1)}+\sum_{\iota=\lambda+1}^{\infty}\frac{1}{\iota^2}$.

Choosing λ so that $\sum_{\iota=\lambda+1}^{\infty}\frac{1}{\iota^2}<\frac{1}{2}\zeta$, we see that, when λ has been so fixed, $\sum_{\iota=1}^{\iota=\lambda}\frac{1}{\iota(\iota+p+1)}<\frac{1}{2}\zeta$, provided p is sufficiently large. Thus the integral is numerically less than ζ, when $m^{(2)}$ is sufficiently large.

It has now been shewn that

$$\int_{(0,\epsilon)}^{(\epsilon,\frac{1}{2}\pi)} \psi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$$

is numerically $< 2\zeta$, provided $m^{(2)}$ is sufficiently large. The integral over $(\epsilon, 0; \frac{1}{2}\pi, \epsilon)$ may be estimated in precisely the same manner.

It has now been proved that the integral

$$\int \psi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$$

over the three cells $(0, 0; \epsilon, \epsilon)$, $(0, \epsilon; \epsilon, c)$, $(\epsilon, 0; c, \epsilon)$ differs from

$$\frac{\pi^2}{4} \psi(+0, +0)$$

by less than an arbitrarily chosen positive number, provided ϵ be chosen sufficiently small, and $m^{(1)}, m^{(2)}$ sufficiently large. Choosing such a fixed value of ϵ, the upper and lower double limits of the integral both differ from $\frac{\pi^2}{4} \psi(+0, +0)$ by less than an arbitrarily chosen positive number.

It can be shewn that

$$\int_{(\epsilon,\epsilon)}^{(c,c)} \psi(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$$

converges to zero, as $m^{(1)} \sim \infty$, $m^{(2)} \sim \infty$. For $\psi(t^{(1)}, t^{(2)}) \operatorname{cosec} t^{(1)} \operatorname{cosec} t^{(2)}$ is summable in the cell $(\epsilon, \epsilon; c, c)$, when $c < \frac{1}{2}\pi$; and the result then follows as in § 460.

It thus appears that, when $0 < c \leqq \frac{1}{2}\pi$, the upper and lower double limits of the integral over $(0, 0; c, c)$ both differ from $\frac{\pi^2}{4} \psi(+0, +0)$ by less than an arbitrarily chosen positive number, and therefore the limit has a unique value. It follows that, for a fixed value of ϵ, the double limit of the integral over the three cells $(0, 0; \epsilon, \epsilon)$, $(0, \epsilon; \epsilon, c)$, $(\epsilon, 0; c, \epsilon)$ is $\frac{\pi^2}{4} \psi(+0, +0)$.

465. If the function $F(t^{(1)}, t^{(2)})$ which denotes

$$f(x^{(1)} + 2t^{(1)}, x^{(2)} + 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} - t^{(2)})$$
$$+ f(x^{(1)} + 2t^{(1)}, x^{(2)} - 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} + 2t^{(2)})$$

is bounded and monotonoid in the area which constitutes a cross-neighbourhood of the point $(x^{(1)}, x^{(2)})$, so that

$$F(t^{(1)}, t^{(2)}) = F_1(t^{(1)}, t^{(2)}) - F_2(t^{(1)}, t^{(2)}),$$

where F_1, F_2 are monotone bounded functions in the domain under consideration, F_1 and F_2 can be so defined that they are monotone in

$$(0, 0, \tfrac{1}{2}\pi, \tfrac{1}{2}\pi).$$

Thus the integral $\frac{1}{\pi^2} \int_{(0,0)}^{(\frac{1}{2}\pi,\frac{1}{2}\pi)} F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}} d(t^{(1)}, t^{(2)})$ converges to $\frac{1}{4} F(+0, +0)$, as $m^{(1)} \sim \infty$, $m^{(2)} \sim \infty$.

We have accordingly the following theorem:

If $f(x^{(1)}, x^{(2)})$, a periodic function, with periods 2π, be summable in the cell $(-\pi, -\pi; \pi, \pi)$, the Fourier's series corresponding to it converges at the point $(x^{(1)}, x^{(2)})$ interior to the cell, to the value $\frac{1}{4}F(+0, +0)$, where $F(t^{(1)}, t^{(2)})$ denotes the function $\Sigma f(x^{(1)} \pm 2t^{(1)}, x^{(2)} \pm 2t^{(2)})$, if the condition is satisfied that $F(t^{(1)}, t^{(2)})$ is bounded and monotonoid in some cross-neighbourhood of the point $(x^{(1)}, x^{(2)})$. At a point of continuity of the function $f(x^{(1)}, x^{(2)})$ it converges to $f(x^{(1)}, x^{(2)})$, if the condition is satisfied.

If $f(x^{(1)}, x^{(2)})$ is of bounded variation in a cross-neighbourhood of the point $(x^{(1)}, x^{(2)})$, two of the four functions $f(x^{(1)} \pm 2t^{(1)}, x^{(2)} \pm 2t^{(2)})$ are monotonoid, and the other two are quasi-monotonoid (§ 462) for values of $(t^{(1)}, t^{(2)})$, in the cross-neighbourhood of $(x^{(1)}, x^{(2)})$. In case the two quasi-monotonoidal functions are monotonoid, which is certainly the case (see § 463) if $f(x^{(1)}, x^{(2)})$ has its derivatives with respect to one of the variables bounded in the cross-neighbourhood, then the function $F(t^{(1)}, t^{(2)})$ is monotonoid, and the four double limits $f(x^{(1)} \pm 0, x^{(2)} \pm 0)$ all exist. We have accordingly the following theorem:

If $f(x^{(1)}, x^{(2)})$ be periodic, of periods 2π, and it be summable in the cell $(-\pi, -\pi; \pi, \pi)$, and if the conditions are satisfied (1), that $f(x^{(1)}, x^{(2)})$ is of bounded variation in some cross-neighbourhood of the point $(x^{(1)}, x^{(2)})$, and (2), that, in that cross-neighbourhood, the partial derivatives of $f(x^{(1)}, x^{(2)})$ with respect to one of the variables are bounded, whether they have everywhere unique values or not, then the double Fourier's series corresponding to $f(x^{(1)}, x^{(2)})$ converges at $(x^{(1)}, x^{(2)})$ to the value

$$\tfrac{1}{4}\{f(x^{(1)}+0, x^{(2)}+0) + f(x^{(1)}-0, x^{(2)}-0) + f(x^{(1)}+0, x^{(2)}-0) + f(x^{(1)}-0, x^{(2)}+0)\},$$

or to $f(x^{(1)}, x^{(2)})$, in case the function is continuous at the point.

In case the function $f(x^{(1)}, x^{(2)})$ is of bounded variation in the whole cell $(-\pi, -\pi; \pi, \pi)$, and the condition is satisfied that one of its partial derivatives (whether a partial differential coefficient everywhere or not) is bounded in the cell, then the double Fourier's series is everywhere convergent in the cell.

A scrutiny of the foregoing investigations suffices to establish the following theorem:

If the function $f(x^{(1)}, x^{(2)})$ is of bounded variation in the cell $(-\pi, -\pi; \pi, \pi)$, and one of the partial derivatives is bounded in the cell, then the double Fourier's series converges uniformly to $f(x^{(1)}, x^{(2)})$ at the points of a closed set in all the points of which the function is continuous. In case the closed set has points on the boundary of the cell, at such points the periodic function obtained by extension to the outside of the cell must be continuous.

466. The following criterion of convergence at a point will be sufficient in many cases, and is simple in application:

The double Fourier's series is convergent at a point $(\xi^{(1)}, \xi^{(2)})$ *if*

$$\frac{\partial f(x^{(1)}, x^{(2)})}{\partial x^{(1)}}, \quad \frac{\partial f(x^{(1)}, x^{(2)})}{\partial x^{(2)}}, \quad \frac{\partial^2 f(x^{(1)}, x^{(2)})}{\partial x^{(1)}\,\partial x^{(2)}}$$

all exist, and are bounded in some cross-neighbourhood of $(\xi^{(1)}, \xi^{(2)})$; *the function* $f(x^{(1)}, x^{(2)})$ *being assumed to be summable in the cell* $(-\pi, -\pi; \pi, \pi)$.

We may consider the cell $(\xi^{(1)}, \xi^{(2)}; \xi^{(1)} + 2t^{(1)}, \xi^{(2)} + 2t^{(2)})$; in this cell the integral

$$\int_{(\xi^{(1)}, \xi^{(2)})}^{(\xi^{(1)}+2t^{(1)}, \xi^{(2)}+2t^{(2)})} \frac{\partial^2 f}{\partial x^{(1)}\,\partial x^{(2)}}\, d(x^{(1)}, x^{(2)})$$

exists, and is monotonoid (see I, § 418) with respect to $(t^{(1)}, t^{(2)})$; its value is

$$f(\xi^{(1)} + 2t^{(1)}, \xi^{(2)} + 2t^{(2)}) - f(\xi^{(1)}, \xi^{(2)} + 2t^{(2)}) - f(\xi^{(1)} + 2t^{(1)}, \xi^{(2)}) + f(\xi^{(1)}, \xi^{(2)}),$$

which is accordingly monotonoid. Since $f(\xi^{(1)}, \xi^{(2)} + 2t^{(2)})$ is an indefinite integral with respect to $t^{(2)}$, it is monotonoid with respect to $t^{(2)}$; similarly $f(\xi^{(1)} + 2t^{(1)}, \xi^{(2)})$ is monotonoid with respect to $t^{(1)}$. It follows that $f(\xi^{(1)} + 2t^{(1)}, \xi^{(2)} + 2t^{(2)})$ is monotonoid with respect to $(t^{(1)}, t^{(2)})$. In a precisely similar manner it can be shewn that $f(\xi^{(1)} + 2t^{(1)}, \xi^{(2)} - 2t^{(2)})$ is monotonoid with respect to $(t^{(1)}, t^{(2)})$. Thus all four functions

$$f(\xi^{(1)} \pm 2t^{(1)}, \xi^{(2)} \pm 2t^{(2)})$$

are monotonoid in each of the cells which constitute the cross-neighbourhood of $(\xi^{(1)}, \xi^{(2)})$; the convergence of the double series at the point $(\xi^{(1)}, \xi^{(2)})$ then follows from the first theorem of § 465.

An investigation has been given by Küstermann (*loc. cit.*) of the conditions of convergence of the double Fourier's series when it is summed diagonally (see § 33).

EXAMPLE

Let $f(x^{(1)}, x^{(2)})$ be defined in $(-\pi, -\pi; \pi, \pi)$ by

$$f(x^{(1)}, x^{(2)}) = x^{(2)} - \pi, \text{ for } 0 < x^{(1)} < \pi,\ 0 < x^{(2)} < x^{(1)};$$
$$f(x^{(1)}, x^{(2)}) = \pi - x^{(1)}, \text{ for } 0 < x^{(1)} < \pi,\ x^{(1)} < x^{(2)} < \pi;$$
$$f(x^{(1)}, x^{(2)}) = f(-x^{(1)}, x^{(2)}) = f(-x^{(1)}, -x^{(2)}) = f(x^{(1)}, -x^{(2)}).$$

The Fourier's series is found to be

$$\frac{12}{\pi}\sum_{n^{(1)}=1} \frac{\cos n^{(1)}x^{(1)}}{n^{(1)2}} - \frac{12}{\pi}\sum_{n^{(2)}=1} \frac{\cos n^{(2)}x^{(2)}}{n^{(2)2}} + \frac{8}{\pi}\sum_{n^{(1)}=1,\, n^{(2)}=1} \frac{\cos n^{(1)}x^{(1)} \cos n^{(2)}x^{(2)}}{n^{(1)2} - n^{(2)2}}.$$

The series is not convergent at the point $(0, 0)$, but the two repeated limits

$$\lim_{n^{(1)}\sim\infty}\ \lim_{n^{(2)}\sim\infty} s_{n^{(1)}, n^{(2)}}(0, 0), \qquad \lim_{n^{(2)}\sim\infty}\ \lim_{n^{(1)}\sim\infty} s_{n^{(1)}, n^{(2)}}(0, 0)$$

exist, and have the values $-\pi$, $+\pi$ respectively. Thus the series converges when summed by rows and columns successively, in either order, but the sums in the two cases are different. There is no analogy in the case of single Fourier's series with this phenomenon.

This example was given by Hardy*, in a slightly different form, and was related by Titchmarsh† with a general discussion of double Fourier's series for functions which are discontinuous along a line.

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467. The function $f(x^{(1)}, x^{(2)})$ being summable in the cell $(-\pi, -\pi; \pi, \pi)$, let us suppose the function to be such that $a_{0,0}, a_{n^{(1)},0}, a_{0,n^{(2)}}, b_{0,n^{(2)}}, c_{n^{(1)},0}$ are all zero. Let $F(x^{(1)}, x^{(2)})$ denote the indefinite integral

$$\int_{(-\pi,-\pi)}^{(x^{(1)},x^{(2)})} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)});$$

then the function $F(x^{(1)}, x^{(2)})$ is continuous and monotonoid.

The function $F(\xi^{(1)} + 2t^{(1)}, \xi^{(2)} + 2t^{(2)}) - F(\xi^{(1)}, \xi^{(2)})$ is equal to

$$\int_{(\xi^{(1)},\xi^{(2)})}^{(\xi^{(1)}+2t^{(1)},\,\xi^{(2)}+2t^{(2)})} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)}) + \int_{(-\pi,\,\xi^{(2)})}^{(\xi^{(1)},\,\xi^{(2)}+2t^{(2)})} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)})$$
$$+ \int_{(\xi^{(1)},\,-\pi)}^{(\xi^{(1)}+2t^{(1)},\,\xi^{(2)})} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)}).$$

The second integral is monotonoid with respect to $t^{(2)}$, and the third is monotonoid with respect to $t^{(1)}$, therefore the first integral is monotonoid with respect to $(t^{(1)}, t^{(2)})$, since the sum of the three is so. Similarly, it is seen that the integral over $(\xi^{(1)}, \xi^{(2)}; \xi^{(1)} - 2t^{(1)}, \xi^{(2)} + 2t^{(2)})$ is monotonoid with respect to $(t^{(1)}, t^{(2)})$. Hence also the integrals over the cells

$$(\xi^{(1)}, \xi^{(2)}; \xi^{(1)} - 2t^{(1)}, \xi^{(2)} - 2t^{(2)}),\ (\xi^{(1)}, \xi^{(2)}; \xi^{(1)} + 2t^{(1)}, \xi^{(2)} - 2t^{(2)})$$

are monotonoid. The continuous periodic function $F(x^{(1)}, x^{(2)})$ is accordingly representable by a double Fourier's series which converges uniformly in every finite cell. If

$$A_{n^{(1)},n^{(2)}},\ B_{n^{(1)},n^{(2)}},\ C_{n^{(1)},n^{(2)}},\ D_{n^{(1)},n^{(2)}}$$

are the typical coefficients in this series, we have

$$A_{n^{(1)},n^{(2)}} = \frac{1}{\pi^2}\int_{(-\pi,-\pi)}^{(\pi,\pi)} F(x^{(1)}, x^{(2)}) \cos n^{(1)}x^{(1)} \cos n^{(2)}x^{(2)}\, d(x^{(1)}, x^{(2)}).$$

The expressions for the other coefficients are obtained by changing one or both of the cosines into sines.

Writing $\pi^2 A_{n^{(1)},n^{(2)}}$ in the form

$$\int_{-\pi}^{\pi} \cos n^{(1)}x^{(1)}\, dx^{(1)} \int_{-\pi}^{\pi} F(x^{(1)}, x^{(2)}) \cos n^{(2)}x^{(2)}\, dx^{(2)},$$

we have

$$\int_{-\pi}^{\pi} F(x^{(1)}, x^{(2)}) \cos n^{(2)}x^{(2)}\, dx^{(2)} = -\frac{1}{n^{(2)}}\int_{-\pi}^{\pi} \frac{\partial F}{\partial x^{(2)}} \sin n^{(2)}x^{(2)}\, dx^{(2)},$$

where, in accordance with I, § 419, $\dfrac{\partial F}{\partial x^{(2)}}$ exists for almost all values of $x^{(2)}$, except when $x^{(1)}$ belongs to an enumerable set of points in the linear interval $(-\pi, \pi)$.

* *Loc. cit.* p. 68. † *Loc. cit.* p. 309.

We now have for $\pi^2 A_{n^{(1)}, n^{(2)}}$ the expression

$$-\frac{1}{n^{(2)}}\int_{-\pi}^{\pi} \sin n^{(2)} x^{(2)}\, dx^{(2)} \int_{-\pi}^{\pi} \frac{\partial F}{\partial x^{(2)}} \cos n^{(1)} x^{(1)}\, dx^{(1)};$$

on integration by parts with respect to $x^{(1)}$, and remembering that $\dfrac{\partial^2 F}{\partial x^{(1)}\, dx^{(2)}}$ exists (I, § 419) almost everywhere in the cell, and is equal to $f(x^{(1)}, x^{(2)})$, we have

$$A_{n^{(1)}, n^{(2)}} = \frac{1}{\pi^2 n^{(1)} n^{(2)}} \int_{(-\pi, -\pi)}^{(\pi, \pi)} f(x^{(1)}, x^{(2)}) \sin n^{(1)} x^{(1)} \sin n^{(2)} x^{(2)}\, d(x^{(1)}, x^{(2)}),$$

or

$$A_{n^{(1)}, n^{(2)}} = \frac{1}{n^{(1)} n^{(2)}} d_{n^{(1)}, n^{(2)}}.$$

It can be shewn that $A_{0, n^{(2)}} = 0$, and that $A_{n^{(1)}, 0} = 0$, in a similar manner, it being assumed that the Fourier's series for $f(x^{(1)}, x^{(2)})$ has no terms which involve one variable only.

The relations

$$B_{n^{(1)}, n^{(2)}} = -\frac{1}{n^{(1)} n^{(2)}} c_{n^{(1)}, n^{(2)}}, \quad C_{n^{(1)}, n^{(2)}} = -\frac{1}{n^{(1)} n^{(2)}} b_{n^{(1)}, n^{(2)}},$$

$$D_{n^{(1)}, n^{(2)}} - \frac{1}{n^{(1)} n^{(2)}} a_{n^{(1)}, n^{(2)}},$$

with the corresponding relations, when $n^{(1)}$ or $n^{(2)}$ is zero, can be obtained by the same method.

It has thus been shewn that the series obtained by integrating twice each term of such a Fourier's series converges uniformly to an indefinite integral.

Conversely, the series obtained by differentiating twice the Fourier's double series which converges uniformly to an indefinite integral is a Fourier's series.

For, if

$$A_{n^{(1)}, n^{(2)}} = \frac{1}{\pi^2} \int_{(-\pi, -\pi)}^{(\pi, \pi)} F(x^{(1)}, x^{(2)}) \cos n^{(1)} x^{(1)} \cos n^{(2)} x^{(2)}\, d(x^{(1)}, x^{(2)}),$$

where $F(x^{(1)}, x^{(2)})$ is an indefinite integral, it can be shewn as before, by two integrations by parts, that

$$n^{(1)} n^{(2)} A_{n^{(1)}, n^{(2)}} = \frac{1}{\pi^2} \int_{(-\pi, -\pi)}^{(\pi, \pi)} \frac{\partial^2 F}{\partial x^{(1)} \partial x^{(2)}} \sin n^{(1)} x^{(1)} \sin n^{(2)} x^{(2)}\, d(x^{(1)}, x^{(2)}),$$

where $\dfrac{\partial^2 F}{\partial x^{(1)} \partial x^{(2)}}$ exists almost everywhere. The function $f(x^{(1)}, x^{(2)})$ being defined to have the values of $\dfrac{\partial^2 F}{\partial x^{(1)} \partial x^{(2)}}$ almost everywhere, it is seen that

$$\frac{1}{\pi^2} \int_{(-\pi, -\pi)}^{(\pi, \pi)} f(x^{(1)}, x^{(2)}) \sin n^{(1)} x^{(1)} \sin n^{(2)} x^{(2)}\, d(x^{(1)}, x^{(2)}) = n^{(1)} n^{(2)} A_{n^{(1)}, n^{(2)}}.$$

The other corresponding results for the other coefficients

$$B_{n^{(1)},n^{(2)}},\ C_{n^{(1)},n^{(2)}},\ D_{n^{(1)},n^{(2)}}$$

may be obtained in a similar manner.

We have thus the following theorem, corresponding to the theorem of § 360:

The necessary and sufficient condition that a double trigonometrical series, without a constant term, and without terms which contain one variable only, shall be a Fourier's double series is that the series obtained by integrating each term with respect to both variables shall converge uniformly to an indefinite integral.

In case the double Fourier's series for $f(x^{(1)}, x^{(2)})$ contains the terms involving a constant and multiples of $\cos n^{(1)}x$, $\sin n^{(1)}x$, $\cos n^{(2)}x$, $\sin n^{(2)}x$, we see that the single Fourier's series corresponding to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(2)},$$

which exists for almost all values of $x^{(1)}$, and is summable with respect to $x^{(1)}$, is

$$a_{0,0} + \sum_{n^{(1)}=1} (a_{n^{(1)},0}\cos n^{(1)}x^{(1)} + c_{n^{(1)},0}\sin n^{(1)}x^{(1)});$$

and the single Fourier's series corresponding to $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(1)}$ is

$$a_{00} + \sum_{n^{(2)}=1} (a_{0,n^{(2)}}\cos n^{(2)}x^{(2)} + b_{0,n^{(2)}}\sin n^{(2)}x^{(2)}).$$

Hence the double Fourier's series corresponding to the function

$$f(x^{(1)}, x^{(2)}) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(2)} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(1)} + \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(1)}\,dx^{(2)}$$

consists of the same terms as that corresponding to $f(x^{(1)}, x^{(2)})$, except that the constant term, and the terms involving one of the variable only, are omitted. The integral of this expression over $(-\pi, -\pi; x^{(1)}, x^{(2)})$ is an indefinite integral of $f(x^{(1)}, x^{(2)})$. Applying the theorem last obtained to this function, we see that the Fourier's series corresponding to

$$\int_{(-\pi,-\pi)}^{(x^{(1)},x^{(2)})} f(x^{(1)}, x^{(2)})\,d(x^{(1)}, x^{(2)}) - \frac{(x^{(2)}+\pi)}{2\pi}\int_{-\pi}^{x^{(1)}} dx^{(1)}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(2)}$$
$$- \frac{(x^{(1)}+\pi)}{2\pi}\int_{-\pi}^{x^{(2)}} dx^{(2)}\int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\,dx^{(1)} + \frac{(x^{(1)}+\pi)(x^{(2)}+\pi)}{4\pi^2}\int_{-\pi}^{(\pi,\pi)} f(x^{(1)}, x^{(2)})$$

is the series $\sum_{n^{(1)}=1,\,n^{(2)}=1} \frac{1}{n^{(1)}n^{(2)}} L_{n^{(1)},n^{(2)}}$, where $L_{n^{(1)},n^{(2)}}$ denotes

$$a_{n^{(1)},n^{(2)}}\sin n^{(1)}x^{(1)}\,.\,\sin n^{(2)}x^{(2)} - b_{n^{(1)},n^{(2)}}\sin n^{(1)}x^{(1)}\,.\,\cos n^{(2)}x^{(2)}$$
$$- c_{n^{(1)},n^{(2)}}\cos n^{(1)}x^{(1)}\,.\,\sin n^{(2)}x^{(2)} + d_{n^{(1)},n^{(2)}}\cos n^{(1)}x^{(1)}\,.\,\cos n^{(2)}x^{(2)}$$

If we apply the known theorem for single Fourier's series (§ 360), we see that the series $\sum_{n^{(1)}=1} \frac{1}{n^{(1)}} L_{n^{(1)},0}$ is uniformly convergent, its sum being

$$\int_{-\pi}^{x^{(1)}} dx^{(1)} \int_{-\pi}^{\pi} f(x^{(1)}, x^{(2)})\, dx^{(2)} - (x^{(1)} + \pi)\, 2\pi a_{0,0}.$$

A similar result holds for the series $\sum_{n^{(2)}=1} \frac{1}{n^{(2)}} L_{0,n^{(2)}}$. We now find that

$$\int_{(-\pi,-\pi)}^{(x^{(1)},x^{(2)})} f(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)}) - (x^{(1)} + \pi)(x^{(2)} + \pi)\, a_{0,0}$$

is the sum of the series

$$\frac{x^{(1)}+\pi}{2\pi} \sum_{n^{(1)}=1} \frac{1}{n^{(1)}} L_{n^{(1)},0} + \frac{x^{(1)}+\pi}{2\pi} \sum_{n^{(2)}=1} \frac{1}{n^{(2)}} L_{0,n^{(2)}} + \sum_{n^{(1)}=1,\, n^{(2)}=1} \frac{1}{n^{(1)} n^{(2)}} L_{n^{(1)},n^{(2)}};$$

the expression converging uniformly.

If we take any cell Δ_1, we find that

$$\int_{(\Delta_1)} \{f(x^{(1)}, x^{(2)}) - s_{n^{(1)},n^{(2)}}(x^{(1)}, x^{(2)})\}\, d(x^{(1)}, x^{(2)})$$

converges to zero, as $n^{(1)}$, $n^{(2)}$ become indefinitely great.

This is the analogue of the theorem given in § 362 for single Fourier's series.

THE CESÀRO SUMMATION OF A DOUBLE FOURIER'S SERIES

468. If the partial summation of a double Fourier's series be taken in accordance with Cesàro's method $(C, 1)$, both with respect to $n^{(1)}$ and with respect to $n^{(2)}$, we obtain the partial Cesàro sum which may be denoted by $s^{(c,c)}_{n^{(1)},n^{(2)}}(x)$. We have as the expression for this partial sum

$$\frac{1}{\pi^2} \int_{(-\pi,-\pi)}^{(\pi,\pi)} \left[\frac{1}{2n^{(1)}} + \sum_{s=1}^{n^{(1)}-1} \frac{n^{(1)}-s}{n^{(1)}} \cos s\,(\xi^{(1)} - x^{(1)})\right]$$
$$\times \left[\frac{1}{2n^{(2)}} + \sum_{s=1}^{n^{(2)}-1} \frac{n^{(2)}-s}{n^{(2)}} \cos s\,(\xi^{(2)} - x^{(2)})\right] f(\xi^{(1)}, \xi^{(2)})\, d(\xi^{(1)}, \xi^{(2)}).$$

which is equal to

$$\frac{1}{4n^{(1)} n^{(2)} \pi^2} \int_{(-\pi,-\pi)}^{(\pi,\pi)} \left\{\frac{\sin \frac{1}{2} n^{(1)} (\xi^{(1)} - x^{(1)})}{\sin \frac{1}{2} (\xi^{(1)} - x^{(1)})}\right\}^2 \left\{\frac{\sin \frac{1}{2} n^{(2)} (\xi^{(2)} - x^{(2)})}{\sin \frac{1}{2} (\xi^{(2)} - x^{(2)})}\right\}^2$$
$$\times f(\xi^{(1)}, \xi^{(2)})\, d(\xi^{(1)}, \xi^{(2)}).$$

and this may be expressed in the form

$$\frac{1}{n^{(1)} n^{(2)} \pi^2} \int_{(0,0)}^{(\frac{1}{2}\pi,\frac{1}{2}\pi)} \left(\frac{\sin n^{(1)} t^{(1)}}{\sin t^{(1)}}\right)^2 \left(\frac{\sin n^{(2)} t^{(2)}}{\sin t^{(2)}}\right)^2 \psi(t^{(1)}, t^{(2)})\, d(t^{(1)}, t^{(2)}),$$

where

$$\psi(t^{(1)}, t^{(2)}) = f(x^{(1)} + 2t^{(1)}, x^{(2)} + 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} - 2t^{(2)})$$
$$+ f(x^{(1)} + 2t^{(1)}, x^{(2)} - 2t^{(2)}) + f(x^{(1)} - 2t^{(1)}, x^{(2)} + 2t^{(2)}).$$

The integral

$$\int_{(\epsilon,\epsilon)}^{(\frac{1}{2}\pi,\frac{1}{2}\pi)} \frac{1}{\pi^2 n^{(1)} n^{(2)}} \psi(t^{(1)}, t^{(2)}) \left(\frac{\sin n^{(1)} t^{(1)}}{\sin t^{(1)}}\right)^2 \left(\frac{\sin n^{(2)} t^{(2)}}{\sin t^{(2)}}\right)^2 d(t^{(1)}, t^{(2)}),$$

where $\psi(t^{(1)}, t^{(2)})$, whether it be bounded or not, is summable in the cell $(\epsilon, \epsilon; \frac{1}{2}\pi, \frac{1}{2}\pi)$, can be seen by means of the general theorem of § 279 to converge to zero, uniformly for all points $(x^{(1)}, x^{(2)})$ at a distance not less than 2ϵ from the boundary of Δ, as $n^{(1)} \sim \infty$, $n^{(2)} \sim \infty$. For the conditions are satisfied that $\frac{1}{n^{(1)} n^{(2)}} \left(\frac{\sin n^{(1)} t^{(1)}}{\sin t^{(1)}}\right)^2 \left(\frac{\sin n^{(2)} t^{(2)}}{\sin t^{(2)}}\right)^2$ is bounded in the cell $(\epsilon, \epsilon; \frac{1}{2}\pi, \frac{1}{2}\pi)$, and that its integral over that rectangle converges to zero, as $n^{(1)}$, $n^{(2)}$ become indefinitely great.

We consider next the integral over the cell $(0, \epsilon; \epsilon, \frac{1}{2}\pi)$. Let it be assumed that $\psi(t^{(1)}, t^{(2)})$ is bounded in the cell $(0, 0; \frac{1}{2}\pi, \frac{1}{2}\pi)$, and that U is the upper boundary of its absolute value; then the integral is numerically not greater than

$$U \int_0^{\epsilon} \frac{1}{n^{(1)}} \left(\frac{\sin n^{(1)} t^{(1)}}{\sin t^{(1)}}\right)^2 dt^{(1)} \cdot \int_{\epsilon}^{\frac{1}{2}\pi} \frac{1}{n^{(2)}} \left(\frac{\sin n^{(2)} t^{(2)}}{\sin t^{(2)}}\right)^2 dt^{(1)};$$

the first integral converges to $\frac{\pi}{2}$, and the second to zero (see § 365), as $n^{(1)}$, $n^{(2)}$ are indefinitely increased. Therefore the integral under consideration converges to zero. The integral over the cell $(\epsilon, 0; \frac{1}{2}\pi, 0)$ may be considered in the same manner.

Lastly, we take

$$\int_{(0,0)}^{(\epsilon,\epsilon)} \psi(t^{(1)}, t^{(2)}) \cdot \frac{1}{n^{(1)} n^{(2)}} \left(\frac{\sin n^{(1)} t^{(1)}}{\sin t^{(1)}}\right)^2 \left(\frac{\sin n^{(2)} t^{(2)}}{\sin t^{(2)}}\right)^2 d(t^{(1)}, t^{(2)}).$$

If the function $f(x^{(1)}, x^{(2)})$ is continuous at the point $(x^{(1)}, x^{(2)})$, ϵ can be so chosen that $\psi(t^{(1)}, t^{(2)})$ differs from $4f(x^{(1)}, x^{(2)})$ by less than an arbitrarily prescribed positive number η; therefore the limit of the integral differs from $f(x^{(1)}, x^{(2)})$ by not more than $\frac{1}{4}\pi^2\eta$, which is arbitrarily small. It follows that, in these circumstances, the Cesàro sum $s^{(c,c)}(x)$ is $f(x^{(1)}, x^{(2)})$.

If the function is not continuous at $(x^{(1)}, x^{(2)})$, but if $\psi(t^{(1)}, t^{(2)})$ has a definite limit, which will in particular be the case if the four limits

$f(x^{(1)}+0, x^{(2)}+0), f(x^{(1)}-0, x^{(2)}-0), f(x^{(1)}+0, x^{(2)}-0), f(x^{(1)}-0, x^{(2)}+0)$

all exist, it is seen as before that the sum $s^{(c,c)}(x^{(1)}, x^{(2)})$ exists, and has the value $\frac{1}{4}\psi(+0, +0)$.

We have now obtained the following result:

If $f(x^{(1)}, x^{(2)})$ be a doubly periodic function, of periods 2π, and be summable in the cell $(-\pi, -\pi; \pi, \pi)$, the double Fourier's series, corresponding to $f(x^{(1)}, x^{(2)})$, converges to $f(x^{(1)}, x^{(2)})$ at any point at which the function is con-

tinuous, provided the point has a cross-neighbourhood in which the function is bounded. Subject to the same condition, the series converges to

$$\tfrac{1}{4}\lim_{h^{(1)}\sim 0,\, k^{(1)}\sim 0}\{f(x^{(1)}+h^{(1)},\, x^{(2)}+h^{(2)})+f(x^{(1)}-h^{(1)},\, x^{(2)}-h^{(2)})$$
$$+f(x^{(1)}+h^{(1)},\, x^{(2)}-h^{(2)})+f(x^{(1)}-h^{(1)},\, x^{(2)}+h^{(2)})\}$$

provided this double limit exists.

It is easily seen that the following theorem holds good as regards uniform convergence:

If the function be continuous at every point of a closed set, and be everywhere bounded, the partial Cesàro sum $s^{(c,c)}_{n^{(1)},n^{(2)}}$ *converges in the closed set uniformly to the value of the function.*

The convergence of the Cesàro sum of double Fourier's series has been investigated by W. H. Young, Küstermann, and C. N. Moore; the last of these has dealt explicitly with cases in which there are lines of discontinuity of the function. The convergence of the sums $s^{(0,c)}_{n^{(1)},n^{(2)}}$, $s^{(c,0)}_{n^{(1)},n^{(2)}}$, in which the summation with respect to one of the variables is taken in the ordinary manner, and that with regard to the other variable in the Cesàro manner, has been considered by W. H. Young (*loc. cit.*).

THE POISSON SUM OF THE DOUBLE SERIES

469. The Poisson method of summation may be applied to the double Fourier's series. The partial sum of the series

$$a_{0,0}+\Sigma\, h^{(1)n_1}h^{(2)n_2}(a_{n^{(1)},n^{(2)}}\cos n^{(1)}x^{(1)}\cos n^{(2)}x^{(2)}+b_{n^{(1)},n^{(2)}}\cos n^{(1)}x^{(1)}\sin n^{(2)}x^{(2)}$$
$$+c_{n^{(1)},n^{(2)}}\sin n^{(1)}x^{(2)}\cos n^{(1)}x^{(2)}+d_{n^{(1)},n^{(2)}}\sin n^{(1)}x^{(1)}\sin n^{(2)}x^{(2)}),$$

where $|h^{(1)}|<1$, $|h^{(2)}|<1$, may be expressed in the form

$$P_{n^{(1)},n^{(2)}}(x^{(1)},x^{(2)})=\frac{1}{4\pi^2}\int_{(-\pi,-\pi)}^{(\pi,\pi)}\frac{1-h^{(1)2}}{1-2h^{(1)}\cos(\xi^{(1)}-x^{(1)})+h^{(1)2}}$$
$$\times\frac{1-h^{(2)2}}{1-2h^{(2)}\cos(\xi^{(2)}-x^{(2)})+h^{(2)2}}f(\xi^{(1)},\xi^{(2)})\,d(\xi^{(1)},\xi^{(2)}).$$

The limit of this integral may be investigated by a method similar to that which has been applied to the Cesàro sum. This has been carried out by Gross* and by Küstermann (*loc. cit.*). A theorem analogous to that of § 410, that the Poisson sum of a single Fourier's series is convergent almost everywhere, has been given by Geiringer (*loc. cit.* p. 135) for the Poisson double sum.

* *Sitzungsber. d. k. Acad. Wien*, vol. CXXIV (1915), p. 1017.

PARSEVAL'S THEOREM FOR THE DOUBLE SERIES

470. It has been shewn in § 467 that, if Δ_1 denote any cell,

$$\lim_{n^{(1)}\sim\infty,\, n^{(2)}\sim\infty} \int_{(\Delta_1)} \{f(x^{(1)}, x^{(2)}) - s_{n^{(1)}, n^{(2)}}(x^{(1)}, x^{(2)})\}\, d(x^{(1)}, x^{(2)}) = 0.$$

Let it now be assumed that $\{f(x^{(1)}, x^{(2)})\}^2$ is summable over the cell $(-\pi, -\pi; \pi, \pi)$; it is then seen, precisely as in § 362, that the series

$$\sum_{n^{(1)}=0}^{\infty} \sum_{n^{(2)}=0}^{\infty} \theta\,(a^2_{n^{(1)}, n^{(2)}} + b^2_{n^{(1)}, n^{(2)}} + c^2_{n^{(1)}, n^{(2)}} + d^2_{n^{(1)}, n^{(2)}}),$$

where $\theta = 1$ when $n^{(1)} > 0$, $n^{(2)} > 0$, and $\theta = 2$ when one of the numbers $n^{(1)}, n^{(2)}$ is zero, and $\theta = 4$ when both are zero, converges to a sum

$$\leq \frac{1}{\pi^2} \int_{(-\pi,\, -\pi)}^{(\pi,\, \pi)} \{f(x^{(1)}, x^{(2)})\}^2\, d(x^{(1)}, x^{(2)}).$$

It can then be proved, exactly as in § 362, that

$$\lim_{n^{(1)}\sim\infty,\, n^{(2)}\sim\infty} \int_{(e)} \{f(x^{(1)}, x^{(2)}) - s_{n^{(1)}, n^{(2)}}(x^{(1)}, x^{(2)})\}\, d(x^{(1)}, x^{(2)}) = 0,$$

where e is any measurable set of points.

As in § 377, it can then be proved that, if $g(x^{(1)}, x^{(2)})$ be any function whose square is summable in $(-\pi, -\pi; \pi, \pi)$,

$$\int_{(-\pi,\, -\pi)}^{(\pi,\, \pi)} g(x^{(1)}, x^{(2)})\, \{f(x^{(1)}, x^{(2)}) - s_{n^{(1)}, n^{(2)}}(x^{(1)}, x^{(2)})\}\, d(x^{(1)}, x^{(2)}) = 0.$$

This is equivalent to the following theorem:

If $f(x^{(1)}, x^{(2)})$, $g(x^{(1)}, x^{(2)})$ be functions whose squares are summable over the cell $(-\pi, -\pi; \pi, \pi)$, then the series

$$\sum_{n^{(1)}=0}^{\infty} \sum_{n^{(2)}=0}^{\infty} \theta\,(a_{n^{(1)}, n^{(2)}} A_{n^{(1)}, n^{(2)}} + b_{n^{(1)}, n^{(2)}} B_{n^{(1)}, n^{(2)}} + c_{n^{(1)}, n^{(2)}} C_{n^{(1)}, n^{(2)}} + d_{n^{(1)}, n^{(2)}} D_{n^{(1)}, n^{(2)}})$$

converges to $$\frac{1}{\pi^2} \int_{(-\pi,\, -\pi)}^{(\pi,\, \pi)} f(x^{(1)}, x^{(2)})\, g(x^{(1)}, x^{(2)})\, d(x^{(1)}, x^{(2)});$$

where $\theta = 1$ if $n^{(1)} > 0$, $n^{(2)} > 0$, and $\theta = 2$ if one of the numbers $n^{(1)}, n^{(2)}$ is zero, and $\theta = 4$ when both are zero. The constants A, B, C, D have reference to the function $g(x^{(1)}, x^{(2)})$.

Also the series

$$\sum_{n^{(1)}=0}^{\infty} \sum_{n^{(2)}=0}^{\infty} \theta\,(a^2_{n^{(1)}, n^{(2)}} + b^2_{n^{(1)}, n^{(2)}} + c^2_{n^{(1)}, n^{(2)}} + d^2_{n^{(1)}, n^{(2)}})$$

converges to $$\frac{1}{\pi^2} \int_{(-\pi,\, -\pi)}^{(\pi,\, \pi)} \{f(x^{(1)}, x^{(2)})\}^2\, d(x^{(1)}, x^{(2)}).$$

There is no difficulty in extending the Riesz-Fischer theorem (§ 379) to the case of double sequences of constants

$$\{a_{n^{(1)},n^{(2)}}\},\ \{b_{n^{(1)},n^{(2)}}\},\ \{c_{n^{(1)},n^{(2)}}\},\ \{d_{n^{(1)},n^{(2)}}\}.$$

For we find that the double limit of

$$\int_{(-\pi,-\pi)}^{(\pi,\pi)} \{s_{n_p^{(1)},n_p^{(2)}}(x^{(1)},x^{(2)}) - s_{n_q^{(1)},n_q^{(2)}}(x^{(1)},x^{(2)})\}^2\, d(x^{(1)},x^{(2)})$$

as $p\sim\infty$, $q\sim\infty$, has the value zero; where $s_{n_p^{(1)},n_p^{(2)}}(x^{(1)},x^{(2)})$ is that partial sum of the double series for which $n^{(1)}\leq n_p^{(1)}$, $n^{(2)}\leq n_p^{(2)}$.

The theory of the average convergence of a sequence, given in § 171, is now applicable. It follows that, having given a double set of numbers $a_{0,0}$, $a_{n^{(1)},n^{(2)}}$, $b_{n^{(1)},n^{(2)}}$, $c_{n^{(1)},n^{(2)}}$, $d_{n^{(1)},n^{(2)}}$, a function $f(x^{(1)},x^{(2)})$ whose square is summable exists, and is unique (except for equivalent functions), such that the given set of numbers, which are assumed to be such that the series of their squares is convergent, are the coefficients in the double Fourier's series corresponding to the function. The detailed proof is similar to that in § 379 for the case of the single Fourier's series.

CHAPTER IX

THE REPRESENTATION OF FUNCTIONS BY FOURIER'S INTEGRALS

FOURIER'S SINGLE INTEGRAL

471. It has been shewn in the course of the investigation of conditions for the convergence of Fourier's series at a point, or in an interval, that

$$\frac{1}{\pi}\int_0^\epsilon \frac{\sin mz}{z}\{f(x+2z)+f(x-2z)\}\,dz$$

converges to the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$,

when the positive number m, which is not necessarily integral, is indefinitely increased, either through a sequence of values, or as a continuous variable; provided $f(x)$ is summable in the interval $(-\pi, \pi)$, and satisfies one of a group of sufficient conditions in the neighbourhood of the point x, at which the existence of $f(x+0)$, $f(x-0)$ is assumed. The number ϵ is such that $0<\epsilon \leqq \frac{1}{2}\pi$.

This is equivalent to the statement that

$$\lim_{u\sim\infty}\frac{1}{\pi}\int_\alpha^\beta f(x')\frac{\sin u(x'-x)}{x'-x}\,dx' = \frac{1}{2}\{f(x+0)+f(x-0)\},$$

where $x-\pi \leqq \alpha < x < \beta \leqq x+\pi$. If x, x' be changed into $\pi x/l$, $\pi x'/l$, and u be changed into $\pi u/l$, and the function $f(\pi x/l)$ be replaced by $f(x)$, we see that the inequality holds for points x within the interval $(-l, l)$, where α, β now satisfy the conditions $x-l \leqq \alpha < x < \beta \leqq x+l$. When x has the value α, or β, the value of the limit is $\frac{1}{2}f(\alpha+0)$, or $\frac{1}{2}f(\beta-0)$, provided the function is such that the limit exists, and also satisfies one of the sufficient conditions already referred to. For a given point x, and for given values of α and β, the number l can always be so chosen that the conditions $x-l \leqq \alpha < x < \beta \leqq x+l$ are satisfied.

Moreover, in a given interval contained within (α, β), in which $f(x)$ is continuous, the continuity being on both sides at the ends of the interval, the convergence of $\frac{1}{\pi}\int_\alpha^\beta f(x')\frac{\sin u(x'-x)}{x'-x}\,dx'$ to the value $f(x)$ is uniform, provided $f(x)$ is of bounded variation in an interval which contains the given interval. Sufficient conditions will now be investigated that, in the integral, we may substitute ∞ and $-\infty$ for β and α respectively.

(1) Let it be assumed that $f(x)$ is summable in every finite interval, and that $\int_A^\infty \left|\frac{f(x)}{x}\right| dx$, $\int_{-\infty}^{-A}\left|\frac{f(x)}{x}\right| dx$ both exist, where A is any positive number.

We have $\left|\int_A^{A'} f(x') \frac{\sin u(x'-x)}{x'-x} dx'\right| \leqq \frac{A}{A-|x|}\int_A^{A'}\left|\frac{f(x')}{x'}\right| dx'$, provided $|x| < A$, $A' > A$. Now A can be so chosen that $\frac{A}{A-|x|}\int_A^{A'}\left|\frac{f(x')}{x'}\right| dx' < \eta$, for all values of A' ($> A$), where η is an arbitrarily chosen number. Similarly a negative number B can be so chosen that

$$\left|\int_{B'}^{B} f(x') \frac{\sin u(x'-x)}{x'-x} dx'\right| < \eta,$$

where $B' < B$, and $|x| < |B|$. If u is not less than some number u_η, we have

$$\left|\int_B^A f(x') \frac{\sin u(x'-x)}{x'-x} dx' - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right| < \eta,$$

where x is within the interval (B, A), and one of the sufficient conditions is satisfied by $f(x)$ in the neighbourhood of the point x.

It follows that, when $u \geqq u_\eta$,

$$\left|\frac{1}{\pi}\int_{B'}^{A'} f(x') \frac{\sin u(x'-x)}{x'-x} dx' - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right| < 3\eta,$$

for $A' > A$, $B' < B$; or

$$\left|\frac{1}{\pi}\int_{-\infty}^{\infty} f(x') \frac{\sin u(x'-x)}{x'-x} dx' - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right| \leqq 3\eta,$$

for $u \geqq u_\eta$. Hence we have

$$\lim_{u\sim\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(x') \frac{\sin u(x'-x)}{x'-x} dx' = \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

subject to the conditions already stated.

Moreover, if $f(x)$ be continuous in a finite interval, the continuity at the end-points being on both sides, and the finite interval is contained in an interval in which $f(x)$ is of bounded variation, the convergence to $f(x)$ is uniform in the finite interval.

It should be observed that the conditions that $f(x)$ is summable in every finite interval, and that $\left|\frac{f(x)}{x}\right|$ is summable in (A, ∞) and in $(-\infty, -A)$, where $A > 0$, are both satisfied, in particular, if $f(x)$ is absolutely summable in the whole infinite interval $(-\infty, \infty)$.

(2) Let it be assumed that $f(x)$ is summable in every finite interval, and that a positive number A exists such that, in (A, ∞) and in $(-\infty, -A)$, $\frac{f(x)}{x}$ is of bounded variation, or in particular monotone and bounded.

Let $\phi(x')$ denote $\frac{f(x')}{x'-x}$, then we have to consider the limit of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x') \sin u(x'-x)\, dx'.$$

Since, in (A, A'), the function $\frac{f(x')}{x'}$ is of bounded variation, it can be expressed as the difference $\psi_1(x') - \psi_2(x')$ of two non-increasing functions, such that the total variation in (A, A') is

$$\{\psi_1(A) - \psi_1(A')\} + \{\psi_2(A) - \psi_2(A')\}.$$

Since this total variation has a finite limit, as $A' \sim \infty$, we may, by taking $\psi_1(\infty)$, $\psi_2(\infty)$ both to be zero, take $\psi_1(A)$, $\psi_2(A)$ to be finite positive numbers independent of A'. Thus $\psi_1(x') - \psi_2(x') = \frac{f(x')}{x'}$, in (A, ∞); and, since $\frac{x'}{x' - x}$ diminishes as x' increases, we may write $\phi(x') = \phi_1(x') - \phi_2(x')$, for (A, ∞), where $\phi_1(x')$, $\phi_2(x')$ are positive monotone diminishing functions. We have

$$\int_A^{A'} \phi_1(x') \sin u(x' - x)\, dx' = \phi_1(A) \int_A^{\alpha} \sin u(x' - x)\, dx',$$

by using Bonnet's form of the second mean value theorem (I, § 422), where α is in the interval (A, A'). The expression on the right-hand side is numerically not greater than $\frac{2}{u}\phi_1(A)$, and this is independent of A', and converges to zero, as $u \sim \infty$.

Thus $\lim\limits_{u\sim\infty} \int_A^{\infty} \phi_1(x') \sin u(x' - x)\, dx' = 0$; and we may substitute $\phi_2(x')$ for $\phi_1(x')$; therefore $\lim\limits_{u\sim\infty} \int_A^{\infty} \frac{f(x')}{x' - x} \sin u(x' - x)\, dx' = 0$; and the same holds for the limits $-A$, $-\infty$, as is seen in a similar manner. It is easily seen that the convergence is uniform for all values of x in an interval interior to $(-A, A)$; for the values of $\phi_1(A)$, $\phi_2(A)$, as x varies in such an interval, will lie between fixed multiples of $\psi_1(A)$, $\psi_2(A)$. The sufficiency of the conditions has now been established that

$$\lim_{u\sim\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \frac{\sin u(x' - x)}{x' - x}\, dx' = \tfrac{1}{2}\{f(x + 0) + f(x - 0)\},$$

it being assumed that a sufficient condition is satisfied by $f(x)$ in the neighbourhood of the point x.

(3) Let $f(x)$ have in (A, ∞) and in $(-\infty, -A)$ a differential coefficient $f'(x)$, such that its indefinite integral in either interval is $f(x)$, and such that $\left|\frac{f'(x)}{x}\right|$ is summable in the intervals, where A is some positive number.

It will be shewn that this is a special case of (2); it is however of somewhat simpler application in particular cases.

If $\int_A^{\infty} \left|\frac{f'(x)}{x}\right| dx$ exists, for some positive value of A, it can be shewn that $\int_A^{\infty} \frac{f(x)}{x^2}\, dx$ also exists; that this is the case was proved* by Hardy.

* See Pringsheim, *Math. Annalen*, vol. LXXI (1911–12), p. 294.

We have, by integration by parts,

$$\int_A^{A'} \frac{|f'(x)|}{x} dx = \frac{1}{A'} \int_A^{A'} |f'(x)|\, dx + \int_A^{A'} \frac{1}{x^2} \left\{ \int_A^x |f'(x)|\, dx \right\} dx.$$

It will be shewn that

$$\lim_{A' \sim \infty} \frac{1}{A'} \int_A^{A'} |f'(x)|\, dx = 0;$$

we have $\displaystyle \frac{1}{A'} \int_A^{A'} |f'(x)|\, dx = \frac{1}{A'} \int_A^{A''} |f'(x)|\, dx + \frac{1}{A'} \int_{A''}^{A'} |f'(x)|\, dx,$

where $A < A'' < A'$; also

$$\int_{A''}^{A'} \frac{|f'(x)|}{x} dx > \frac{1}{A'} \int_{A''}^{A'} |f'(x)|\, dx,$$

hence $\displaystyle \frac{1}{A'} \int_A^{A'} |f'(x)|\, dx < \frac{1}{A'} \int_A^{A''} |f'(x)|\, dx + \int_{A''}^{A'} \frac{|f'(x)|}{x} dx.$

Let $A' \sim \infty$, we have then

$$\lim_{A' \sim \infty} \frac{1}{A'} \int_A^{A'} |f'(x)|\, dx \leq \int_{A''}^{\infty} \frac{|f'(x)|}{x} dx,$$

and the integral on the right-hand side is arbitrarily small, if A'' be taken large enough; therefore the limit has the value zero.

We now have

$$\int_A^{\infty} \frac{|f'(x)|}{x} dx = \int_A^{\infty} \frac{1}{x^2} \left\{ \int_A^x |f'(x)|\, dx \right\} dx.$$

From this it follows that

$$\int_A^{\infty} \frac{|f'(x)|}{x} dx \geq \int_A^{\infty} \frac{1}{x^2} \left| \int_A^x f'(x)\, dx \right| dx = \int_A^{\infty} \frac{|f(x) - f(A)|}{x^2} dx;$$

and hence $\displaystyle \int_A^{\infty} \frac{|f(x)|}{x^2} dx \leq \int_A^{\infty} \frac{|f'(x)|}{x} dx + \frac{|f(A)|}{A},$

from which the absolute summability of $\dfrac{f(x)}{x^2}$ in (A, ∞) is clear. Since $\dfrac{d}{dx}\dfrac{f(x)}{x} = -\dfrac{f(x)}{x^2} + \dfrac{f'(x)}{x}$, it follows from the absolute summability of $\dfrac{f'(x)}{x}$, and consequently of $\dfrac{f(x)}{x^2}$, in (A, ∞), that $\dfrac{f(x)}{x}$ has bounded variation in (A, ∞). Hence the condition in (2) is satisfied in case that in (3) is satisfied.

The following theorem has now been established:

If $f(x)$ be summable in every finite interval, and one of the sufficient conditions for the convergence of Fourier's series at a point to

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\},$$

or for its uniform convergence in an interval, be satisfied, then

$$\lim_{u \sim \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \frac{\sin u(x'-x)}{x'-x} dx'$$

has the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ *at the point* x, *or converges uniformly in an interval to the value of* $f(x)$, *provided one of the following additional conditions be satisfied:*

(1) *If a positive number* A *exists such that* $\int_A^\infty \frac{|f(x)|}{x}dx$ *and* $\int_{-\infty}^{-A}\frac{|f(x)|}{x}dx$ *exist as finite numbers.*

(2) *If a positive number* A *exists such that* $\frac{f(x)}{x}$ *is of bounded variation in* (A, ∞) *and in* $(-\infty, -A)$; *or in particular if it is bounded and monotone in those intervals.*

(3) *If* $f(x)$ *have, for some positive value of* A, *in* (A, ∞) *and in* $(-\infty, -A)$ *a differential coefficient* $f'(x)$ *such that its indefinite integral in the intervals is* $f(x)$, *and such that* $\frac{|f'(x)|}{x}$ *is summable in the two intervals.*

This is known as Fourier's representation of a function by means of a single integral.

The condition (1) was given by Hobson* and by Pringsheim†; the condition (2) was given by Pringsheim, and (3) was also given by Pringsheim, but contained, as given by him, the redundant condition that $\frac{|f(x)|}{x^2}$ must be summable in the two intervals (A, ∞), $(-\infty, -A)$.

472. If, in the theorem of § 471, we assume that $f(x)$ has the value zero in the interval $(-\infty, A)$, where $A > 0$, and we let $x = 0$, then, provided one of the conditions of the theorem is satisfied, we have

$$\lim_{u\sim\infty}\int_A^\infty f(x')\frac{\sin ux'}{u}dx' = 0.$$

By a slight modification of the proofs in § 471 it can be shewn that

$$\lim_{u\sim\infty}\int_A^\infty f(x')\frac{\cos u(x'-x)}{u}dx' = 0,$$

provided x is not in the interval (A, ∞), and thus, by taking $x = 0$, that $\lim_{u\sim\infty}\int_A^\infty f(x')\frac{\cos ux'}{u}dx' = 0$, the alternative conditions satisfied by $f(x')$ being the same as before; we have in fact only to change $\sin u(x'-x)$ into $\cos u(x'-x)$. In this manner we obtain the following theorem:

The integrals $\int_A^\infty f(x')\frac{\sin ux'}{x'}dx'$, $\int_A^\infty f(x')\frac{\cos ux'}{x'}dx'$ *converge to zero, as* $u\sim\infty$, *if* $f(x)$ *be summable in every finite interval* (A, A'), *where* $A > 0$, *and if one of the following conditions be satisfied:*

(1) *That* $\int_A^\infty \left|\frac{f(x)}{x}\right|dx$ *exists.*

* *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 373.

† *Math. Annalen*, vol. LXVIII (1909–10), p. 384.

(2) *That* $\dfrac{f(x)}{x}$ *is of bounded variation in an interval* (A', ∞), *where* A' *is some number* $\geqq A$.

(3) *That* $f(x)$ *have in some interval* (A', ∞), *where* $A' \geqq A$, *a differential coefficient* $f'(x)$, *of which* $f(x)$ *is an indefinite integral, and that* $\left|\dfrac{f'(x)}{x}\right|$ *is summable in* (A', ∞).

FOURIER'S REPEATED INTEGRAL

473. Since $$\frac{\sin u(x'-x)}{x'-x} = \int_0^u \cos v(x'-x)\,dv,$$
the single integral $$\int_{-\infty}^{\infty} f(x') \frac{\sin u(x'-x)}{x'-x}\,dx'$$
may be written in the form
$$\int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\,dv;$$
and therefore the theorem of § 471 may be taken to refer to
$$\lim_{u\sim\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\,dv.$$

It will now be shewn that, subject to certain sufficient conditions satisfied by $f(x')$, the order of integration may be changed without altering the value of the integral, so that the limit then becomes
$$\frac{1}{\pi} \int_0^{\infty} dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\,dx',$$
which is known as Fourier's double integral representation of the function, although it is in reality a repeated integral representation, the order in which cannot be reversed, because $\frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\,dx' \int_0^{\infty} \cos v(x'-x)\,dv$ does not exist, as $\int_0^{\infty} \cos v(x'-x)\,dv$ has no definite value.

(1) Let it be assumed that $f(x)$ is absolutely summable in $(-\infty, \infty)$.

Let $$\int_{\alpha}^{\beta} f(x') \cos v(x'-x)\,dx'$$
be denoted by $\psi(\alpha, \beta, v)$, and let
$$\int_{-\infty}^{\infty} f(x') \cos v(x'-x)\,dx',$$
which exists, on account of the absolute summability of $f(x')$, be denoted by $\psi(v)$.

It can be shewn that $\int_0^u \psi(v)\,dv = \lim_{\substack{\alpha\sim-\infty\\ \beta\sim\infty}} \int_0^u \psi(\alpha, \beta, v)\,dv$; for, since $|\psi(\alpha, \beta, v)|$ is less than a fixed positive number, independent of α and β, by the theorem of § 225 the equality holds when α and β have continuous values which diverge to $-\infty$ and $+\infty$ respectively.

We have therefore

$$\int_0^u dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\,dx' = \lim_{\substack{\beta\sim\infty\\ \alpha\sim-\infty}} \int_0^u dv \int_\alpha^\beta f(x') \cos v(x'-x)\,dx'$$

$$= \lim_{\substack{\beta\sim\infty\\ \alpha\sim-\infty}} \int_\alpha^\beta dx' \int_0^u f(x') \cos v(x'-x)\,dx' = \int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\,dx,$$

from which the required result follows, by letting u diverge to ∞.

(2) Let $f(x)$ be such that, for some positive number A, $f(x)$ is of bounded variation in (A, ∞) and in $(-\infty, -A)$, and that it converges to zero, as $x \sim \infty$, and as $x \sim -\infty$.

It will be clearly sufficient to assume that, in the interval (A, ∞), $f(x)$ is monotone non-increasing, and converges to zero; the general case will then be deduced by taking $f(x)$ to be the difference of two such functions.

If $A_2 > A_1 > A$, we have

$$\int_{A_1}^{A_2} f(x') \cos v(x'-x)\,dx' = f(A_1) \int_{A_1}^{A_2'} \cos v(x'-x)\,dx',$$

where A_2' is in the interval (A_1, A_2); and thus the integral on the left-hand side is numerically less than $\frac{2}{v} f(A_1)$, which is arbitrarily small, provided A_1 is large enough. Since this holds for all values of A_2 ($> A_1$), the existence of the integral $\int_A^\infty f(x') \cos v(x'-x)\,dx'$ is assured.

We have $\int_A^\infty f(x') \cos v(x'-x)\,dx' = f(A) \int_A^\alpha \cos v(x'-x)\,dx'$,

where α is in the interval (A, ∞); hence

$$\left|\int_A^\infty f(x') \cos v(x'-x)\,dx'\right| < \frac{2f(A)}{v},$$

where $v > 0$.

A similar result holds when $-\infty, -A$ are the limits of integration.

The difference of $\int_{u_0}^u dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\,dx'$

and $$\int_{u_0}^u dv \int_{-A}^{A} f(x') \cos v(x'-x)\,dx',$$

where $u > u_0 > 0$, is less than $\epsilon \int_{u_0}^u \frac{dv}{v}$, provided A is sufficiently large, where ϵ is arbitrarily chosen; therefore

$$\int_{u_0}^u dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\,dx' = \lim_{A\sim\infty} \int_{u_0}^u dv \int_{-A}^{A} f(x') \cos v(x'-x)\,dx'$$

$$= \int_{-\infty}^{\infty} dx' \int_{u_0}^u f(x') \cos v(x'-x)\,dv.$$

For all values of u_0 (> 0), in an interval $(0, a)$, $\left|\int_{u_0}^u f(x') \cos v(x'-x)\,dv\right|$

is less than $u\,|f(x')|$, which is a summable function of x' in $(-\infty, \infty)$; by the theorem of § 225,

$$\lim_{u_0 \sim 0} \int_{-\infty}^{\infty} dx' \int_{u_0}^{u} f(x') \cos v(x'-x)\, dv = \int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\, dv.$$

Hence, we have

$$\int_0^u dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\, dx' = \int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\, dv,$$

from which we have

$$\int_0^{\infty} dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\, dx' = \lim_{u \sim \infty} \int_{-\infty}^{\infty} dx' \int_0^u f(x') \cos v(x'-x)\, dv,$$

and thus the repeated integral on the left-hand side has the value

$$\frac{\pi}{2}\{f(x+0)+f(x-0)\},$$

provided $f(x)$ satisfies a sufficient condition in the neighbourhood of the point x.

(3) Let it be assumed that $f(x)$ converges to zero, as $x \sim \infty$, and as $x \sim -\infty$, and that, for some value of A, in (A, ∞), $(-\infty, -A)$ it has a differential coefficient $f'(x)$ which is absolutely summable in these intervals, and of which $f(x)$ is an indefinite integral.

The total variation of $f(x)$ in the interval (A, ∞) is (see I, § 415) $\int_A^{\infty} |f'(x)|\, dx$, and is therefore finite. Thus the conditions of (2) are satisfied.

It has now been proved that:

If $f(x)$ be summable in every finite interval, then

$$\frac{1}{\pi}\int_0^{\infty} dv \int_{-\infty}^{\infty} f(x') \cos v(x'-x)\, dx'$$

has the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point x, if a sufficient condition for the convergence of Fourier's series is satisfied; provided one of the following additional conditions be satisfied:

(1), *$f(x)$ is absolutely summable in $(-\infty, \infty)$.*

(2), *$f(x)$ converges to zero, as $x \sim \infty$, or $x \sim -\infty$, and there exists a positive number A such that $f(x)$ has bounded variation in the intervals (A, ∞), $(-\infty, -A)$.*

(3), *$f(x)$ converges to zero, as $x \sim \infty$, or $x \sim -\infty$, and a positive number A exists such that, in (A, ∞) and in $(-\infty, -A)$, $f(x)$ has a differential coefficient $f'(x)$ which is absolutely summable in these intervals, and of which $f(x)$ is an indefinite integral.*

474. If we assume that, outside the interval (α, β), the value of $f(x)$ is zero, we see that

$$\frac{1}{\pi}\int_0^{\infty} dv \int_{\alpha}^{\beta} f(x') \cos v(x'-x)\, dx'$$

has the value $\quad \frac{1}{2}\{f(x+0)+f(x-0)\},$

at an interior point of the interval (α, β), provided one of the sufficient conditions is satisfied in the neighbourhood of the point x. At the point α it has the value $\frac{1}{2}f(\alpha+0)$, and at the point β the value $\frac{1}{2}f(\beta-0)$, provided $f(x)$ satisfies a sufficient condition in the neighbourhood of either point. If x is exterior to the interval (α, β), the value of the repeated integral is zero.

If we assume that $f(x)$ is zero for all negative values of x, we have

$$\frac{1}{\pi}\int_0^\infty dv \int_0^\infty f(x')\cos v(x'-x)\,dx' = f(x),$$

where $f(x)$ is taken to have the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point of ordinary discontinuity; it being assumed that the requisite conditions are satisfied. Also

$$\frac{1}{\pi}\int_0^\infty dv \int_0^\infty f(x')\cos v(x'+x)\,dx' = 0;$$

hence we have

$$\frac{2}{\pi}\int_0^\infty \cos vx\,dv \int_0^\infty f(x')\cos vx'dx' = f(x) \qquad \text{......(1)},$$

$$\frac{2}{\pi}\int_0^\infty \sin vx\,dv \int_0^\infty f(x')\sin vx'\,dx' = f(x) \qquad \text{......(2)}.$$

The expressions (1) and (2) are known as Fourier's cosine and sine integrals for the representation of a function. It is clear that (1) affords a representation, subject to the stated conditions, of an even function $f(x)$ in the interval $(-\infty, \infty)$; and (2) affords a representation of an odd function.

THE SUMMABILITY (ϕ) OF A FOURIER'S REPEATED INTEGRAL

475. The method of summability (ϕ), given in § 266, may be applied to $\int_0^\infty du \int_A^B f(x')\cos u(x'-x)\,dx'$. This integral will be, in accordance with the method, replaced by

$$\lim_{k\sim 0}\int_0^\infty \phi(ku)\,du \int_A^B f(x')\cos u(x'-x)\,dx',$$

where $\phi(u)$ is a function which satisfies the conditions laid down in § 266; and this limit may exist in cases in which the original integral is not convergent.

Denoting by $I(\xi)$ the integral $\int_0^\infty \phi(u)\cos\xi u\,du$, we may write

$$\int_0^\infty \phi(ku)\cos u(x'-x)\,du$$

in the form $\frac{1}{k}I\left(\frac{x'-x}{k}\right)$, or $nI\{n(x'-x)\}$, where $n=\frac{1}{k}$.

Since $\phi(ku) f(x')$ is absolutely summable in the domain $(0, A; \infty, B)$, $\phi(ku) f(x') \cos u(x'-x)$ is absolutely summable, and therefore (see § 240) we have

$$\int_0^\infty \phi(ku)\,du \int_A^B f(x') \cos u(x'-x)\,dx' = \int_A^B nI\{n(x'-x)\} f(x')\,dx'.$$

We have therefore to investigate the value of

$$\lim_{n\sim\infty} \int_A^B f(x') F(x'-x, n)\,dx',$$

where $F(t, n) = nI(nt)$; and this may be done by means of the general convergence theorem of § 279. We first shew that the conditions (1) and (2) of that theorem are satisfied.

We have $\quad F(x'-x, n) = n\int_0^\infty \phi(u) \cos nu(x'-x)\,du;$

assuming that $|x'-x| \geqq \mu$, the expression on the right-hand side is $\frac{1}{x'-x}\int_0^\infty \phi(u)\,n(x'-x) \cos nu(x'-x)\,du$; and, in virtue of a theorem given in § 335, Ex. (2), the conditions of which are satisfied by $\phi(u)$, we have

$$|F(x'-x, n)| < \frac{K}{\mu},$$

where K is a fixed number independent of n and $x'-x$; thus the condition (1) is satisfied. In order to shew that the condition (2) is satisfied, we have

$$\int_\alpha^\beta F(x'-x, n)\,dx' = \int_\alpha^\beta n\,dx' \int_0^\infty \phi(u) \cos nu(x'-x)\,du$$
$$= \int_0^\infty \phi(u)\left[\frac{\sin nu(\beta-x) - \sin nu(\alpha-x)}{u}\right] du.$$

Since $\int_0^\infty \phi(u)\frac{\sin \lambda u}{u}\,du$ converges to $\frac{\pi}{2}\phi(+0)$, as $\lambda \sim \infty$, we have

$$\left|\int_0^\infty \phi(u)\frac{\sin \lambda u}{u}\,du - \frac{\pi}{2}\phi(+0)\right| < \epsilon,$$

provided $\lambda \geqq \lambda_\epsilon$, hence $\left|\int_\alpha^\beta F(x'-x, n)\,dx'\right| < 2\epsilon$, provided $n \geqq n_\epsilon$, some number dependent on ϵ, for all values of x in a finite interval which has no points in common with the interval $(A-\mu, B+\mu)$. Thus the condition (2) of the theorem of § 279 is satisfied. It follows that

$$\lim_{k\sim 0} \int_0^\infty \phi(ku)\,du \int_A^B f(x') \cos u(x'-x)\,dx' = 0,$$

for all points x not in the closed interval (A, B); moreover, the convergence is uniform for all points x in a finite interval exterior to (A, B).

It will now be shewn that the conditions (a) and (b) of the theorem in § 292 are satisfied.

We have

$$\int_0^\mu F(t, n)\,dt = \int_0^\mu dt \int_0^\infty n\phi(u)\cos nut\,du = \int_0^\infty \phi(u)\frac{\sin nu\mu}{u}\,du;$$

and the integral on the right-hand side converges to $\frac{\pi}{2}\phi(+0)$; therefore the condition (*a*) of § 292 is satisfied.

Let us next consider $\int_0^\mu |F(t, n)|\,dt$, which is

$$\int_0^\mu n\left|\int_0^\infty \phi(u)\cos ntu\,du\right|dt, \text{ or } \int_0^{n\mu}\left|\int_0^\infty \phi(u)\cos tu\,du\right|dt.$$

It can be shewn that $\int_0^\infty\left|\int_0^\infty \phi(u)\cos tu\,du\right|dt$ exists as a definite number; it then follows that $\int_0^\mu |F(t, n)|\,dt$ is bounded with respect to (μ, n), and thus that the condition (*b*) of § 292 is satisfied.

Using the properties of the function $\phi(u)$, we see, by two integrations by parts, that $\int_0^\infty \phi(u)\cos tu\,du = \frac{1}{t^2}\int_0^\infty \phi''(u)(1 - \cos ut)\,du$. Since $\phi''(u)$ is positive for sufficiently large values of u, we see that, for all values of t, the integral on the left-hand side is numerically less than a fixed multiple of $\frac{1}{t^2}$; moreover it is a continuous function of t, therefore it is absolutely summable in the interval $(0, \infty)$ of t.

We have now established the following theorem:

If $\phi(u)$ satisfies the conditions laid down in § 266, then

$$\lim_{k\sim 0}\frac{1}{\pi}\int_0^\infty \phi(ku)\,du\int_A^B f(x')\cos u(x' - x)\,dx'$$

has the value zero, if x is exterior to the interval (A, B); at any point interior to (A, B), at which $f(x + 0)$, $f(x - 0)$ exist, it has the value

$$\tfrac{1}{2}\{f(x + 0) + f(x - 0)\}.$$

In any interval interior to (A, B) in which $f(x)$ is continuous, the continuity at the end-points being on both sides, the convergence to $f(x)$ is uniform. At A and B it has the values $\frac{1}{2}f(A + 0)$, $\frac{1}{2}f(B - 0)$ provided these limits exist.

476. We proceed to the extension of the last theorem to the case in which A and B are replaced by $-\infty$, ∞ respectively.

(1) Let it be assumed that $f(x)$ is absolutely summable in the indefinitely great interval $(-\infty, \infty)$. As in § 475, $\phi(ku) f(x')\cos u(x' - x)$

is absolutely summable in the domain $(0, -\infty; \infty, \infty)$, and we have accordingly

$$\int_0^\infty \phi(ku)\,du \int_{-\infty}^\infty f(x') \cos u(x'-x)\,dx' = \int_{-\infty}^\infty f(x') F(x'-x, n)\,dx',$$

where $F(t, n)$ denotes $nI(nt)$, or $n\int_0^\infty \phi(u)\cos ntu\,du$, which may also be expressed as $\int_0^\infty \phi\left(\frac{u}{n}\right)\cos tu\,du$.

By the theorem of § 280, since $|f(x')|$ is absolutely summable, and the other conditions of the theorem are satisfied, we see that

$$\int_0^\infty \phi(ku)\,du \int_B^\infty f(x')\cos u(x'-x)\,dx',$$

and the corresponding expression in which B, ∞ are replaced by $-\infty, A$, converge to zero, as $k \sim 0$, provided x is interior to the interval (A, B). Moreover, the convergence is uniform for all points x in an interval interior to (A, B). It then follows that

$$\lim_{k\sim 0}\int_0^\infty \phi(ku) \int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

has the properties stated in the last theorem for the case in which the limits of the integration with respect to x' are finite.

(2) Let it be assumed that A and B can be so chosen that, in $(-\infty, A)$ and in (B, ∞) the function $f(x)$ has bounded variation and converges to 0, as $x \sim \infty$, and as $x \sim -\infty$; or in particular that $f(x)$ decreases steadily to zero, as x increases from B to ∞, and as x decreases from A to $-\infty$. It is sufficient to consider this special case, since, in the general case, $f(x)$ is representable as the difference of two functions, each of which has this property.

Let us consider

$$\int_0^\infty \phi(ku)\,du \int_B^\infty f(x')\cos u(x'-x)\,dx',$$

where $f(x')$ is non-increasing in the interval (B, ∞) and converges to zero, as $x' \sim \infty$. The integral with respect to u will be divided into two parts, taken over the intervals $(0, u_1)$ and (u_1, ∞); where, for $0 < k \leq 1$, u_1 is so chosen that $\phi(ku)$ is monotone in the interval $(0, u_1)$.

Taking first the integral

$$\int_{u_1}^\infty \phi(ku)\,du \int_B^\infty f(x')\cos u(x'-x)\,dx';$$

since
$$\left|\int_B^\beta f(x')\cos u(x'-x)\,dx'\right| < \frac{2f(B)}{u}.$$

for all values of β ($> B$); and since $\dfrac{\phi(ku)}{u}$ is summable in the interval (u_1, ∞), we have

$$\int_{u_1}^{\infty} \phi(ku)\,du \int_B^{\infty} f(x')\cos u(x'-x)\,dx'$$
$$= \lim_{\beta\sim\infty} \int_{u_1}^{\infty} \phi(ku) \int_B^{\beta} f(x')\cos u(x'-x)\,dx'$$
$$= \int_B^{\infty} f(x')\,dx' \int_{u_1}^{\infty} \phi(ku)\cos u(x'-x)\,du.$$

The inversion of the order of integration in the last step is justified, since $f(x')\,\phi(ku)\cos u(x'-x)$ is absolutely summable over the domain $(B, u_1;\, \beta, \infty)$, of (x, u).

The integral on the right-hand side may be written, by taking $x' = x + \xi k$, $u = \dfrac{v}{k}$, in the form

$$\int_{\frac{B-x}{k}}^{\infty} f(x+\xi k)\,d\xi \int_{ku_1}^{\infty} \phi(v)\cos \xi v\,dv,$$

which is less than $\quad f(B)\displaystyle\int_{\frac{B-x}{k}}^{\infty} \left| \int_0^{\infty} \phi(v)\cos\xi v\,dv \right| d\xi,$

and this converges to zero, as $k \sim 0$, and uniformly for all values of x in a fixed finite interval not abutting on the interval (B, ∞). Moreover

$$\left| \int_{u_1}^{\infty} \phi(ku)\,du \int_B^{\infty} f(x')\cos u(x'-x)\,dx' \right| < \eta,$$

for all sufficiently large values of B, and for all values of k.

Next, we have

$$\int_0^{u_1} \phi(ku)\,du \int_B^{\infty} f(x')\cos u(x'-x)\,dx'$$
$$= \phi(0)\int_0^{u'} du \int_B^{\infty} f(x')\cos u(x'-x)\,dx'$$
$$+ \phi(u_1)\int_{u'}^{u_1} du \int_B^{\infty} f(x')\cos u(x'-x)\,dx',$$

where u' is a number in the interval $(0, u_1)$. It will be shewn that the order of integration in the repeated integrals on the right-hand side may be reversed.

We have, as before, if $0 < \epsilon < u_1$,

$$\int_{\epsilon}^{u_1} du \int_B^{\infty} f(x')\cos u(x'-x)\,du = \int_B^{\infty} f(x') \int_{\epsilon}^{u_1} \cos u(x'-x)\,dx'$$
$$= \int_B^{\infty} f(x') \left[\frac{\sin u_1(x'-x) - \sin\epsilon(x'-x)}{x'-x} \right] dx'.$$

Also

$$\int_B^\infty f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx' = \left[\int_B^{B_1}+\int_{B_1}^\infty\right] f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx'$$

and

$$\left|\int_{B_1}^\infty f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx'\right| = \left|f(B_1)\int_{B_1}^{\beta'}\frac{\sin\epsilon(x'-x)}{x'-x}dx'\right|$$

$$= \left|f(B_1)\int_{\epsilon(B_1-x)}^{\epsilon(\beta'-x)}\frac{\sin t}{t}dt\right| < \pi f(B_1),$$

where $B_1 \leqq \beta'$. Since B_1 may be taken to be such that $f(B_1)$ is arbitrarily small, we see that, for all positive values of ϵ, and for all values of x in an interval that does not abut upon the interval (B, ∞), we have

$$\left|\int_B^\infty f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx' - \int_B^{B_1} f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx'\right| < \eta$$

for a fixed value of B_1, and for all positive values of ϵ. Let $\epsilon \sim 0$, then we have $\overline{\lim}_{\epsilon\sim 0}\int_B^\infty f(x')\frac{\sin\epsilon(x'-x)}{x'-x}dx'$ numerically not greater than the arbitrary number η; and thus the common value of the limits is zero. It follows that

$$\int_0^{u_1}du\int_B^\infty f(x')\cos u(x'-x)\,dx'$$

exists as
$$\lim_{\epsilon\sim 0}\int_\epsilon^{u_1}du\int_B^\infty f(x')\cos u(x'-x)\,dx',$$

and is equal to
$$\int_B^\infty f(x')\frac{\sin u_1(x'-x)}{x'-x}dx'.$$

We now see that

$$\int_0^{u_1}\phi(ku)\,du\int_B^\infty f(x')\cos u(x'-x)\,dx' = \phi(0)\int_B^\infty \frac{\sin u'(x'-x)}{x'-x}f(x')\,dx'$$
$$+\phi(u_1)\int_B^\infty\frac{\sin u_1(x'-x)-\sin u'(x'-x)}{x'-x}f(x')\,dx'.$$

Now
$$\int_B^\infty\frac{\sin u'(x'-x)}{x'-x}f(x')\,dx' = f(B)\int_B^{B}\frac{\sin u'(x'-x)}{x'-x}dx',$$

hence the integral on the left-hand side is numerically less than $\pi f(B)$, which can be made arbitrarily small, by taking B large enough. The same remark applies to the other integrals.

It follows that

$$\int_0^{u_1}\phi(ku)\,du\int_B^\infty f(x')\cos u(x'-x)\,dx'$$

is numerically less than the arbitrarily chosen positive number η, provided B is chosen sufficiently large; and this for all values of k such that $0 < k \leqq 1$,

and for all values of x in an interval which does not abut on the point B. It has now been shewn that, if B is sufficiently large,

$$\left|\int_0^\infty \phi(ku)\,du \int_B^\infty f(x')\cos u(x'-x)\,dx'\right| < 2\eta.$$

If $|A|$ be sufficiently large, a similar statement may be made as regards the expression in which the integration with respect to x' is over the interval $(-\infty, A)$.

The numbers A and B having been fixed so that these conditions are satisfied, we see that

$$\int_0^\infty \phi(ku)\,du \int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

differs from
$$\int_0^\infty \phi(ku)\,du \int_A^B f(x')\cos u(x'-x)\,dx'$$

by less than 4η, where A and B are properly chosen; and this for all values of k such that $0 < k \leqq 1$.

As the second expression satisfies the conditions of the theorem of § 475, it follows that

$$\lim_{k\sim 0}\frac{1}{\pi}\int_0^\infty \phi(ku)\int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

differs from $\frac{1}{2}\{f(x+0)+f(x-0)\}$ by less than 4η, at any point at which $f(x+0)$ and $f(x-0)$ exist. Since η is arbitrary, we now obtain the following theorem:

If $\phi(u)$ satisfies the conditions in § 266, and if either (1), *$f(x)$ is absolutely summable in the interval $(-\infty, \infty)$, or* (2), *$f(x)$ is summable in every finite interval, converges to 0, as $x \sim \infty$, and as $x \sim -\infty$, and has bounded variation in intervals (B, ∞), $(-\infty, A)$, where A, B are properly chosen numbers, then*

$$\lim_{k\sim 0}\frac{1}{\pi}\int_0^\infty \phi(ku)\,du \int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

has the value $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at any point at which $f(x+0)$, $f(x-0)$ exist. Moreover, the convergence to $f(x)$ is uniform in any finite interval in which $f(x)$ is continuous, the continuity at the end-points of the interval being on both sides.

This theorem was given* by Hardy, who obtained a more general result applicable to the case in which $f(x')$ is replaced by $f(x')\cos ax'$.

The cases in which $\phi(u) = e^{-u}$, $\phi(u) = e^{-u^2}$ are of importance in problems of Mathematical Physics. We obtain in fact the following theorem:

If either (1), *$f(x)$ is absolutely summable in the interval $(-\infty, \infty)$, or* (2), *$f(x)$ is summable in every finite interval, and has bounded variation in*

* *Camb. Phil. Trans.* vol. XXI (1912), p. 427.

the intervals (B, ∞), $(-\infty, A)$, *where* A, B *are properly chosen, and converges to zero as* $x \sim \infty$, *and as* $x \sim -\infty$, *then*

$$\frac{1}{\pi}\int_0^\infty e^{-ku}\,du\int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

and

$$\frac{1}{\pi}\int_0^\infty e^{-ku^2}\,du\int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

both converge, as $k \sim 0$, *to the value* $\frac{1}{2}\{f(x+0)+f(x-0)\}$, *at any point* x *at which* $f(x+0)$, $f(x-0)$ *both exist. They converge to* $f(x)$ *uniformly in any finite interval in which* $f(x)$ *is continuous, the continuity at the end-points of the interval being on both sides.*

This theorem was given* by Sommerfeld for the case in which the integration with respect to x' is over a finite interval.

477. Let the function $f(x')$, assumed to satisfy one of the conditions of the theorem in § 476, be expressed as the sum of two functions $f_1(x')$ and $f_2(x')$, where $f_1(x')$ has the value $f(x')$ in the interval $(x-\mu,\ x+\mu)$, for a fixed value of x, and has the value zero outside that interval. The function $f_2(x')$ has the value zero in the interval $(x-\mu,\ x+\mu)$, and the value $f(x')$ outside that interval. Applying the theorem of § 476, it is seen that

$$\lim_{k\sim 0}\frac{1}{\pi}\int_0^\infty \phi(ku)\,du\int_{-\infty}^\infty f_2(x')\cos u(x'-x)\,dx'$$

has the value zero at all points interior to the interval $(x-\mu,\ x+\mu)$. It thus appears that

$$\frac{1}{\pi}\int_0^\infty \phi(ku)\,du\int_{-\infty}^\infty f(x')\cos u(x'-x)\,dx'$$

will converge to $f(x)$ if

$$\frac{1}{\pi}\int_0^\infty \phi(ku)\,du\int_{x-\mu}^{x+\mu} f(x')\cos u(x'-x)\,dx'$$

converges to $f(x)$.

The condition that this should be the case is that

$$\lim_{k\sim 0}\int_0^\infty \phi(ku)\,du\int_0^\mu \psi(t)\cos ut\,dt = 0,$$

where $\psi(t)$ denotes $f(x+t)+f(x-t)-2f(x)$.

Let it now be assumed that, for the point x,

$$\lim_{t\sim 0}\frac{1}{t}\int_0^t |\psi(t)|\,dt = 0,$$

from which it follows that, if ϵ be an arbitrarily chosen positive number,

$$\int_0^t |\psi(t)|\,dt < \epsilon t,\ \text{ for }\ 0 \leqq t \leqq t_1,$$

where t_1 depends upon ϵ.

* See his Dissertation, *Die willkürlichen Funktionen in der Math. Physik*, Königsberg, 1901. See also Hardy, *Camb. Phil. Trans.* vol. XXI, p. 39.

We have

$$\int_0^\infty \phi(ku)\,du \int_0^\mu \psi(t)\cos ut\,dt = \int_0^{n\mu} \psi\left(\frac{t}{n}\right) dt \int_0^\infty \phi(u)\cos ut\,du,$$

where $k = \frac{1}{n}$; this may be expressed in the form

$$\int_0^1 \psi\left(\frac{t}{n}\right) dt \int_0^\infty \phi(u)\cos ut\,du + \int_1^{n\mu} \psi\left(\frac{t}{n}\right) dt \int_0^\infty \phi(u)\cos ut\,du.$$

The first repeated integral is numerically less than

$$n\int_0^\infty |\,\phi(u)\,|\,du \int_0^{\frac{1}{n}} |\,\psi(t)\,|\,dt,$$

which is less than $\epsilon \int_0^\infty |\,\phi(u)\,|\,du$, provided $n > \frac{1}{t_1}$.

The second repeated integral is numerically less than

$$k'\int_1^{n\mu} \left|\psi\left(\frac{t}{n}\right)\right| \frac{dt}{t^2},$$

where k' is a fixed number, and this is equivalent to

$$\frac{k'}{n}\int_{\frac{1}{n}}^{\mu} |\,\psi(t)\,|\,\frac{dt}{t^2};$$

denoting $\int_0^t |\,\psi(t)\,|\,dt$ by $\Phi(t)$, we have, integrating by parts,

$$\frac{k'}{n}\int_{\frac{1}{n}}^{\mu} |\,\psi(t)\,|\,\frac{dt}{t^2} = \frac{k'}{n}\left[\frac{\Phi(\mu)}{\mu^2} - n^2\Phi\left(\frac{1}{n}\right) + 2\int_{\frac{1}{n}}^{\mu} \frac{\Phi(t)}{t^3}\,dt\right].$$

We have $n\Phi\left(\frac{1}{n}\right) < \epsilon$, if $n > \frac{1}{t_1}$; also

$$\int_{\frac{1}{n}}^{\mu} \frac{\Phi(t)}{t^3}\,dt = \int_{\frac{1}{n}}^{t_1} \frac{\Phi(t)}{t^3}\,dt + \int_{t_1}^{\mu} \frac{\Phi(t)}{t^3}.$$

The first integral on the right-hand side is less than $\epsilon\int_{\frac{1}{n}}^{t_1} \frac{dt}{t^2}$, or than $n\epsilon$, and the second is less than $\frac{1}{2t_1^2}\Phi(\mu)$; it follows that $\frac{1}{n}\int_{\frac{1}{n}}^{\mu} \frac{\Phi(t)}{t^3}\,dt$ is numerically less than $\epsilon + \frac{1}{2t_1^2 n}\Phi(\mu)$, or than 2ϵ, provided n be sufficiently large.

It has now been shewn that, if n be sufficiently large, or k sufficiently small, $\int_0^\infty \phi(ku)\,du \int_0^\mu \psi(t)\cos ut\,dt$ is numerically less than a fixed multiple of ϵ, provided $\lim\limits_{t\sim 0} \frac{1}{t}\int_0^t |\,\psi(t)\,|\,dt = 0$. Since ϵ is arbitrarily small, the limit, as $k \sim 0$, of the repeated integral is zero. The following theorem has thus been established:

The integral $\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(x') \cos u(x'-x)\, dx'$ *is summable* (ϕ), *and has* $f(x)$ *for its sum* (ϕ), *at any point at which*

$$\lim_{t\sim 0} \frac{1}{t}\int_0^t |f(x+t)+f(x-t)-2f(x)|\, dt = 0.$$

This is the case almost everywhere in the interval $(-\infty, \infty)$. *It is assumed that* $f(x)$ *satisfies one or other of the conditions laid down in the last theorem.*

In particular

$$\lim_{k\sim 0} \frac{1}{\pi}\int_0^\infty e^{-ku} \int_{-\infty}^\infty f(x') \cos u(x'-x)\, dx' = f(x),$$

$$\lim_{k\sim 0} \frac{1}{\pi}\int_0^\infty e^{-ku^2} \int_{-\infty}^\infty f(x') \cos u(x'-x)\, dx' = f(x)$$

almost everywhere in the interval $(-\infty, \infty)$.

This theorem includes that of § 476, since

$$\lim_{t\sim 0} \frac{1}{t}\int_0^t |f(x+t)+f(x-t)-2f(x)|\, dt = 0$$

at any point x at which $f(x+0)$, $f(x-0)$ exist, provided

$$f(x) = \tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

THE SUMMABILITY (C, r) OF FOURIER'S REPEATED INTEGRAL

478. The sum (C, r) of the integral $\int_0^\infty \psi(u)\, du$ has been defined in § 264 as the limit of $\int_0^n \left(1-\frac{u}{n}\right)^r \psi(u)\, du$, as the number n diverges through continuous values to ∞, whenever that limit exists.

We shall accordingly consider the integral

$$\int_0^n \left(1-\frac{u}{n}\right)^r du \int_A^B f(x') \cos u(x'-x)\, dx',$$

where $r > 0$.

The order of the successive integrations may be reversed, so that we have to consider $\int_A^B f(x') F(x'-x, n)\, dx'$, where $F(x'-x, n)$ denotes $\int_0^n \left(1-\frac{u}{n}\right)^r \cos u(x'-x)\, du$. In order to evaluate the limit, as $n \sim \infty$, of the integral, the theorem of § 290 may be applied.

On integration by parts, we have

$$F(t, n) = \int_0^n \frac{r}{n}\left(1-\frac{u}{n}\right)^{r-1} \frac{\sin ut}{t}\, du$$

$$= \frac{r}{t}\int_0^1 (1-u)^{r-1} \sin nut\, du.$$

Hence $\mid F(t, n) \mid < \frac{r}{t}\int_0^1 (1-u)^{r-1}\,du < \frac{1}{t}$. Thus if $t \geqq \mu$, $\mid F(t, n) \mid$ is bounded with respect to (t, n); therefore the condition (1) of the theorem is satisfied.

To shew that the condition (2) is satisfied, we have

$$\int_\alpha^\beta F(t, n)\,dt = \int_\alpha^\beta dt \int_0^n \left(1 - \frac{u}{n}\right)^r \cos ut\,du = \int_0^n \left(1 - \frac{u}{n}\right)^r \frac{\sin \beta u - \sin \alpha u}{u}\,du$$
$$= \int_0^1 (1-u)^r \frac{\sin n\beta u - \sin n\alpha u}{u}\,du.$$

Since $\displaystyle\lim_{\gamma \sim \infty} \int_0^1 (1-u)^r \frac{\sin \gamma u}{u}\,du = \frac{\pi}{2}$, $\displaystyle\int_\alpha^\beta F(t, n)\,du$

converges to zero, as $n \sim \infty$, uniformly for all values of x that are not in the interval $(\alpha - \mu, \beta + \mu)$, as $n \sim \infty$; where α and β are both $> x + \mu$, or both $< x - \mu$. Hence the condition (2) is satisfied.

To shew that the condition (a) of § 292 is satisfied, we have

$$\int_0^\mu F(t, n)\,dt = \int_0^1 (1-u)^r \frac{\sin nu\mu}{u}\,du;$$

and this converges to $\frac{1}{2}\pi$, as $n \sim \infty$.

We have also

$$\int_0^\mu \mid F(t, n) \mid dt < \int_0^\mu n\,dt \left| \int_0^1 (1-u)^r \cos nut\,du \right|;$$

the expression on the right-hand side is equivalent to

$$\int_0^{\mu n} dt \left| \int_0^1 (1-u)^r \cos ut\,du \right|,$$

which is less than

$$\int_0^1 dt \left| \int_0^1 (1-u)^r \cos ut\,du \right| + \int_1^\infty dt \left| \int_0^1 (1-u)^r \cos ut\,du \right|.$$

The first term is less than $\frac{1}{r+1}$, and the second is less than $K\int_1^\infty \frac{dt}{t^{r+1}}$, or than $\frac{K}{r}$, where K is a fixed number (see § 371); it is here assumed that $r \leqq 1$. If $r > 1$, the second integral is less than $K\int_1^\infty \frac{dt}{t^2}$, or than K. It thus follows that $\int_0^\mu \mid F(t, n) \mid dt$ is less than a number which is independent of μ and n; hence the condition (b) of the theorem in § 292 is satisfied.

It now follows that $\displaystyle\int_A^B f(x')\,dx' \int_0^n \left(1 - \frac{u}{n}\right)^r \cos u(x' - x)\,dx'$ converges, as $n \sim \infty$, to $\frac{1}{2}\{f(x+0) + f(x-0)\}$, at any point x at which $f(x+0)$ and $f(x-0)$ exist. Moreover, the convergence to $f(x)$ is uniform in any interval interior to (A, B) in which the function is continuous, the continuity at the end-points being on both sides.

It has now been proved that

The integral $\dfrac{1}{\pi}\displaystyle\int_0^\infty du \int_A^B f(x')\cos u(x'-x)\,dx'$ *is summable* (C, r), *where* $r > 0$, *at any point* x, *within* (A, B) *at which* $f(x+0), f(x-0)$ *exist, the sum* (C, r) *being* $\frac{1}{2}\{f(x+0)+f(x-0)\}$. *At any point* x *exterior to* (A, B) *the sum* (C, r) *is zero.*

479. If ξ is any point within the interval (A, B), we may divide $f(x')$ into the sum of two functions $f_1(x')$, $f_2(x')$, such that $f_1(x') = f(x')$ in the interval $(\xi-\mu, \xi+\mu)$ and is elsewhere zero. The function $f_2(x')$ is zero in the interval $(\xi-\mu, \xi+\mu)$, and has the value $f(x')$ elsewhere. By the theorem just established the sum (C, r) of

$$\int_0^\infty du \int_A^B f_2(x')\cos u(x'-x)\,dx'$$

is zero at all interior points of the interval $(\xi-\mu, \xi+\mu)$. Thus the summability (C, r) of $\dfrac{1}{\pi}\displaystyle\int_0^\infty du \int_A^B f(x')\cos u(x'-x)\,dx'$ at the point ξ depends upon that of $\dfrac{1}{\pi}\displaystyle\int_0^\infty du \int_{\xi-\mu}^{\xi+\mu} f(x')\cos u(x'-x)\,dx'$.

In case $f(x')$ has the constant value $f(\xi)$ in the interval $(\xi-\mu, \xi+\mu)$, the sum (C, r) at ξ is $f(\xi)$. Thus the condition that the repeated integral is summable (C, r) at the point ξ, and has $f(\xi)$ for its sum (C, r), is that

$$\int_0^n \left(1-\frac{u}{n}\right)^r du \int_0^\mu [f(\xi+t)+f(\xi-t)-2f(\xi)]\cos ut\,dt$$

should converge to zero, as $n \sim \infty$.

The expression is equivalent to

$$\int_0^1 (1-u)^r du \int_0^{n\mu} \Phi\left(\frac{t}{n}\right)\cos ut\,dt,$$

where $\Phi(t)$ denotes $f(\xi+t)+f(\xi-t)-2f(\xi)$; and this is equivalent to

$$\int_0^{n\mu} \Phi\left(\frac{t}{n}\right) dt \int_0^1 (1-u)^r \cos ut\,dt;$$

and this may be expressed by

$$\int_0^1 \Phi\left(\frac{t}{n}\right) dt \int_0^1 (1-u)^r \cos ut\,du + \int_1^{n\mu} \Phi\left(\frac{t}{n}\right) dt \int_0^1 (1-u)^r \cos ut\,du.$$

The first of these integrals is numerically less than $\dfrac{1}{r+1}\displaystyle\int_0^1 \left|\Phi\left(\frac{t}{n}\right)\right| dt$; if now it be assumed that $\displaystyle\lim_{t\sim 0}\frac{1}{t}\int_0^t |\Phi(t)|\,dt = 0$, we have $\displaystyle\int_0^t |\Phi(t)|\,dt < \epsilon t$, provided t is sufficiently small; and so $\displaystyle\int_0^1 \left|\Phi\left(\frac{t}{n}\right)\right| dt = n\int_0^{\frac{1}{n}} |\Phi(t)|\,dt < \epsilon$, if n is sufficiently large. Thus the first integral is $< \epsilon$, for all sufficiently large values of n.

The second integral is less than $K\int_1^{n\mu}\left|\Phi\left(\frac{t}{n}\right)\right|\frac{dt}{t^{r+1}}$, where K is a fixed number (see § 371), or than $\frac{K}{n^r}\int_{\frac{1}{n}}^{\mu}|\Phi(t)|\frac{dt}{t^{r+1}}$, where $r \leqq 1$, and by integration by parts this becomes

$$\frac{K}{n^r}\left\{\left[\frac{1}{t^{r+1}}\int_0^t|\Phi(t)|\,dt\right]_{\frac{1}{n}}^{\mu}+(r+1)\int_{\frac{1}{n}}^{\mu}\frac{1}{t^{r+2}}\int_0^t|\Phi(t)|\,dt\right\},$$

or

$$\frac{K}{n^r}\left\{\frac{1}{\mu^{r+1}}\Psi(\mu)-n^{r+1}\Psi\left(\frac{1}{n}\right)+(r+1)\int_{\frac{1}{n}}^{\mu}\frac{\Psi(t)}{t^{r+2}}\,dt\right\},$$

where $\Psi(t)=\int_0^t|\Phi(t)|\,dt$. Since $\Psi(t)<\epsilon t$, for $t<t_1$, we have

$$\int_{\frac{1}{n}}^{t_1}\frac{\Psi(t)}{t^{r+2}}\,dt<\epsilon\int_{\frac{1}{n}}^{t_1}\frac{dt}{t^{r+1}}\,dt<\frac{\epsilon}{r}\left(n^r-\frac{1}{t_1^{\,r}}\right),$$

and thus $\frac{1}{n^r}\int_{\frac{1}{n}}^{t_1}\frac{\Psi(t)}{t^{r+2}}\,dt<\frac{\epsilon}{r}$; also $\frac{1}{n^r}\int_{t_1}^{\mu}\frac{\Psi(t)}{t^{r+2}}\,dt$ is less than $\frac{\Psi(\mu)}{n^r}\cdot\frac{1}{r+1}\,\frac{1}{t_1^{\,r+1}}$ and this is arbitrarily small, for all sufficiently large values of n. It is thus seen that, provided n is sufficiently large, the second integral is less than an arbitrarily chosen positive number η. In case $r>1$, a slight change in the calculation is required.

It has now been proved that, provided

$$\int_0^t|f(\xi+t)+f(\xi-t)-2f(\xi)|\,dt$$

has, at $t=0$, a differential coefficient equal to zero,

$$\frac{1}{\pi}\int_0^n\left(1-\frac{u}{n}\right)^r du\int_A^B f(x')\cos u(x'-\xi)\,dx'$$

converges at the point ξ to $f(\xi)$. The condition is satisfied almost everywhere in (A, B).

It has thus been established that:

The sum (C, r), for $r>0$, of $\frac{1}{\pi}\int_0^\infty du\int_A^B f(x')\cos u(x'-\xi)\,dx'$ exists, and has the value $f(\xi)$, at any point ξ, interior to (A, B), at which

$$\frac{1}{t}\int_0^t|f(\xi+t)+f(\xi-t)-2f(\xi)|\,dt$$

converges to zero, with t; and this is the case almost everywhere in (A, B).

480. In the theorems of §§ 478, 479, A and B may be replaced by $-\infty, \infty$, provided either (1), $|f(x)|$ is summable in the interval $(-\infty, \infty)$, or (2), $f(x)$ converges to zero, as $x\sim\infty$, and as $x\sim-\infty$, and it is of bounded variation in neighbourhoods (B', ∞), $(-\infty, A')$ of these points.

In case (1), the expression

$$\int_0^n \left(1 - \frac{u}{n}\right)^r \int_B^\infty f(x') \cos u (x' - x)\, dx'$$

is numerically $< \eta$, if B be sufficiently large, where η is an arbitrarily chosen positive number; and the similar statement holds for

$$\int_0^n \left(1 - \frac{u}{n}\right)^r \int_{-\infty}^A f(x') \cos u (x' - x)\, dx'.$$

That this is the case follows from the fact that

$$\left| \left(1 - \frac{u}{n}\right)^r f(x') \cos u (x' - x) \right|$$

is summable in the domain $(u, B; n, \infty)$, exactly as in § 476.

In case (2), the proof in § 476 is applicable, if we take $\phi(u) = (1-u)^r$, for $0 \leqq u \leqq 1$, and $\phi(u) = 0$, for $u > 1$, to shew that the above integrals are numerically less than η, when B and $-A$ are sufficiently large.

At a point x within (A, B) at which the sum (C, r) exists, and is equal to $f(x)$, we have

$$\left| \frac{1}{\pi} \int_0^n \left(1 - \frac{u}{n}\right)^r \int_A^B f(x') \cos u (x' - x)\, dx' - f(x) \right| < \eta,$$

provided n is sufficiently large; hence

$$\left| \frac{1}{\pi} \int_0^n \left(1 - \frac{u}{n}\right)^r \int_{-\infty}^\infty f(x') \cos u (x' - x)\, dx' - f(x) \right| < 3\eta,$$

for all sufficiently large values of n. Since η is arbitrary, it follows that

$$\lim_{n \sim \infty} \frac{1}{\pi} \int_0^n \left(1 - \frac{u}{n}\right)^r \int_{-\infty}^\infty f(x') \cos u (x' - x)\, dx' = f(x).$$

The following theorem has now been established:

The sum (C, r), for $r > 0$, of $\frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(x') \cos u (x' - x)\, dx'$ exists, and has the value $f(x)$, at any point x, at which $\frac{1}{t} \int_0^t |f(x+t) + f(x-t) - 2f(x)|\, dt$ converges with t to zero. This holds at every point of continuity, at every point of ordinary discontinuity of the function at which

$$f(x) = \tfrac{1}{2}\{f(x+0) + f(x-0)\},$$

and almost everywhere in the whole interval $(-\infty, \infty)$. The convergence to $f(x)$ is uniform in any interval in which $f(x)$ is continuous, the continuity at the end-points being on both sides.

The theorem is subject to one or other of the conditions (1), *that $|f(x)|$ is summable in the interval $(-\infty, \infty)$, or* (2), *that $f(x)$ is summable in every finite interval, and converges to zero at ∞ and $-\infty$, and is of bounded variation in some neighbourhood of each of these points.*

FOURIER TRANSFORMS

481. It has been shewn in § 474 that, subject to certain conditions,

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty f(x')\cos ux'\,dx'$$

in the interval $(0, \infty)$. This may be expressed by the two equations

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty F(u)\cos ux\,du \qquad \ldots\ldots(1),$$

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(u)\cos ux\,du \qquad \ldots\ldots(2),$$

which connect the two functions $f(x)$, $F(x)$. When (1) and (2) hold good they express symmetrical relations between the two functions $f(x)$, $F(x)$, and each of these functions may be said to be the *Fourier cosine transform* of the other. The repeated integral formula has been shewn to hold good when $|f(x)|$ is summable in the interval $(0, \infty)$, at a point x in the neighbourhood of which $f(x)$ satisfies one of a set of sufficient conditions; in particular when in such neighbourhood the function is of bounded variation, and $f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$. It also holds good when $f(x)$ is summable in every finite interval contained in $(0, \infty)$, converges to zero at ∞, and is of bounded variation in some neighbourhood (A, ∞) of the point ∞, provided a sufficient condition is satisfied in a neighbourhood of the point x. The formula holds for every point x, provided $f(x)$ is of bounded variation in the whole interval $(0, \infty)$, and $f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$, except that, at the point 0, $f(0) = \frac{1}{2}f(+0)$.

The integral in (2) exists, and is a continuous function of x, at any point $x > 0$, when $|f(x)|$ is summable in $(0, \infty)$, or when $f(x)$, summable in every finite interval, converges to zero at ∞, and is of bounded variation in some interval (A, ∞) (see Ex. (4), § 229). A complete theory of transforms should enable us to infer the properties of the function $F(x)$ from those of $f(x)$. This can be carried out in case $\{f(x)\}^2$ is summable in $(0, \infty)$, and more generally, when $|f(x)|^q$ is summable in $(0, \infty)$, for some value of q such that $1 < q \leq 2$. This theory has been given* by Titchmarsh, and is a particular case of a more general theory, due† to Plancherel, of transforms applicable to the case of orthogonal functions of any type. The theory as developed by Titchmarsh is independent of, and considerably simpler than, the general theory of Plancherel, which was applied by him to the case $q = 2$; and Titchmarsh's investigations form the basis of the account given below.

* *Proc. Camb. Phil. Soc.* vol. XXI (1923), p. 463; and *Proc. Lond. Math. Soc.* (2), vol. XXIII (1924), p. 279.

† *Rendiconti di Palermo*, vol. XXX (1910), p. 289; and *Math. Annalen*, vol. LXXVI (1915), p. 315.

If the theory of the repeated Fourier's integral, as given in §§ 473–480, be regarded from the point of view of the formulae (1), (2), there is complete formal symmetry, but as no resemblance has been made manifest between the properties of the two functions $f(x)$, $F(x)$, there is logical asymmetry. This is remedied, so far as the case permits, by the theory of Plancherel-Titchmarsh. In the case $q = 2$, the symmetry established by that theory is complete; when $1 < q < 2$, complete symmetry does not hold, but this lies in the nature of the case, and is consequently not a defect of the theory. Various formal resemblances will be exhibited in the results of this theory with corresponding theorems in the theory of Fourier's series.

In these cases the integrals (1), (2) do not in general exist, and it will be shewn that, instead of them, the formulae

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\int_0^\infty \frac{\sin ux}{u} F(u)\,du \qquad \text{......(1)}',$$

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\int_0^\infty \frac{\sin ux}{u} f(u)\,du \qquad \text{......(2)}'$$

must be substituted. The expressions (1)′, (2)′ reduce to (1), (2) whenever the differentiation under the integral sign can be effected. It will appear that (2)′ has a meaning for almost every value of x, and that the integrability of $|F(x)|^{\frac{q}{q-1}}$ follows as a consequence of that of $|f(x)|^q$, when $1 < q \leqq 2$. The whole theory is applicable to the corresponding Fourier sine transforms, in which the reciprocal relation is expressed by

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\int_0^\infty \frac{1 - \cos ux}{u} F(u)\,du,$$

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx}\int_0^\infty \frac{1 - \cos ux}{u} f(u)\,du.$$

482. Let us consider the integral $\int_a^b f(u)\cos ux\,du$, where $|f(u)|^q$ is summable in $(0, \infty)$, for some value of q such that $1 < q \leqq 2$; the numbers a and b will be taken to be such that $0 < a < b$.

The interval (a, b) may be divided into the parts

$$\left(a, \frac{m+1}{\lambda}\right), \quad \left(\frac{m+1}{\lambda}, \frac{m+2}{\lambda}\right) \dots \left(\frac{n}{\lambda}, \frac{n+1}{\lambda}\right), \quad \left(\frac{n+1}{\lambda}, b\right);$$

where m, n. and λ are positive integers, such that

$$\frac{m}{\lambda} < a < \frac{m+1}{\lambda}, \quad \frac{n+1}{\lambda} < b < \frac{n+2}{\lambda}.$$

Let h_s denote $\int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} f(u)\,du$; then, if ϕ_n denotes $\sum_{s=1}^{s=n} h_s \cos\frac{sx}{\lambda}$, we have

$$\int_a^b f(u)\cos ux\,du - (\phi_n - \phi_m) = \int_a^{\frac{m+1}{\lambda}} f(u)\cos ux\,du + \int_{\frac{n+1}{\lambda}}^b f(u)\cos ux\,du$$
$$+ \sum_{s=m+1}^{s=n} \int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} f(u)\left\{\cos xu - \cos\frac{sx}{\lambda}\right\} du.$$

The first and second integrals on the right-hand side are less numerically than the integrals of $|f(u)|$ in the intervals $\left(a, \frac{m+1}{\lambda}\right)$, $\left(\frac{n+1}{\lambda}, b\right)$, and therefore converge to zero, as $\lambda \sim \infty$, uniformly for all values of x.

The last expression on the right-hand side is less, numerically, than

$$2\sin\frac{x}{2\lambda}\,\Sigma\int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} |f(u)|\,du,$$

or than
$$2\sin\frac{x}{2\lambda}\int_a^b |f(u)|\,du,$$

and therefore converges, as $\lambda \sim \infty$, to zero, uniformly for all values of x in an interval (x_1, x_2), where $0 < x_1 < x_2$. It follows that

$$\int_a^b \cos ux\, f(u)\,du = \lim_{\lambda\sim\infty}(\phi_n - \phi_m),$$

the convergence being uniform for $x_1 \leq x \leq x_2$.

We have, employing a known inequality theorem (I, § 435),

$$|h_s|^q \leq \left[\int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} |f(u)|^q\,du\right]\left[\int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} dx\right]^{q-1} \leq \lambda^{1-q}\int_{\frac{s}{\lambda}}^{\frac{s+1}{\lambda}} |f(u)|^q\,du.$$

If we apply to the finite Fourier's series

$$\phi_n - \phi_m = \sum_{s=m+1}^{s=n} h_s\cos\frac{sx}{\lambda},$$

the Theorem II in § 392, we have

$$\int_0^{\pi\lambda} |\phi_n - \phi_m|^{\frac{q}{q-1}}\,dx = \lambda\int_0^{\pi}\left|\sum_{s=m+1}^{s=n} h_s\cos sx\right|^{\frac{q}{q-1}} dx \leq \tfrac{1}{2}\pi\lambda\left(\sum_{s=m+1}^{s=n}|h_s|^q\right)^{\frac{1}{q-1}}$$
$$\leq \tfrac{1}{2}\pi\left[\int_{\frac{m+1}{\lambda}}^{\frac{n+1}{\lambda}} |f(x)|^q\,dx\right]^{\frac{1}{q-1}}.$$

If $0 < x_1 < x_2 < \pi\lambda$, we have now,

$$\int_{x_1}^{x_2} |\phi_n - \phi_m|^{\frac{q}{q-1}}\,dx \leq \tfrac{1}{2}\pi\left[\int_a^b |f(x)|^q\right]^{\frac{1}{q-1}}.$$

As $\lambda \sim \infty$, we thus have, since $\phi_n - \phi_m$ converges uniformly to

$$\int_a^b f(u) \cos ux du,$$

$$\int_{x_1}^{x_2} \left[\int_a^b f(u) \cos ux du\right]^{\frac{q}{q-1}} dx \leqq \tfrac{1}{2}\pi \left[\int_a^b |f(x)|^q\right]^{\frac{1}{q-1}},$$

for $0 < x_1 < x_2$.

Letting $x_1 \sim 0$, $x_2 \sim \infty$, we now have

$$\int_0^\infty \left|\int_a^b f(u) \cos ux du\right|^{\frac{q}{q-1}} dx \leq \tfrac{1}{2}\pi \left[\int_a^b |f(x)|^q\right]^{\frac{1}{q-1}}.$$

If a and b diverge in any manner to ∞, we have

$$\lim_{a\sim\infty,\, b\sim\infty} \int_0^\infty \left|\int_a^b f(u) \cos ux du\right|^{\frac{q}{q-1}} dx = 0.$$

Let
$$F_a(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a f(u) \cos ux du;$$

we then have
$$\lim_{a\sim\infty,\, b\sim\infty} \int_0^\infty |F_b(x) - F_a(x)|^{\frac{q}{q-1}} dx = 0;$$

and thus $\{F_a(x)\}$ is convergent on the average, when a has the values in a divergent sequence. From an extension of the theorem in § 177 we infer* that there exists a function $F(x)$, defined uniquely, almost everywhere, such that $|F(x)|^{\frac{q}{q-1}}$ is summable in $(0, \infty)$, and that

$$\lim_{a\sim\infty} \int_0^\infty |F(x) - F_a(x)|^{\frac{q}{q-1}} dx = 0.$$

It will be seen that this function $F(x)$ is the transform of $f(x)$.

483. Since

$$\left|\int_0^\infty g(x)\{F(x) - F_a(x)\} dx\right| \leq \left[\int_0^\infty |g(x)|^q dx\right]^{\frac{1}{q}} \left[\int_0^\infty |F(x) - F_a(x)|^{\frac{q}{q-1}}\right]^{\frac{q-1}{q}},$$

where $g(x)$ is any function such that $|g(x)|^q$ is summable in $(0, \infty)$, it follows that

$$\lim_{a\sim\infty} \int_0^\infty g(x)\{F(x) - F_a(x)\} dx = 0.$$

Taking x to have any finite value, we have

$$\lim_{a\sim\infty} \int_0^x \{F(t) - F_a(t)\} dt = 0;$$

and since

$$\int_0^x F_a(t) dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x dt \int_0^a f(u) \cos ut du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a \frac{\sin ux}{u} f(u) du,$$

we have
$$\int_0^x F(t) dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin ux}{u} f(u) du;$$

* It was assumed by Titchmarsh that F. Riesz's theorem in § 177 holds when the interval of integration is infinite. In the case $q = 2$ this has been proved in §§ 171, 172; by a modification of the proof there given, the general theorem can be established for values of $q > 1$.

and therefore, almost everywhere

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin ux}{u} f(u)\, du.$$

It has now been shewn that:

If $|f(x)|^q$, *where* $1 < q \leq 2$, *is summable in* $(0, \infty)$, *the function*

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin ux}{u} f(u)\, du$$

is such that $|F(x)|^{\frac{q}{q-1}}$ *is integrable over* $(0, \infty)$.

This theorem is the analogue of Parseval's theorem (§ 378).

Denoting $\int_0^x f(t)\, dt$ by $\phi(x)$, the continuous function $\phi(x)$ has bounded variation in the finite interval $(0, a)$; and thus

$$\phi(x) = \frac{2}{\pi} \int_0^\infty \sin ux\, du \int_0^a \sin ux' \phi(x')\, dx',$$

provided $0 < a$. Since

$$\int_0^a \sin ux' \,.\, \phi(x')\, dx' = -\frac{\cos ua}{u} \phi(a) + \int_0^a \frac{\cos ux'}{u} f(x')\, dx',$$

we have
$$\phi(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin ux}{u} du \int_0^a f(x') \cos ux' dx',$$

because
$$\int_0^\infty \frac{\cos ua}{u} \sin ux\, du = 0, \text{ when } x < a.$$

It follows that

$$\int_0^x f(x)\, dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin ux}{u} F_a(u)\, du, \quad (x < a);$$

and since
$$\lim_{a \sim \infty} \int_0^\infty \frac{\sin ux}{u} \{F(u) - F_a(u)\}\, du = 0,$$

$\left|\frac{\sin ux}{u}\right|^p$ being summable over $(0, \infty)$, we find that

$$\int_0^x f(x)\, dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin ux}{u} F(u)\, du.$$

It then follows that, for almost all values of x

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin ux}{u} F(u)\, du.$$

It has accordingly been shewn that:

If $|f(x)|^q$ *is summable over* $(0, \infty)$, *for some value of* q *such that* $1 < q \leq 2$, *there exists a function* $F(x)$ *such that* $|F(x)|^{\frac{q}{q-1}}$ *is summable over* $(0, \infty)$, *which satisfies the relation*

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin ux}{u} F(u)\, du.$$

This theorem is the analogue of the Riesz-Fischer theorem (§ 379).

Titchmarsh has shewn* that neither of the above theorems is in general valid if $q > 2$. He has also shewn that $|f(x)|^q$ may be integrable, and that, however small ϵ may be, the integrals of

$$|F(x)|^{\frac{q}{q-1}-\epsilon}, \quad |F(x)|^{\frac{q}{q-1}+\epsilon}$$

may be both divergent. It should be observed that a function $\phi(x)$ may be summable in $(0, \infty)$, but $\{\phi(x)\}^{1-\eta}$ need not be summable in $(0, \infty)$; for example if $\phi(x) = \frac{1}{x^{1+k}}$, where $k > 0$, in the neighbourhood of ∞, $\{\phi(x)\}^{1-\eta}$ is not summable if $\eta \geqq \frac{k}{1+k}$. The summability of $|\phi(x)|^p$ in the interval $(0, \infty)$ does not necessarily involve that of $|\phi(x)|^{p'}$ for any value of $p' \neq p$.

484. Let $F(x)$, $G(x)$ be the transforms of $f(x)$ and $g(x)$. We have then

$$\begin{aligned}\int_0^b g(x)\,F_a(x)\,dx &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^b g(x)\,dx \int_0^a f(u)\cos ux\,dx\\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^a f(u)\,du \int_0^b g(x)\cos ux\,dx\\ &= \int_0^a f(x)\,F_b(x)\,dx.\end{aligned}$$

Since $$\lim_{a\sim\infty} \int_0^b g(x)\{F(x) - F_a(x)\}\,dx = 0,$$

and $$\lim_{b\sim\infty} \int_0^a f(x)\{G(x) - G_b(x)\}\,dx = 0,$$

we now obtain the important relation

$$\int_0^\infty f(x)\,G(x)\,dx = \int_0^\infty g(x)\,F(x)\,dx \qquad \text{......(A).}$$

In the case $q = 2$, we may put $g(x) = F(x)$, but we cannot do this when $q < 2$, because $g(x)\,F(x)$ is then not necessarily summable over $(0, \infty)$ We thus have

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{F(x)\}^2\,dx, \text{ when } q = 2.$$

The corresponding relation for the case $1 < q < 2$ is

$$\int_0^\infty |F(x)|^{\frac{q}{q-1}}\,dx < \frac{\pi}{2}\left[\int_0^\infty |f(x)|^q\,dx\right]^{\frac{1}{q-1}}.$$

485. It will be proved that:

If $|f(x)|^q$ *is summable over* $(0, \infty)$, *for some value of* q *such that* $1 < q \leqq 2$, *then* $\int_0^z f(x)\cos ux\,dx = o(\log z)$, *for almost all values of* u.

* *Proc. Lond. Math. Soc* (2), vol XXIII (1924), p. 286.

If $F(x)$ be the transform of $f(x)$, and $g(x)$ be the transform of the function defined by $G(x) = \cos ux$, when $0 \leq x \leq z$, $G(x) = 0$, when $z < x$.

We have

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^z \cos u\xi \cos x\xi \, d\xi = \frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{\sin z(x+u)}{x+u} + \frac{\sin z(x-u)}{x-u}\right).$$

On account of the relation (A) we have

$$\int_0^z f(x) \cos ux \, dx = \frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(x) \left\{\frac{\sin z(x+u)}{x+u} + \frac{\sin z(x-u)}{x-u}\right\} du$$
$$= \frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^\infty F(\xi + u) \frac{\sin z\xi}{\xi} \, d\xi.$$

The integral on the right-hand side may be divided into three parts, taken over the intervals $(-\delta, \delta)$, $(-\infty, -\delta)$, (δ, ∞) respectively.

It is known that $\displaystyle\int_{-\delta}^{\delta} F(\xi + u) \frac{\sin z\xi}{\xi} \, d\xi = o(\log z)$,

for almost all values of u (see § 406).

We have next

$$\left| \int_\delta^\infty F(\xi + u) \frac{\sin z\xi}{\xi} \, d\xi \right| \leq \left\{\int_\delta^\infty |F(\xi + u)|^{\frac{q}{q-1}} d\xi\right\}^{\frac{q-1}{q}} \left\{\int_\delta^\infty \left|\frac{\sin z\xi}{\xi}\right|^q d\xi\right\}^{\frac{1}{q}}$$
$$< \left\{\int_{-\infty}^\infty |F(\xi)|^{\frac{q}{q-1}} d\xi\right\}^{\frac{q-1}{q}} \frac{1}{\{(q-1)\,\delta^{q-1}\}^{\frac{1}{q}}} = 0(1) = o(\log z).$$

Since the remaining integral, over $(-\infty, -\delta)$, may be treated in the same manner, it has now been shewn that

$$\int_0^z f(x) \cos ux \, du = o(\log z).$$

486. It will now be shewn that:

If $|f(x)|^q$ be summable in the interval $(0, \infty)$, for some value of q such that $1 < q \leq 2$, the integral $\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(x) \cos ux \, dx$ converges $(C, 1)$ to $f(u)$ for almost all values of u. The convergence is uniform in an interval in which $f(u)$ is continuous, the continuity at the end-points being assumed to be on both sides.

We have to consider the expression

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^z \left(1 - \frac{x}{z}\right) F(x) \cos ux \, dx.$$

Let $g(x) = \left(1 - \dfrac{x}{z}\right) \cos ux$, for $0 \leq x \leq z$, and $g(x) = 0$, for $x > z$; we have then to consider $\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty g(x) F(x) \, dx$, which is, in virtue of (A),

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty G(x) f(x) \, dx.$$

We have $$G(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^z \left(1 - \frac{\xi}{z}\right) \cos \xi x \cos u\xi \, d\xi$$
$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{z} \left[\frac{\sin^2 \frac{1}{2}z(x+u)}{(x+u)^2} + \frac{\sin^2 \frac{1}{2}z(x-u)}{(x-u)^2}\right].$$

The expression to be considered thus becomes
$$\frac{2}{\pi z} \int_{-\infty}^{\infty} f(u+\xi) \frac{\sin^2 \frac{1}{2}z\xi}{\xi^2} d\xi;$$
and the integral may be divided, as in § 485, into three parts.

It is seen as in the case of
$$\int_\delta^\infty F(\xi + u) \frac{\sin z\xi}{\xi} d\xi$$
that
$$\int_\delta^\infty f(u+\xi) \frac{\sin^2 \frac{1}{2}z\xi}{\xi^2} d\xi$$
is numerically less than a fixed number independent of u and z; therefore, when multiplied by $\frac{2}{\pi z}$, the expression converges to zero, as $z \sim \infty$, uniformly for all values of u. The integral over $(-\infty, -\delta)$ has the same property. We have then to consider
$$\frac{2}{\pi z} \int_{-\delta}^{\delta} f(u+\xi) \frac{\sin^2 \frac{1}{2}z\xi}{\xi^2} d\xi$$
which is equivalent to
$$\frac{2}{\pi z} \int_0^\delta \{f(u+\xi) + f(u-\xi) - 2f(u)\} \frac{\sin^2 \frac{1}{2}z\xi}{\xi^2} d\xi + \frac{4f(u)}{\pi z} \int_0^\delta \left(\frac{\sin \frac{1}{2}z\xi}{\xi}\right)^2 d\xi.$$

It has been shewn in § 368 that the first part of this expression converges to zero, as $z \sim \infty$, at every point at which
$$\lim_{\xi \sim 0} \frac{1}{\xi} \int_0^\xi |f(u+\xi) + f(u-\xi) - 2f(u)| \, d\xi = 0,$$
and also that, in any interval in which $f(u)$ is continuous, the continuity at the end-points of the interval being on both sides, the convergence is uniform. Since the second part of the expression is
$$\frac{2}{\pi} f(u) \int_0^{\frac{1}{2}z\delta} \left(\frac{\sin \theta}{\theta}\right)^2 d\theta,$$
it converges to $f(u)$, as $z \sim \infty$.

It has now been shewn that
$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^z \left(1 - \frac{x}{z}\right) F(x) \cos ux \, dx$$
converges almost everywhere to $f(u)$, or that
$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(x) \cos ux \, du$$
is almost everywhere summable $(C, 1)$.

It follows that, *if* $|f(x)|^q$ *is summable in* $(0, \infty)$, *and consequently* $|F(x)|^{\frac{q}{q-1}}$ *is so also,* $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\cos ux\,dx$ *converges* $(C, 1)$ *to* $F(x)$ *for almost all values of* u, *the convergence being uniform in an interval in which* $f(u)$ *is continuous, the continuity at the end-points being assumed.*

For, in the proof of the foregoing theorem no use is made of the fact that $q < 2$, and the proof is accordingly applicable when $F(x)$ and $f(x)$ are interchanged.

487. Since $F(x)$ is the transform of $f(x)$, we see that $F(x)$ is, for almost all values of x, the sum $(C, 1)$ of $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(u)\cos ux\,du$. It thus follows that

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty f(x')\cos ux'\,dx'.$$

where both the integrations are understood to be taken with the meaning $(C, 1)$. The repeated integration has, in this sense, a meaning for almost all values of x, and in particular at every point of continuity, or of ordinary discontinuity of $f(x)$. The whole theory is applicable to the sine integrals.

We have thus obtained the following theorem:

If $|f(x)|^q$ *be summable in the interval* $(0, \infty)$ *for some value of* q *such that* $1 < q < 2$, *the two expressions*

$$\frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty f(x')\cos ux'\,dx',$$

$$\frac{2}{\pi}\int_0^\infty \sin ux\,du \int_0^\infty f(x')\sin ux'\,dx'$$

have the value $f(x)$, *for almost all values of* x, *provided all the integrals are understood to be taken* $(C, 1)$. *In particular they represent the value of* $f(x)$ *at any point of continuity* $x\ (> 0)$, *and in every interval of continuity their convergence to* $f(x)$ *is uniform, the continuity at the end-points being assumed to be on both sides.*

488. From the theorem in § 265 which expresses the necessary and sufficient condition that an integral $\int_0^\infty \phi(x)\,dx$ which is convergent $(C, 1)$ should exist in the ordinary sense, that is $(C, 0)$, we see that the necessary and sufficient condition that, at a point u, at which

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\cos ux\,dx$$

exists $(C, 1)$, the integral should be convergent is that

$$\int_0^z xf(x)\cos ux\,du = o(z).$$

Let $G(x) = 0$, for $0 \leq x < a$, $a > 1$, $= f(x)\log x$, for $a \leq x < \infty$, where $|f(x)\log x|^q$ is summable in $(0, \infty)$, for some value of q such that $1 < q \leq 2$. From the theorem in § 485, we have

$$\int_0^z G(x) \cos ux\,dx = o(\log z),$$

for almost all values of u.

Now

$$\begin{aligned}\int_a^z xf(x)\cos ux\,dx &= \int_a^z \frac{x}{\log x} G(x) \cos ux\,dx \\ &= \frac{z}{\log z}\int_a^z G(x)\cos ux\,dx - \int_a^z dx\,\frac{d}{dx}\left(\frac{x}{\log x}\right)\int_a^x G(t)\cos ut\,dt \\ &= \frac{z}{\log z}\,o(\log z) + \int_a^z O\left(\frac{1}{\log x}\right) o(\log x)\,dx \\ &= o(z) + \int_a^z o(1)\,dx = o(z),\end{aligned}$$

and this holds good for almost all values of u.

We have also

$$\left|\int_0^a xf(x)\cos ux\,dx\right| < \int_0^a |xf(x)|\,dx$$
$$< \left\{\int_0^a \left|\frac{x}{\log x}\right|^{\frac{q}{q-1}}\right\}^{\frac{q-1}{q}} \left\{\int_0^a |f(x)\log x|^q\right\}^{\frac{1}{q}},$$

hence the expression on the left-hand side is bounded, and is therefore $o(\log z)$. It follows that

$$\int_0^z xf(x)\cos ux\,dx = o(\log z),$$

and consequently $$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\cos ux\,dx$$

converges, in the ordinary sense, to $F(u)$, for almost all values of u.

From the summability of $|f(x)\log x|^q$ in $(0, \infty)$, that of $|f(x)|^q$ is not a necessary consequence, as is seen by considering the functions in the neighbourhood of the point $x = 1$.

We have thus obtained the following theorem:

If $|f(x)|^q$ and $|f(x)\log x|^q$ are summable in $(0, \infty)$, for some value of q such that $1 < q \leq 2$, then the transform $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\cos ux\,dx$ is convergent in the ordinary sense for almost all values of u, and the function $F(u)$ to which it converges is such that $|F(u)|^{\frac{q}{q-1}}$ is summable in $(0, \infty)$.

The particular case of this theorem which arises when $q = 2$, that, if $|f(x)|^2$ and $|f(x)\log x|^2$ are summable in $(0, \infty)$, $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)\cos ux\,dx$ converges, for almost all values of u, to a function whose square is summable was obtained* by Plancherel, who stated it in the form that, if $a > 1$, and $f(x)$ is defined in (a, ∞), and is such that $\int_a^\infty \{f(x)\log x\}^2\,dx$ is finite, then $\lim_{z\sim\infty}\int_a^z f(x)\cos ux\,dx$ converges for almost all values of u in the interval, and represents a function $F(u)$ such that $\int_{-\infty}^\infty \{F(u)\}^2\,du$ exists. This theorem is analogous to that for Fourier's series given in § 409.

* *Math. Annalen*, vol. LXXVI (1915), p. 324. See also *Math. Annalen*, vol. LXXIV (1913), p. 578, for an earlier theorem.

CHAPTER X

SERIES OF NORMAL ORTHOGONAL FUNCTIONS

489. If (a, b) be a finite, or infinite, interval, and $\{\phi_n(x)\}$ be a sequence of functions such that $\{\phi_n(x)\}^2$ is, for every value of n, summable in (a, b), and such that $\int_a^b \phi_n(x)\,\phi_{n'}(x)\,dx$ has the value zero, for every pair of unequal values of n and n', the system $\{\phi_n(x)\}$ is said to be an orthogonal system of functions for the interval (a, b). If $\int_a^b \{\phi_n(x)\}^2\,dx$ has a value different from unity, that value can be made to be unity by multiplying $\phi_n(x)$ by the factor $1\Big/\left\{\int_a^b \{\phi_n(x)\}^2\,dx\right\}^{\frac{1}{2}}$. When this is done for each value of n, we have $\int_a^b \{\phi_n(x)\}^2\,dx = 1$, where $\phi_n(x)$ is the new value of the function.

A system of orthogonal functions for the interval (a, b) is said to form a system of normal orthogonal functions when $\int_a^b \{\phi_n(x)\}^2\,dx = 1$, for all the values of n.

The system is such that no function $\phi_n(x)$ is expressible as a linear function of a finite number of the other functions, for if we assume that

$$\phi_n(x) = c_1\phi_{p_1}(x) + c_2\phi_{p_2}(x) + \ldots + c_r\phi_{p_r}(x),$$

where n is not equal to any of the finite set $p_1, p_2, \ldots p_r$, we have

$$\int_a^b \phi_n(x)\,\phi_{p_1}(x)\,dx = c_1\int_a^b \{\phi_{p_1}(x)\}^2\,dx = c_1 > 0,$$

which is not in accordance with the property of orthogonality. Thus it has been shewn that the system $\{\phi_n(x)\}$ is such that the functions are linearly independent.

A sequence $\{\psi_n(x)\}$ of functions such that $\{\psi_n(x)\}^2$ is summable in (a, b) is said to be a *complete sequence of functions* in (a, b), if no function $F(x)$ whose square is summable in (a, b) exists and is such that

$$\int_a^b F(x)\,\psi_n(x)\,dx = 0,$$

for all values of n.

In particular, the set $\{\phi_n(x)\}$ of normal orthogonal functions in (a, b) is complete if no function $F(x)$ whose square is summable in (a, b) exists and is such that $\int_a^b F(x)\,\phi_n(x)\,dx = 0$, for every value of n.

If $\{\psi_n(x)\}$ be a complete sequence of linearly independent functions for the interval (a, b), a normal orthogonal and complete system of functions $\{\phi_n(x)\}$ can be so determined that $\phi_n(x)$ is a linear function of

$$\psi_1(x),\ \psi_2(x),\ \dots\ \psi_n(x).$$

The system $\{\phi_n(x)\}$ may be said to be formed from $\{\psi_n(x)\}$ by the process of orthogonalization.

Let
$$\phi_1(x) = \frac{\psi_1(x)}{\left\{\int_a^b \{\psi_1(x)\}^2\,dx\right\}^{\frac{1}{2}}},$$

$$\phi_2(x) = \frac{\psi_2(x) - \phi_1(x)\int_a^b \psi_2(z)\,\phi_1(z)\,dz}{\left\{\int_a^b \left(\psi_2(x) - \phi_1(x)\int_a^b \psi_2(z)\,\phi_1(z)\,dz\right)^2 dx\right\}^{\frac{1}{2}}};$$

we then have $\int_a^b \{\phi_1(x)\}^2\,dx = 1$; $\int_a^b \phi_1(x)\,\phi_2(x)\,dx = 0$,

and
$$\int_a^b \{\phi_2(x)\}^2\,dx = 1.$$

Generally we take

$$\phi_n(x) = \frac{\psi_n(x) - \sum_{k=1}^{k=n-1} \phi_k(x)\int_a^b \psi_n(z)\,\phi_k(z)\,dz}{\left\{\int_a^b \left(\psi_n(x) - \sum_{k=1}^{n-1} \phi_k(x)\int_a^b \psi_m(z)\,\phi_k(z)\,dz\right)^2 dx\right\}^{\frac{1}{2}}};$$

it can then be easily verified that

$$\int_a^b \{\phi_n(x)\}^2\,dx = 1, \quad \int_a^b \phi_n(x)\,\phi_m(x)\,dx = 0, \text{ for } m < n.$$

It will be observed that the denominator in $\phi_n(x)$ cannot vanish, for otherwise $\phi_n(x)$ would be a linear function of $\psi_1(x), \psi_2(x), \dots \psi_{n-1}(x)$.

Since $\phi_n(x)$ is a linear function of $\psi_1(x), \psi_2(x), \dots \psi_n(x)$, the completeness of $\{\phi_n(x)\}$ is a consequence of the completeness of $\{\psi_n(x)\}$.

A simple case is that in which the sequence $\{\psi_n(x)\}$, for the interval $(-1, 1)$, consists of the sequence $1, x, x^2, \dots$. Then $\phi_n(x)$ is a polynomial of degree n; and it is easily verified that $\phi_n(x) = \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} P_n(x)$, where $P_n(x)$ is the nth Legendre's function.

For the interval $(-\pi, \pi)$,

$$\frac{1}{(2\pi)^{\frac{1}{2}}},\ \frac{1}{\pi^{\frac{1}{2}}}\cos x,\ \frac{1}{\pi^{\frac{1}{2}}}\sin x, \dots,\ \frac{1}{\pi^{\frac{1}{2}}}\cos nx,\ \frac{1}{\pi^{\frac{1}{2}}}\sin nx, \dots$$

forms a complete set of normal orthogonal functions which are employed in Fourier's series.

The two independent sets of normal orthogonal functions, for the interval $(0, \pi)$,

$$\frac{1}{\pi^{\frac{1}{2}}},\quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos x,\quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos 2x, \ldots \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos nx, \ldots,$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x,\quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin 2x, \ldots \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin nx, \ldots$$

are employed in the Fourier's cosine and sine series respectively.

THE CONVERGENCE OF THE SERIES OF ORTHOGONAL FUNCTIONS

490. The theory of complete systems of normal orthogonal functions is closely connected with the theory of integral equations and of the linear differential equations with which they are associated, but this relation will not be considered here. There exists no general theory, independent of the form of the functions $\phi_n(x)$, relating to the convergence at a point, of a series $a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_n\phi_n(x) + \ldots$, where the coefficients a_n are expressed in terms of a function $f(x)$, summable in the interval (a, b), by $a_n = \int_a^b f(x)\,\phi_n(x)\,dx$, in analogy with the case of Fourier's series. In fact, in the general case, the behaviour of the series at a point, as regards convergence, oscillation, or divergence, does not depend, as in the case of Fourier's series, only upon the properties of the function $f(x)$ in an arbitrarily small neighbourhood of the point. There exists however a theory of the convergence, almost everywhere in the interval (a, b), of the series corresponding to functions $f(x)$, such that $\{f(x)\}^2$ is summable in (a, b), independent of the particular set of orthogonal functions $\{\phi_n(x)\}$ employed in forming the coefficients $\int_a^b f(x)\,\phi_n(x)\,dx$ of the series corresponding to $f(x)$; an account of this theory will be given below.

It has been shewn* by Steinhaus that a function $f(x)$ can be determined, and a set of normal orthogonal functions $\{\phi_n(x)\}$ defined, for an interval (a, b), such that the series $a_1\phi_1(x) + a_2\phi_2(x) + \ldots$, corresponding to $f(x)$, is nowhere convergent in the interval (a, b).

It has been shewn† by Banach that a function $f(x)$ and a set of normal orthogonal functions $\{\phi_n(x)\}$ can be so constructed that the series corresponding to $f(x)$ is everywhere convergent in the finite interval (a, b), to which $\{\phi_n(x)\}$ refer, but that its sum is everywhere different from the value of $f(x)$.

* *Proc. Lond. Math. Soc.* (2), vol. XX (1921), p. 123.

† *Ibid.* (2), vol. XXI (1923), p. 95.

Let $\{\psi_n(x)\}$ denote the set of normal orthogonal functions in (a, b) defined by

$$\psi_1(x) = \frac{1}{(b-a)^{\frac{1}{2}}}, \quad \psi_2(x) = \left(\frac{2}{b-a}\right)^{\frac{1}{2}} \cos\left(2\pi\frac{x-a}{b-a}\right),$$

$$\psi_3(x) = \left(\frac{2}{b-a}\right)^{\frac{1}{2}} \sin\left(2\pi\frac{x-a}{b-a}\right), \ldots;$$

that is the Fourier's system for (a, b).

Let $\bar{f}(x)$ denote a function summable in (a, b), and ≥ 0, but such that $\{\bar{f}(x)\}^2$ is not summable in (a, b); and let

$$\alpha_n = -\int_a^b \bar{f}(x)\,\psi_n(x)\,dx' \Big/ \int_a^b \bar{f}(x)\,dx.$$

Let it be assumed that, if possible, a function $F(x)$, such that $\{F(x)\}^2$ is summable in (a, b), exists and is such that

$$\int_a^b \{\alpha_n + \psi_n(x)\}\,F(x)\,dx = 0, \text{ for } n = 1, 2, 3, \ldots; \quad \int_a^b \{F(x)\}^2\,dx > 0.$$

We have then (see § 492) $\sum\limits_{n=1}^{\infty}\left\{\int_a^b F(x)\,\psi_n(x)\,dx\right\}^2 = \int_a^b \{F(x)\}^2\,dx;$

and therefore $$\sum_{n=1}^{\infty}\left\{\alpha_n \int_a^b F(x)\,dx\right\}^2 = \int_a^b \{F(x)\}^2\,dx > 0,$$

hence $\int_a^b F(x)\,dx \neq 0$, and $\sum\limits_{n=1} \alpha_n^2$ is convergent; therefore

$$\sum_{n=1}\left\{\int_a^b \bar{f}(x)\,\psi_n(x)\,dx\right\}^2$$

is convergent, from which it follows (see § 493), that $\int_a^b \{\bar{f}(x)\}^2\,dx$ exists and

$$= \sum_{n=1}\left\{\int_a^b \bar{f}(x)\,\psi_n(x)\,dx\right\}^2,$$

which is contrary to the hypothesis as to $\bar{f}(x)$. It follows that no such function as $F(x)$ can exist, and therefore the system $\{\alpha_n + \psi_n(x)\}$ is complete. From this system we can define by orthogonalization a complete system $\{\phi_n(x)\}$ of normal orthogonal functions. Since $\int_a^b \bar{f}(x)\,\phi_n(x)\,dx$ is a linear function of

$$\int_a^b \bar{f}(x)\,(\psi_1(x) + \alpha_1)\,dx, \quad \int_a^b \bar{f}(x)\,(\psi_2(x) + \alpha_2)\,dx, \ldots \int_a^b \bar{f}(x)\,(\psi_n(x) + \alpha_n)\,dx,$$

all of which are zero, it follows that

$$\int_a^b \bar{f}(x)\,\phi_n(x)\,dx = 0, \text{ for } n = 1, 2, 3, \ldots.$$

Consequently the series corresponding to $\bar{f}(x)$ vanishes identically, and if we add to $\bar{f}(x)$ a function $\chi(x)$ such that the series

$$c_1\phi_1(x) + c_2\phi_2(x) + \ldots,$$

corresponding to $\chi(x)$ everywhere converges to $\chi(x)$, the function $f(x)$ defined by $\chi(x)+\bar{f}(x)$ will have $c_1, c_2, \ldots$ for the coefficients in the series corresponding to it, and the series will be everywhere convergent but will nowhere converge to $f(x)$.

It has thus been shewn that:

A set of normal orthogonal functions $\{\phi_n(x)\}$, *for a finite interval* (a, b), *can be defined, and a function* $\bar{f}(x)$, *summable in* (a, b), *and everywhere positive, can be defined, such that the constants* $\int_a^b \bar{f}(x)\,\phi_n(x)\,dx$ *are all zero. A series* $c_1\phi_1(x)+c_2\phi_2(x)+\ldots$, *which is everywhere convergent, can be so defined that it does not at any point converge to a certain function* $f(x)$ *for which* $\int_a^b f(x)\,\phi_n(x)\,dx = c_n$, *for* $n = 1, 2, 3, \ldots$.

Another example of normal functions which have this property has been given*, in a case in which the interval is infinite, by Looman.

THE FAILURE OF CONVERGENCE AT A PARTICULAR POINT

491. The nth partial sum of the series corresponding to $f(x)$ in an interval (a, b), for which $\{\phi_n(x)\}$ is a set of orthogonal functions, is given by

$$s_n(x) = \int_a^b f(x')\,F_n(x', x)\,dx',$$

where

$$F_n(x', x) = \sum^{r=n} \phi_r(x')\,\phi_r(x).$$

It can be shewn that, *if* α *be a point in* (a, b) *such that* $\int_a | F_n(x, \alpha) |\,dx$ *is unbounded, then a function* $f(x)$, *continuous in* (a, b), *can be so defined that the series of orthogonal functions corresponding to it is non-convergent at the point* α.

This theorem† was given by Haar.

Denoting $\int_a^b | F_n(x, \alpha) |\,dx$ by ω_n, since ω_n is unbounded, a partial sequence $\omega_{\nu_1}, \omega_{\nu_2}, \ldots$ belonging to the sequence $\{\omega_n\}$ can be determined so as to be divergent.

Let $v_{\nu_p}(x) = 1, -1$, or 0, according as $F_{\nu_p}(x, \alpha)$ is positive, negative, or zero; we have then

$$| F_{\nu_p}(x, \alpha) | = v_{\nu_p}(x)\,F_{\nu_p}(x, \alpha),$$

and therefore

$$\omega_{\nu_p} = \int_a^b v_{\nu_p}(x)\,F_{\nu_p}(x, \alpha)\,dx;$$

the functions $v_{\nu_p}(x)$ have ω_{ν_p} for the ν_pth partial sums of the orthogonal series corresponding to them.

* *Proc. Lond. Math. Soc.* (2), vol. XXII (1924), *Records*, p. xxxix.

† *Math. Annalen*, vol. LXIX (1910), p. 336.

We next construct a sequence of continuous functions $f_{\nu_1}(x), f_{\nu_2}(x), \ldots$ all of which are in absolute value $\leqq 1$, and are such that

$$\int_a^b (v_{\nu_p}(x) - f_{\nu_p}(x))^2\, dx < \delta_p,$$

for $p = 1, 2, 3, \ldots$, where $\{\delta_p\}$ is a sequence of positive numbers. This construction can be made by means of a theorem given in I, § 433, in which, when $f(x)$ is bounded, with U for its upper boundary, $\phi(x)$ can always be so chosen that $\phi(x) \leqq U$, and $\int_a^b \{f(x) - \phi(x)\}^2\, dx < \epsilon$. For, if in an interval $\phi(x) \geqq U$, or $\leqq -U$, we can replace $\phi(x)$ by the continuous function which has the value U or $-U$ whenever $\phi(x) \geqq U$ or $\leqq -U$, and is elsewhere unaltered.

We have

$$\left| \int_a^b f_{\nu_p}(x) F_{\nu_p}(x, a)\, dx \right|$$
$$= \left| \int_a^b v_{\nu_p}(x) F_{\nu_p}(x, a)\, dx - \int_a^b \{v_{\nu_p}(x) - f_{\nu_p}(x)\} F_{\nu_p}(x, a)\, dx \right|$$
$$\geqq \omega_{\nu_p} - \left[\int_a^b \{F_{\nu_p}(x, a)\}^2\, dx \,.\, \int_a^b \{v_{\nu_p}(x) - f_{\nu_p}(x)\}^2\, dx \right]^{\frac{1}{2}}.$$

Let δ_p be so chosen that

$$\tfrac{1}{2}\omega_{\nu_p} > \left\{ \delta_p \int_a^b \{F_{\nu_p}(x, a)\}^2\, dx \right\}^{\frac{1}{2}},$$

we have then
$$\left| \int_a^b f_{\nu_p}(x) F_{\nu_p}(x, a)\, dx \right| > \tfrac{1}{2}\omega_{\nu_p}.$$

The ν_1th partial sum of the series corresponding to $\phi^{(1)}(x) \equiv f_{\nu_1}(x)$ at $x = a$ is greater than $\frac{1}{2}\omega_{\nu_1}$. If the series corresponding to this function is convergent at $x = a$, choose the number $G^{(1)}$ so that, at a, the nth partial sum of the series is, for every value of n, $< G^{(1)}$. Let $\nu^{(1)} = \nu_1$; from the sequence $\nu_2, \nu_3, \ldots$, we choose $\nu^{(2)}$ so that $\omega_{\nu^{(2)}} > 6.4\,(G^{(1)} + 1)$. Take $\phi^{(2)}(x) = f_{\nu^{(1)}}(x) + \frac{1}{4} f_{\nu^{(2)}}(x)$. If the series corresponding to $\phi^{(2)}(x)$ converges at $x = a$, a number $G^{(2)}$ can be so determined that, for every value of n, the nth partial sum of the series, at the point a, is $< G^{(2)}$. Take then $\omega_{\nu^{(3)}} > 6.4^2\,(G^{(2)} + 2)$, and form the function

$$\phi^{(3)}(x) = f_{\nu^{(1)}}(x) + \frac{1}{4} f_{\nu^{(2)}}(x) + \frac{1}{4^2} f_{\nu^{(3)}}(x).$$

Proceeding in this manner, we form a function

$$\phi^{q-1}(x) = f_{\nu^{(1)}}(x) + \frac{1}{4} f_{\nu^{(2)}}(x) + \ldots + \frac{1}{4^{q-2}} f_{\nu^{(q-1)}}(x).$$

If the series corresponding to $\phi^{(q-1)}(x)$ is convergent at $x = a$, the nth partial sum of the series corresponding to $\phi^{(q-1)}(x)$ is, at the point a,

$< G^{(q-1)}$, for every value of n. We then choose an index $\nu^{(q)}$ out of the sequence $\{\nu_p\}$ such that

$$\omega_{\nu^{(q)}} > 6.4^{q-1}(G^{(q-1)} + q - 1).$$

If this process does not come to an end by the ascertainment of a value of q for which the series corresponding to $\phi^{(q)}(x)$ does not converge at the point α, we consider the function $\phi(x)$, given by the infinite series

$$f_{\nu^{(1)}}(x) + \frac{1}{4}f_{\nu^{(2)}}(x) + \dots + \frac{1}{4^{q-1}}f_{\nu^{(q)}}(x) + \dots.$$

It can be shewn that the function $\phi(x)$ is continuous, and that the series of orthogonal functions corresponding to it does not converge at the point α.

The series for $\phi(x)$ converges uniformly in (a, b), since all the functions $f_{\nu^{(q)}}(x)$ are in absolute value < 1; thus $\phi(x)$ is continuous in (a, b).

In order to calculate the $\nu^{(q)}$th partial sum of the series corresponding to $\phi(x)$ at the point α, we take

$$\phi(x) = \left\{f_{\nu^{(1)}}(x) + \dots + \frac{1}{4^{q-2}}f_{\nu^{(q-1)}}(x)\right\} + \frac{1}{4^{q-1}}f_{\nu^{(q)}}(x) + \left\{\frac{1}{4^q}f_{\nu^{(q+1)}}(x) + \dots\right\}.$$

The expression in the first bracket has for the $\nu^{(q)}$th partial sum at the point α, of the series corresponding to it, a value which is numerically less than $G^{(q-1)}$. The expression in the second bracket is numerically $\leq \frac{1}{3.4^{q-1}}$, and the $\nu^{(q)}$th partial sum of the series corresponding to it has, at the point α, a value less than $\frac{\omega_{\nu^{(q)}}}{3.4^{q-1}}$. It now follows that the $\nu^{(q)}$th partial sum of $\phi(x)$, at the point α, is in absolute value,

$$> \frac{\omega_{\nu^{(q)}}}{2.4^{q-1}} - G^{(q-1)} - \frac{\omega_{\nu^{(q)}}}{3.4^{q-1}},$$

or greater than $\frac{\omega_{\nu^{(q)}}}{6.4^{q-1}} - G^{(q-1)}$. In consequence of the relation

$$\omega_{\nu^{(q)}} > 6.4^{q-1}(G^{q-1} + q - 1),$$

we now see that the $\nu^{(q)}$th partial sum of $\phi(x)$, at the point α, is $> q - 1$. It follows that the sequence of the partial sums of $\phi(x)$, at the point α, of indices $\nu^{(1)}, \nu^{(2)}, \dots$ increases indefinitely. Therefore the series corresponding to $\phi(x)$ is not convergent at the point α, and in fact either diverges, or oscillates infinitely.

EXTENSION OF THE THEOREMS OF PARSEVAL AND RIESZ-FISCHER

492. Let it be assumed that $\{f(x)\}^2$ is summable in the finite, or infinite, interval (a, b), for which $\{\phi_n(x)\}$ is a system of normal orthogonal functions. Let a_n denote the coefficient $\int_a^b f(x)\phi_n(x)\,dx$, of $\phi_n(x)$ in the series corresponding to $f(x)$.

We have

$$\int_a^b \left[f(x) - \sum_{r=1}^{r=n} a_r \phi_r(x)\right]^2 dx = \int_a^b [f(x)]^2 dx - \sum_{r=1}^{r=n} a_r^2;$$

it follows that, for all values of n, $\sum_{r=1}^{r=n} a_r^2$ is not greater than $\int_a^b [f(x)]^2 dx$; and thus that the series $\sum_{r=1} a_r^2$ converges to a number that is $\leq \int_a^b [f(x)]^2 dx$.

Denoting by $f_n(x)$ the partial sum $\sum_{r=1}^{r=n} a_r \phi_r(x)$, we have

$$\int_a^b \{f_p(x) - f_q(x)\}^2 dx = \sum_{r=p+1}^{r=q} a_r^2;$$

and from this it follows that

$$\lim_{p\sim\infty,\, q\sim\infty} \int_a^b \{f_p(x) - f_q(x)\}^2 dx = 0.$$

Thus the sequence $\{f_n(x)\}$ is convergent on the average in (a, b); and the theorems of §§ 171, 172 are therefore applicable to the sequence. There exists a function $\bar{f}(x)$, whose square is summable in (a, b), to which the sequence $\{f_n(x)\}$ converges on the average, and so that

$$\lim_{n\sim\infty} \int_a^b \{\bar{f}(x) - f_n(x)\}^2 dx = 0.$$

If $g(x)$ be another function whose square is summable in (a, b), we have

$$\int_a^b g(x)\{\bar{f}(x) - f_n(x)\} dx \leq \left[\int_a^b \{g(x)\}^2 dx \int_a^b \{\bar{f}(x) - f_n(x)\}^2 dx\right]^{\frac{1}{2}},$$

and it then follows that

$$\lim_{n\sim\infty} \int_a^b g(x)\{\bar{f}(x) - f_n(x)\} dx = 0.$$

Let $g(x) = \phi_m(x)$, then

$$\int_a^b \bar{f}(x)\phi_m(x) dx = \lim_{n\sim\infty} \int_a^b f_n(x)\phi_m(x) dx = \lim_{n\sim\infty} \int_a^b \sum_{r=1}^{r=n} c_r \phi_r(x)\phi_m(x) dx;$$

and thus
$$\int_a^b \bar{f}(x)\phi_m(x) dx = c_m.$$

We now have
$$\int_a^b \{f(x) - \bar{f}(x)\}\phi_m(x) dx = 0,$$

for every value of m; the square of the function $f(x) - \bar{f}(x)$ is summable, and therefore, if the set of orthogonal functions is complete, $f(x)$ and $\bar{f}(x)$ have the same value almost everywhere in (a, b), and therefore $\{f_m(x)\}$ converges on the average to $f(x)$.

It now follows from the results given in § 172 that

$$\int_a^b \{f(x)\}^2 dx = \lim_{m\sim\infty} \int_a^b \{f_m(x)\}^2 dx = \sum_{r=0}^{\infty} a_r^2,$$

and that
$$\lim_{m\sim\infty} \int_a^b \{f(x) - f_m(x)\}^2 dx = 0.$$

We have now $$\lim_{n\sim\infty}\int_a^b g(x)\{f(x)-f_n(x)\}\,dx=0,$$
where $g(x)$ is any function such that $\{g(x)\}^2$ is summable in (a, b).

Therefore
$$\int_a^b f(x)\,g(x)\,dx=\lim_{n\sim\infty}(a_1a_1'+a_2a_2'+\dots+a_na_n'),$$
where $a_1', a_2', \dots$ are the coefficients in the series corresponding to $g(x)$.

We have now established the following theorem which is a generalization of Parseval's theorem in the theory of Fourier's series (see § 378):

If $\{\phi_n(x)\}$ be a complete set of normal orthogonal functions for the finite, or infinite, interval (a, b); and $\{a_n\}$, $\{a_n'\}$ be the sets of coefficients corresponding to two functions $f(x)$, $g(x)$ whose squares are summable in (a, b), then

$$\sum_{n=1} a_n a_n' \textit{ converges to } \int_a^b f(x)\,g(x)\,dx,$$
$$\sum_{n=1} a_n^2 \textit{ converges to } \int_a^b \{f(x)\}^2\,dx.$$

493. Let $$c_1\phi_1(x)+c_2\phi_2(x)+\dots+c_n\phi_n(x)+\dots$$
be a series such that $\sum\limits_{n=1} c_n^2$ is convergent, where $\{\phi_n(x)\}$ is a set of normal orthogonal functions for the interval (a, b).

Denoting the partial sums of the series by $s_n(x)$, we have
$$\lim_{p\sim\infty,\,q\sim\infty}\int_a^b[s_p(x)-s_q(x)]^2\,dx=\lim_{p\sim\infty,\,q\sim\infty}(c_{p+1}^2+c_{p+2}^2+\dots+c_q^2)$$
$$=0,$$
where $q>p$. It follows from § 172 that there exists a function whose square is summable in (a, b) such that
$$\lim_{n\sim\infty}\int_a^b\{f(x)-s_n(x)\}^2\,dx=0,$$
$$\int_a^b\{f(x)\}^2\,dx=\lim_{n\sim\infty}\int_a^b\{s_n(x)\}^2\,dx=\sum_{n=1}^{\infty}c_n^2.$$
Also since $$\lim_{n\sim\infty}\int_a^b\{f(x)-s_n(x)\}\,\phi_m(x)\,dx=0,$$
we have $$\int_a^b f(x)\,\phi_m(x)\,dx=c_m.$$

In case the set $\{\phi_m(x)\}$ is complete, we see that $f(x)$ is unique, except for equivalent functions; for, if there were two such functions, their difference would be orthogonal to all the functions $\phi_m(x)$.

We have thus obtained the following theorem which is a generalization of the Riesz-Fischer theorem obtained in § 379:

If $\{\phi_n(x)\}$ is a set of normal orthogonal functions for the finite, or infinite, interval (a, b); and $c_1, c_2, \dots c_n, \dots$ be a set of constants such that
$$c_1^2+c_2^2+\dots+c_n^2+\dots$$

is convergent, there exists a function $f(x)$ whose square is summable in (a, b), for which $c_n = \int_a^b f(x)\,\phi_n(x)$, $(n = 1, 2, 3, \ldots)$. Moreover $f(x)$ is unique (except for equivalent functions) in case the orthogonal system $\{\phi_n(x)\}$ is complete.

The following extensions of the theorems obtained above for the case in which $\{f(x)\}^2$ is summable in the interval for which $\{\phi_n(x)\}$ is a system of orthogonal functions have already been established in §§ 392, 393 by a method which is applicable, not only to the case of the particular set of orthogonal functions employed in Fourier's series, but to the case of any bounded set of orthogonal functions for a finite interval.

If $\{\phi_n(x)\}$ be a complete system of orthogonal functions for a finite interval (a, b), such that $|\phi_n(x)| \leq M$, for all the values of n and x, and $f(x)$ be such that $|f(x)|^q$ is summable in (a, b), for some value of q such that $1 < q \leq 2$; and if $\sum_{n=1} a_n\phi_n(x)$ be the series corresponding to $f(x)$, then the series $\sum_{n=1} |a_n|^{\frac{q}{q-1}}$ is convergent, and its sum is

$$\leq M^{\frac{2-q}{q}} \left\{\int_a^b |f(x)|^q\,dx\right\}^{\frac{1}{q-1}}.$$

If the series $\sum_{n=1} |a_n|^q$ is convergent for some value of q such that $1 < q \leq 2$, the numbers a_n are the coefficients in the series corresponding to a function $f(x)$ such that $|f(x)|^{\frac{q}{q-1}}$ is convergent, and

$$\sum_{n=1} |a_n|^q \geq \frac{1}{M^{2-q}}\left\{\int_a^b |f(x)|^{\frac{q}{q-1}}\,dx\right\}^{q-1}.$$

It has been pointed out* by F. Riesz that the following theorem is contained in both these theorems, and that conversely both theorems may be deduced from it by means of a limiting process:

If the system $x_1, x_2, \ldots x_n$ passes over into the system $\xi_1, \xi_2, \ldots \xi_n$, by means of an orthogonal substitution, of determinant $+1$, and if all the coefficients of the substitution are, in absolute value, $\leq M$, then

$$\left\{\sum_{k=1}^{k=n} |\xi_k|^\beta\right\}^{\frac{1}{\beta}} < M^{\frac{2-\alpha}{\alpha}} \left[\sum_{k=1}^{k=n} |x_k|^\alpha\right]^{\frac{1}{\alpha}},$$

where $$1 < \alpha < \beta, \text{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

THE CONVERGENCE OF SERIES OF ORTHOGONAL FUNCTIONS

494. A series of investigations has been made relating to the convergence, almost everywhere, of the series of type $\sum_{n=1} a_n\phi_n(x)$ in the finite interval, taken for convenience to be $(0, 1)$, for which $\{\phi_n(x)\}$ form a system of orthogonal functions.

* *Math. Zeitschr.* vol. XVIII (1923), p. 124.

It was first shewn* by Weyl that the convergence of the series $\sum_{n=1} n^{\frac{1}{3}} a_n^2$ is sufficient to ensure that the series $\sum_{n=1} a_n \phi_n(x)$ is convergent almost everywhere in the interval $(0, 1)$. Weyl shewed further that, when $|\phi_n(x)|$ is bounded with respect to (n, x), the convergence of $\sum_{n=1} n^{\frac{1}{3}} a_n^2$ is sufficient to ensure the same result. The more general theorem was proved† by Hobson that, if $\sum_{n=1} n^k a_n^2$ be convergent for some value of k (> 0), the series $\sum_{n=1} a_n \phi_n(x)$ converges almost everywhere in the interval $(0, 1)$, whether the functions $\phi_n(x)$ be bounded or not. It was next proved‡ by Plancherel, by re-arranging Hobson's proof, that the convergence of $\sum_{n=1} a_n^2 (\log n)^3$ is sufficient for the convergence of the series almost everywhere.

Lastly, it was proved by§ Rademacher and by‖ Menchoff that, if $\sum_{n=1} a_n^2 (\log n)^2$ is convergent, then $\sum_{n=1} a_n \phi_n(x)$ converges almost everywhere; this result includes all the preceding theorems.

Moreover, it is final, in the sense that $(\log n)^2$ cannot be replaced by any function of n which is $o\{(\log n)^2\}$, so long as the system $\{\phi_n(x)\}$ is not specialized.

We proceed to establish the theorem that:

If the constants a_n are such that $\sum_{n=1} a_n^2 (\log n)^2$ is convergent, where $\{\phi_n(x)\}$ is a system of orthogonal functions for the interval $(0, 1)$, then the series $\sum_{n=1} a_n \phi_n(x)$ converges almost everywhere in $(0, 1)$.

Let
$$s(x, n) = \sum_{\nu=1}^{n} a_\nu \phi_\nu(x).$$

(1) It will be shewn that, if $\sum_{n=1} a_n^2 \log n$ is convergent, $s(x, 2^n)$ is convergent, as $n \sim \infty$, for almost all values of x in $(0, 1)$. We have

$$\int_0^1 [\{s(x, 2^{r+m}) - s(x, 2^r)\}^2 + \{s(x, 2^{r+m}) - s(x, 2^{r+1})\}^2 + \dots + \{s(x, 2^{r+m}) - s(x, 2^{r+m-1})\}^2]\, dx$$

$$= \sum_{2^r+1}^{2^{r+m}} c_t^2 + \sum_{2^{r+1}+1}^{2^{r+m}} c_t^2 + \dots + \sum_{2^{r+m-1}+1}^{2^{r+m}} c_t^2$$

$$= \sum_{2^r+1}^{2^{r+1}} c_t^2 + 2 \sum_{2^{r+1}+1}^{2^{r+2}} c_t^2 + \dots + m \sum_{2^{r+m-1}+1}^{2^{r+m}} c_t^2 < \sum_{2^r+1}^{2^{r+m}} c_t^2 \log t;$$

where the logarithm is taken for convenience to have the base 2.

* *Math. Annalen*, vol. LXVII (1909), p. 225.
† *Proc. Lond. Math. Soc.* (2), vol. XII (1912), p. 297.
‡ *Comptes Rendus*, vol. CLVII (1913), p. 539.
§ *Math. Annalen*, vol. LXVIII (1922), p. 112.
‖ *Fundamenta Math.* vol. IV (1923), p. 82.

Choosing r so that $\sum_{2^r+1}^{\infty} c_t^2 \log t < \delta^3$, we have

$$\int_0^1 \sum_{p=0}^{m-1} \{s(x, 2^{r+m}) - s(x, 2^{r+p})\}^2 dx < \delta^3.$$

The set of points in which the integrand is $\geqq \delta^2$ has its measure $< \delta$, and therefore that in which it is $< \delta^2$ has measure $> 1 - \delta$.

Hence, in a set of points H_m, of measure $> 1 - \delta$,

$$|s(x, 2^{r+m}) - s(x, 2^{r+p})| < \delta,$$

for $p = 0, 1, 2, \dots m - 1$; and thus, in H_m,

$$|s(x, 2^q) - s(x, 2^{q'})| < 2\delta,$$

for all pairs of values of q and q' such that

$$r \leqq q \leqq r + m, \qquad r \leqq q' \leqq r + m.$$

Consider the sets $H_m, H_{m+1}, H_{m+2}, \dots$, each of which is of measure $> 1 - \delta$.

A point of the set H_{m+1} belongs to H_m, so that each set contains the next. It follows that a set H, common to all the sets $H_m, H_{m+1}, \dots$ exists, and is of measure $\geqq 1 - \delta$.

In the set H we have

$$|s(x, 2^q) - s(x, 2^{q'})| < 2\delta$$

for all pairs of values of q and q' such that $r \leqq q$, $r \leqq q'$. Let η be an arbitrarily chosen positive number, and let δ_1, δ_2 denote a sequence such that $\delta_1 + \delta_2 + \dots$ converges to the value η. Corresponding to each value δ_n, of δ, there is a value r_n, of r, and a set $H^{(n)}$, such that

$$|s(x, 2^q) - s(x, 2^{q'})| < 2\delta_n,$$

in that set, for all pairs of values of q and q' such that $r_n \leqq q$, $r_n \leqq q'$; the measure of $H^{(n)}$ is $\geqq 1 - \delta_n$. The sets $H^{(n)}$ have in common a set K, of measure $\geqq 1 - \sum_{n=1}^{\infty} \delta_n \geqq 1 - \eta$.

In the set K we have $|s(x, 2^q) - s(x, 2^{q'})| < 2\delta_n$ provided $r_n \leqq q$, $r_n \leqq q'$, for all values of n; therefore, in the set K, $s(x, 2^t)$ is uniformly convergent, as $t \sim \infty$. Since η is arbitrarily small, it follows that $s(x, 2^t)$ is convergent for almost all values of x, as $t \sim \infty$.

It should be remarked that the special sequence $s(x, 2^n)$ which has been shewn to be convergent, subject to the condition that $\sum_{n=1} a_n^2 \log n$ is convergent, is such that the sequence $\{2^n\}$ is independent of the particular system of normal functions.

(2) Any integer n, such that $2^m < n < 2^{m+1}$ is of the form $2^m + K . 2^l$, where l has one of the values $0, 1, 2, \dots m - 1$, and K is an odd integer.

Let $\chi(l, s) = 2^m + s.2^l$, where l, s may have all values such that

$$\left.\begin{array}{l} 0 \leqq l < m \\ 0 \leqq s < 2^{m-l} \end{array}\right\}$$

Let $$D(x, l, s) = s(x, \chi(l, s+1)) - s(x, \chi(l, s)),$$

$$\int_0^1 \{D(x, l, s)\}^2\, dx = \sum_{\chi(l,s)+1}^{\chi(l,s+1)} a_n^2.$$

We have

$$\begin{aligned} s(x, n) - s(x, 2^m) &= \{s(x, 2^m + 2^l) - s(x, 2^m)\} \\ &+ \{s(x, 2^m + 2.2^l) - s(x, 2^m + 2^l)\} \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots \\ &+ \{s(x, 2^m + K.2^l) - s(x, 2^m + \overline{K-1}\, 2^l)\} \\ &= \sum_{s=0}^{s=K-1} D(x, l, s), \end{aligned}$$

where l has a fixed value for a given value of n, such that $n = 2^m + K.2^l$.

(3) Let $e(l, s)$ be the set of all points x such that $|D(x, l, s)| \geqq \delta$, where δ is a prescribed positive number; we have then

$$m\{e(l, s)\} \leqq \frac{1}{\delta^2} \sum_{\chi(l,s)+1}^{\chi(l,s+1)} a_n^2.$$

(4) Let q be a prescribed positive integer, and let $E(\delta)$ be the set of all points x such that the number of the different functions $D(x, l, s)$ for which $0 \leqq s < 2^{m-l}$, $0 \leqq l < m$ which are such that $|D(x, l, s)| \geqq \delta$, at x, is $\geqq q$; then

$$m\,|E(\delta)| \leqq \frac{m}{q\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2.$$

Let $E(l, s) \equiv D\{E(\delta), e(l, s)\}$, the set common to $E(\delta)$ and $e(l, s)$, then $m\,[E(l, s)| \leqq me(l, s)$. Also let $E = \Sigma E(l, s)$, the summation being taken for all l and s such that $0 \leqq s < 2^{m-l}$, $0 \leqq l < m$.

Each point of E belongs to a number of the sets $E(l, s)$ of which the number in the sum $\Sigma E(l, s)$ is $\geqq q$.

Let $f_{l,s}(x)$ be the characteristic function of $E(l, s)$; that is the function which has the value 1 at all points of $E(l, s)$, and is elsewhere zero; and let $f(x)$ be the characteristic function of E.

We have $\quad f(x) \leqq \dfrac{1}{q} \Sigma f_{l,s}(x), \quad \displaystyle\int_0^1 f(x) \leqq \frac{1}{q} \Sigma \int_0^1 f_{l,s}(x)\, dx,$

$$m(E) \leqq \frac{1}{q} \Sigma\, mE(l, s) \leqq \frac{1}{q\delta^2} \Sigma \sum_{\chi(l,s)+1}^{\chi(l,s+1)} a_n^2 \leqq \frac{m}{q\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2.$$

(5) Let $G_m(\delta)$ denote the set of all points x such that

$$|s(x, n') - s(x, n)| < 4\delta$$

for all pairs of values of n and n' such that $2^m \le n < n' < 2^{m+1}$. It will be shewn that $\sum_{m=1}^{\infty} m\{C(G_m)\}$ is a convergent series.

Let $$k_p = \frac{m}{p^4}, \text{ for } p = 1, 2, 3, \ldots.$$

Let e_p be the set of all points x for which the number of functions $D(x, l, s)$ which satisfy the condition

$$\frac{\delta}{k_{p-1}} \le |D(x, l, s)| < \frac{\delta}{k_p} \text{ is } < \frac{k_p}{p^2}.$$

Employing (4) and writing $\frac{\delta}{k_{p-1}}$ for δ, and $\frac{k_p}{p^2}$ for q, we see that

$$mC(e_p) \le \frac{m\,p^2}{\delta^2 k_p} k_{p-1}^2 \sum_{2^m+1}^{2^{m+1}} a_n^2 \le \frac{8m^2p^6}{\delta^2 (p-1)^8} \sum_{2^m+1}^{2^{m+1}} a_n^2.$$

Let E_m be the common part of all the sets $e_2, e_3, e_4, \ldots$; $C(E_m)$ is the set of points each of which belongs to one at least of the sets $C(e_2), C(e_3), \ldots$ and thus

$$mC(E_m) \le \sum_{p=2}^{\infty} mC(e_p) < \frac{2^6 m^2}{\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2 \sum_{p=2}^{\infty} \frac{1}{(p-1)^2} < \frac{2^7 m^2}{\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2.$$

It can be shewn that E_m is a part of G_m.

Let x be a point of E_m; it thus belongs to all the sets e_p $(p = 1, 2, 3, \ldots)$.

If N_p $(p = 2, 3, \ldots)$ be the number of different functions $D(x, l, s)$ for which $\frac{\delta}{k_{p-1}} \le |D(x, l, s)| < \frac{\delta}{k_p}$, then $N_p < \frac{k_p}{p^2}$.

If $p \ge 2$, there are N_p different functions $D(x, l, s)$, all such that

$$\frac{\delta}{k_{p-1}} \le |D(x, l, s)| < \frac{\delta}{k_1},$$

and for all the other functions $D(x, l, s)$, we have $|D(x, l, s)| < \frac{\delta}{m}$.

If n is any integer such that $2^m \le n < 2^{m+1}$

$$|s(x, n) - s(x, 2^m)| \le \sum_{s=0}^{K-1} |D(x, l, s)|.$$

Among the K terms $|D(x, l, s)|$ there are at most N_p terms for which

$$\frac{\delta}{k_{p-1}} \le |D(x, l, s)| < \frac{\delta}{k_p}, \quad p = 2, 3, \ldots,$$

all the other terms, of which the number cannot exceed m, are $< \frac{\delta}{m}$.

Therefore

$$|s(x, n) - s(x, 2^m)| \le \delta + \sum_{p=2}^{\infty} \frac{\delta}{k_p} N_p < \delta + \delta \sum_{p=2}^{\infty} \frac{1}{p^2} < 2\delta,$$

and thus $$|s(x, n') - s(x, n)| < 4\delta.$$

Therefore each point of E_m belongs to $G_m(\delta)$; E_m is a part of G_m, and consequently $C(G_m)$ is a part of $C(E_m)$.

Hence $$m\{C(G_m)\} < \frac{2^7 m^2}{\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2 < \frac{2^7}{\delta^2} \sum_{2^m+1}^{2^{m+1}} a_n^2 (\log n)^2;$$

and thus $\Sigma m\{C(G_m)\}$ is a convergent series.

The set of points, each of which belongs to an infinite number of the sets $C(G_m)$, has measure zero; for, if ϵ be an arbitrary positive number, each point belongs to one or more of the sets $C(G_r)$, $C(G_{r+1})$, ..., the sum of the measures of which is $< \epsilon$; thus the set has measure $< \epsilon$. Since ϵ is arbitrary, this measure is zero. The measure of the set H, each part of which belongs to all the sets $G_1, G_2, \ldots$, from and after some fixed one of the sets, dependent on the point, has measure 1. Thus there is a set of points of measure 1 such that, for each such point, for all values of n and n'

$$|s(x, n') - s(x, n)| < 4\delta,$$

where $2^m < n < n' < 2^{m+1}$; and for all values of m, from and after a fixed one.

Also, since $|s(x, 2^m) - s(x, 2^{m'})| < \delta$, if m and m' are both $>$ some fixed value of m, it follows that $|s(x, n') - s(x, n)| < 9\delta$, for all values of n and n' greater than 2^m, for some fixed value of m, in a set H_δ such that $m(H_\delta) = 1$.

Lastly, giving δ the values in a sequence which converges to zero, we see that there exists a set of measure 1, in which $s(x, n)$ is convergent.

Menchoff has also proved that, if $W(n)$ is such that

$$\lim_{n\sim\infty} \frac{W(n)}{(\log n)^2} = 0,$$

there exists a set of normal functions such that $\Sigma a_n \phi_n(x)$ converges nowhere in the interval $(0, 1)$, although $\Sigma W(n) a_n^2$ is convergent. It thus appears that the theorem cannot be replaced by one in which $(\log n)^2$ is replaced by a lower power of $\log n$ than the square. This may however be the case for a special set of orthogonal functions; as for example in the case of Fourier's series, for which it has been shewn in § 409 that $(\log n)^2$ may be replaced by $(\log n)^q$, where $q > 1$.

A proof was sketched by Weyl*, and given fully by Hobson†, that the series $\sum_{n=1} a_n \phi_n(x)$, corresponding to a function, of which the square is summable in the finite interval for which $\{\phi_n(x)\}$ forms a normal orthogonal system, is summable $(C, 1)$, almost everywhere in the interval, if the series $\sum_{n=1} a_n^2 \log n$ is convergent. The wider theorem has been established‡ by

* *Math. Annalen*, vol. LXVII (1909), p. 241.

† *Proc. Lond. Math. Soc.* (2), vol. XIV (1915), p. 428.

‡ *Math. Zeitschr.* vol. XXIII (1925), p. 263.

Kaczmarz that the convergence of the series $\sum_{n=1} a_n^2 (\log n)^{\frac{1}{2}}$ is sufficient. He has also shewn that, for any function whose square is summable in the domain $\Sigma a_n \phi_n (x)$ is either summable $(C, 1)$ almost everywhere, or else its Poisson sum does not almost everywhere exist.

It has however been announced* by Menchoff that he has obtained the following complete theorem:

If $\sum_{n=1}^{\infty} a_n^2 (\log\log n)^2$ is convergent, then the series $\sum_{n=1}^{\infty} a_n \phi_n (x)$ is summable (C, k), for $k > 0$, almost everywhere in the interval for which the orthogonal functions exist; and consequently the series is almost everywhere summable by Poisson's method: If $\omega(n)$ satisfies the condition $\lim_{n \sim \infty} \frac{\omega(n)}{(\log\log n)^2} = 0$, there exists a series which is not summable at any point by the method of Poisson, while $\Sigma \omega (n) a_n^2$ is convergent.

495. The following theorems have been established† by Menchoff:

If $\sum_{n=1} | a_n |^{2-\lambda}$ is convergent, for some positive value of λ such that $2 > \lambda > \frac{4}{3}$, then the series $\sum_{n=1} a_n \phi_n (x)$ converges almost everywhere in the finite interval $(0, 1)$ for which $\{\phi_n (x)\}$ is a sequence of orthogonal functions.

This is a particular case of the more general theorem that:

If $\omega (u)$ is a positive function of u which increases with u, and the series

$$\sum_{n=1} \left[\omega\left(\log\log \frac{1}{|a_n|}\right)\right]^2 \left(\log \frac{1}{|a_n|}\right)^2 a_n^2,$$

$$\sum_{n=1} \frac{1}{\omega\left(\log\log \frac{1}{|a_n|}\right)},$$

are both convergent, then $\Sigma a_n \phi_n (x)$ converges almost everywhere.

The logarithms are taken to the base 2. It will be observed that the convergence of the second series implies the divergence of $\log\log \frac{1}{|a_n|}$, and this involves the convergence of a_n to zero.

The first theorem is included in the second because, if $\sum_{n=1} | a_n |^{2-\lambda}$ is convergent, it is possible to choose the function $\omega (u)$ so as to satisfy the conditions of the second theorem, the convergence of the two series being then a consequence of the convergence of $\sum_{n=1} | a_n |^{2-\lambda}$.

* *Comptes Rendus*, vol. CLXXX (1925), p. 2011.

† *Comptes Rendus*, vol. CLVIII (1924), p. 802. The condition $\lambda > \frac{4}{3}$ is not stated there, but it appears to be necessary in order that the first theorem may be included in the second.

In fact the ratio of the general term of the first series in the second theorem to that in the series of the first theorem is

$$\left[\omega\left(\log\log\frac{1}{|a_n|}\right)\right]^2\left(\log\frac{1}{|a_n|}\right)^2\left(\frac{1}{|a_n|}\right)^{-\lambda};$$

and writing $\log\dfrac{1}{|a_n|} = z$, this becomes $[\omega(\log z)]^2 z^2 2^{-\lambda z}$. If now we take $\omega(u) = 2^{p2^u}$, the series $\Sigma\dfrac{1}{\omega(\log z)}$ becomes $\sum\limits_{n=1} |a_n|^p$, which is convergent if $p \geqq 2-\lambda$; also $z^2 2^{-\lambda z}.2^{2pz}$ converges to zero, as $z \sim \infty$, if $\lambda > 2p$. If then $\lambda > \frac{4}{3}$, $\omega(u)$ can be so chosen that the first series of the second theorem converges if the series $\Sigma |a_n|^{2-\lambda}$ converges.

It will be assumed that $\sum\limits_{n=1}^{\infty} a_n^2 < 1$; this involves no loss of generality.

We proceed to group the constants a_n. The group Γ_p consists of those constants a_n which are such that $2^{2^p} \leq \dfrac{1}{|a_n|} < 2^{2^{p+1}}$. We denote the values of n which belong to Γ_p by $n(p, 1)$, $n(p, 2), \ldots, n(p, N_p)$; where each one of these values of n, in Γ_p, is less than the next; the number of values of n in Γ_p is accordingly N_p.

Since $\sum\limits_{n=1}^{\infty} a_n^2$ is less than 1, and $|a_{n(p,s)}| < \dfrac{1}{2^{2^p}}$, it follows that

$$N_p\frac{1}{2^{2^{p+1}}} < 1, \quad \text{or} \quad N_p < 2^{2^{p+1}}.$$

Choosing a positive number δ, we define the set of points $E_p(\delta)$ to be that set of points x, in $(0, 1)$, for which

$$\left|\sum_{t=s}^{t=s'} a_{n(p,t)}\phi_{n(p,t)}(x)\right| < \delta,$$

for all integers s and s' such that $1 \leq s \leq s' \leq N_p$. When $N_p = 0$, we take $E_p(\delta)$ to be the interval $(0, 1)$; then $m[C\{E_p(\delta)\}] = 0$.

We take $\omega(p)$ to denote the least value of $\omega\left(\log\log\dfrac{1}{|a_{n(p,s)}|}\right)$, for the values $s = 1, 2, 3, \ldots, N_p$; and we take the number δ to be $\dfrac{1}{\omega(p)}$. Since $\dfrac{1}{\omega(p)}$ is the greatest of the numbers $\dfrac{1}{\omega\left(\log\log\dfrac{1}{|a_{n(p,s)}|}\right)}$, it follows from the assumption of the convergence of the series of which $\dfrac{1}{\omega\left(\log\log\dfrac{1}{|a_n|}\right)}$ is the general term that $\sum\limits_{p=1}\dfrac{1}{\omega(p)}$ is convergent.

It has been shewn in § 494 that, if G_n denote the set of points for which $\left|\sum_{n=n'}^{n=n''} c_n\phi_n(x)\right| < \delta$, for all values of n' and n'' such that $2^m \leqq n' < n'' < 2^{m+1}$, the measure $m[C(G_m)]$, of the complement of G_m, is less than $k\dfrac{m^2}{\delta^2}\sum_{2^m+1}^{2^{m+1}} c_n^2$, where k is an absolute constant. In the last expression m^2 may be replaced by the square of the logarithm of any one of the indices n which occurs in the summation.

Let $C_m = 0$, except when m has a value $n(p, s)$, in which case

$$C_{n(p,s)} = a_{n(p,s)};$$

we then see that

$$m[C(E_p)] < \frac{k}{\delta^2}(\log N_p)^2 \sum_{s=1}^{s=N_p} a_{n(p,s)}^2, \text{ if } N_p > 1.$$

Since $\log N_p < 2^{p+1}$, and $\delta = 1/\omega(p)$, where

$$\omega(p) \leqq \omega\left(\log\log\frac{1}{|a_{n(p,s)}|}\right),$$

and

$$2^p \leqq \log\frac{1}{|a_{n(p,s)}|} \leqq 2^{p+1},$$

we have

$$m[C(E_p)] < k'\sum_{s=1}^{s=N_p}\left\{\omega\left(\log\log\frac{1}{|a_{n(p,s)}|}\right)\right\}^2\left(\log\frac{1}{|a_{n(p,s)}|}\right)^2 a_{n(p,s)}^2,$$

where k' is an absolute constant. The same inequality holds when $N_p = 1$.

From the assumed convergence of the series found by taking $p = 1, 2, 3, \ldots$, and adding the series together, we see that the series $\sum_{p=1} m[C(E_p)]$ is convergent, and consequently $\lim_{p\to\infty} m[C(E_p)] = 0$. Therefore the set F, of points, each of which belongs to an infinite number of the sets $C(E_p)$, has its measure zero.

Take any two indices n', n'' such that $n' < n''$, and let p' and p'' be the smallest and greatest of those values of p for which the condition $2^{2^p} \leqq \dfrac{1}{|a_n|} < 2^{2^{p+1}}$ is satisfied for at least one value of n such that $n' \leqq n \leqq n''$. It is clear that p' and p'' diverge as n' does so. Corresponding to each point x, of $C(F)$, there is a minimum value $n(x)$, of n, for which the condition $\left|\sum_{n=n'}^{n=n''} a_n\phi_n(x)\right| < \sum_{p=p'}^{p=p''}\dfrac{1}{\omega(p)}$ is satisfied for all values of n' and n'' such that $n'' > n' > n(x)$. Since $\sum\dfrac{1}{\omega(p)}$ is convergent, it follows that $\left|\sum_{n=n'}^{=n''} a_n\phi_n(x)\right| < \epsilon$, provided n' is large enough; therefore the series $\sum_{n=1} a_n\phi_n(x)$ converges at a point x, of $C(F)$; and since $C(F)$ has measure unity, it follows that $\sum_{n=1} a_n\phi_n(x)$ converges almost everywhere in the interval $(0, 1)$.

SERIES OF STURM-LIOUVILLE FUNCTIONS

496. The class of normal orthogonal functions known as the Sturm-Liouville functions has the special property that the series of which the terms consist of multiples of these functions behave in the same manner, in relation to convergence, divergenee, or oscillation at a point as Fourier's series do. A brief sketch of the investigations relating to these functions will be given here.

If g, k, l are functions of x which are positive and continous, and do not vanish, in an interval (a, b), of x, and r is a parameter, the functions are the solutions of the differential equation

$$\frac{d}{dx}\left(k\frac{dV}{dx}\right) + (gr - l)\, V = 0,$$

which are such that $\frac{dV}{dx} - hV = 0$, for $x = a$, and $\frac{dV}{dx} + Hv = 0$, for $x = b$, where h, H are positive constants, and the parameter r is so determined that a solution of the differential equation exists which satisfies the conditions at a and b. It is convenient to assume that l and $(gk)^{-\frac{1}{4}}$ have bounded variation in (a, b).

The differential equation reduces to the form

$$\frac{d^2U}{dz^2} + (\rho^2 - l_1)\, U = 0$$

by means of the transformation

$$z = \int_a^x \left(\frac{g}{k}\right)^{\frac{1}{2}} dx, \quad \theta = (gk)^{-\frac{1}{4}}, \quad V = \theta U, \quad r = \rho^2,$$

where l_1 has the value

$$\frac{1}{\theta\,(gk)^{\frac{1}{2}}}\left\{l\left(\frac{k}{g}\right)^{\frac{1}{2}}\theta - \frac{d\,(gk)^{\frac{1}{2}}}{dz}\frac{d\theta}{dz} - (gk)^{\frac{1}{2}}\frac{d^2\theta}{dz^2}\right\},$$

and the conditions at the end-points of the interval become

$$\frac{dU}{dz} - h'U = 0, \text{ for } z = 0, \text{ and } \frac{dU}{dz} + H'u = 0, \text{ for } z = \pi,$$

where the assumption, involving no real loss of generality, is made that

$$\int_a^b \left(\frac{g}{k}\right)^{\frac{1}{2}} dx = \pi.$$

An asymptotic form of the normal functions was obtained* by Liouville, and a more precise asymptotic expression of them, sufficient for the purpose of the investigation of the series, was obtained† by Hobson. This expression is

$$\phi_n(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos nz \left\{1 + \frac{\alpha(z, n)}{n^2}\right\} + \sin nz \left\{\frac{\beta(z)}{n} + \frac{\alpha(z, n)}{n^2}\right\},$$

* *Liouville's Journal*, vol. II (1837), p. 24.

† *Proc. Lond. Math. Soc.* (2), vol. VI (1908), p. 379.

where $\beta(z)$ denotes functions which are bounded with respect to z, and $\alpha(z, n)$ denotes functions which are bounded with respect to (z, n). It is known from the general theory of the functions connected with differential equations that the system $\{\phi_n(z)\}$ is complete.

It was shewn (*loc. cit.*) by Hobson that the series

$$\sum_{r=1} \phi_n(z) \int_0^\pi \phi_n(z') f(z')\, dz'$$

corresponding to a function $f(z)$, summable in $(0, \pi)$, converges to

$$\tfrac{1}{2}\{f(z+0) + f(z-0)\}$$

at any interior point of the interval $(0, \pi)$ if, in some neighbourhood of the point z, $f(z)$ is of bounded variation. Also it was shewn that, in any interval in which $f(z)$ is continuous, and which is interior to an interval in which the function has bounded variation, the convergence of the series to $f(z)$ is uniform. It was also shewn that, at 0 and π, the series converges to $f(+0)$, $f(\pi - 0)$ if at 0 and π there are neighbourhoods in which $f(z)$ is of bounded variation.

The more general theorem was established by Haar*, and by Mercer†, independently of one another, that:

The series behaves at any point, as regards convergence, divergence, or oscillation, in the same manner as the Fourier's cosine series corresponding to $f(z)$ behaves at the same point.

A proof of this result was given‡ by Hobson, based upon a consideration of the function

$$\sum_{r=1}^{r=n} \phi_r(z)\, \phi_r(z') - \frac{2}{\pi} \sum_{r=1}^{r=n} \cos rz \cos rz' = F(z, z', n).$$

It can be shewn that the function $F(z, z', n)$ satisfies the conditions of the general convergence theorem of § 279, and thus that

$$\lim_{n\sim\infty} \int_0^\pi f(z')\, F(z, z', n)\, dz' = 0;$$

the convergence to the limit being uniform in the interval $(0, \pi)$ of z.

The result stated above is now deducible from this result.

By considering the function

$$\sum_{r=1}^{r=n} \left(1 - \frac{r-1}{n}\right) \phi_r(z)\, \phi_r(z') - \frac{2}{\pi} \sum_{r=1}^{r=n} \left(1 - \frac{r-1}{n}\right) \cos rz \cos rz'$$

in a similar manner, the theorem, due to Haar (*loc. cit.*), can be established, that the summation of the series of Sturm-Liouville functions by the method of arithmetic means is the same as that for the Fourier's cosine series; and thus that the Cesàro sum of the Sturm-Liouville series, corresponding to a summable function, exists for almost all values of z.

* *Math. Annalen*, vol. XLIX (1910), p. 355.

† *Phil. Trans.* vol. CCXI (A) (1912), p. 111.

‡ *Proc. Lond. Math. Soc.* (2), vol. XII (1912), p. 170.

CORRECTIONS AND ADDITIONS TO VOLUME I

Page 104. Line 17 from the foot, *for* "*closed*" *read* "*perfect.*" Line 15 from the foot, *for* "closed" *read* "perfect."

Page 105. Line 3, *for* "S" *read* "G."

Page 110. Line 10 from the foot, *for* "$0<\zeta<\epsilon$" *read* "$0<\zeta-X<\epsilon$."

Page 131. Line 13 from the foot, *after* "D_{n_2}" *insert* "none of which contains a point of G_2."

Page 143. Line 6, *after* "If P be a point of G_1" *add* "and P' be a point of G_2."

Page 144. Line 17, *for* "G" *read* "O."

Page 179. Line 22, *for* "D_{n_1}, D_{n_2} ..., each of which contains an enumerable set" *read* "D'_{n_1}, D'_{n_2} ..., each of which contains a set of points of G of measure zero." Line 24, *for* "enumerable" *read* "of measure zero." Line 10 from the foot, *for* "of measure zero" *read* "of measure >0."

Page 180. Line 19 from the foot, *for* "$>c\,(>0)$" *read* "$>c/\lambda$, *where* λ *is a sufficiently large number, independent of* c." Line 7 from the foot, *for* "the two systems of nets" *read* "the systems of nets the measure of whose meshes is $<\lambda a^p$."

Page 181. Lines 15, 19, *for* "c" *read* "c/λ." Line 10 from the foot, *for* "$>a$" *read* "$>a/\lambda$."

Line 2 from the foot, *for* "$am\,(d_{n_p})$" *read* "$\frac{a}{\lambda}m\,(d_{n_p})$."

Page 182. Lines 4, 16, *for* "$am\,(E_a)$" *read* "$\frac{a}{\lambda}m\,(E_a)$."

Page 276. Line 3, *after* "*another positive number*" *insert* "η."

Page 277. Line 5, *for* "$|\,\phi\,(a)\,|$" *read* "$|\,\phi\,(x)\,|$."

Page 337. Line 8 from the foot, the second part of the enumeration of the theorem should *read* ***Moreover, if the first limit has no unique value, the upper and lower values of the second limit are in the interval bounded by the upper and lower values of the first limit.***

It may happen that $\dfrac{f\,(a+h)}{F\,(a+h)}$ has a unique value, whilst this is not the case for $\dfrac{f'\,(a+h)}{F'\,(a+h)}$.

Page 338. In the second part of the statement of the theorem, the same amendment is required as on p. 337.

Page 371. Line 14, *for* "$f\,(x)-k$" *read* "$f\,(x)-kx$."

Page 390. § 307. The theorem is correct only for the two open quadrants for which $x+h>x$, $y+k>y$; $x-h<x$, $y-k<y$. The proof for the other two quadrants is invalid, because it is impossible to choose the numbers h_1, k_1, h_1', k_1' as stated on page 391.

A theorem has been given by R. C. Young (*L'enseignement mathématique*, 1924–25, p. 79) from which it follows that, for a quasi-monotone function of any of the four types specified in § 255, the limit exists for all four quadrants

If (x, y), $(x+h, y+k)$ denote two points A, B, the expression

$$f\,(x+h,\,y+k)-f\,(x,\,y+k)-f\,(x+h,\,y)+f\,(x,\,y)$$

may be denoted by Δ_A^B, and will be taken to be $\geqq 0$, for all pairs of points A, B, for which h and k are positive, in some given domain. For any cell (P, Q) contained in (A, B), $\Delta_P^Q \leqq \Delta_A^B$; for the cell (P, Q) may be one of a number of cells into which (A, B) is divided, and Δ_A^B is equal to the sum for all these cells (α, β) of Δ_α^β, and Δ_α^β is by hypothesis $\geqq 0$. If (A, Q_1), (A, Q_2) ... (A, Q_n) ... be a sequence of cells such that Q_n is in the cell (A, Q_{n-1}), for all values of n, the sequence $\{\Delta_A^{Q_n}\}$ is monotone non-in-

creasing, and therefore converges to a limit, as $n \sim \infty$. Let $\{P_n\}$, $\{Q_n\}$ be any two sequences of points, each converging to A; a sequence

$$P_{n_1}, Q_{m_1}, P_{n_2}, Q_{m_2}, \ldots P_{n_\iota}, Q_{m_\iota}, \ldots$$

can be so determined that Q_{m_ι} is in the cell (A, P_{n_ι}), and P_{n_ι} in the cell $(AQ_{m_{\iota-1}})$, for all values of ι; then the sequence $\Delta_A^{P_{n_1}}, \Delta_A^{Q_{m_1}}, \Delta_A^{P_{n_2}}, \ldots$ has a unique limit. If $\{\Delta_A^{P_n}\}$, $\{\Delta_A^{Q_n}\}$ both have definite limits, it follows that these limits have the same value. Hence $\lim \Delta_A^P$ has a definite value as P converges in any manner to A, in the positive quadrant. Also $f(x, y+k)$, $f(x+h, y)$ being monotone in k and h respectively, each being either non-increasing or non-diminishing, it follows that $f(x+h, y+k)$ has a definite limit as h and k converge to zero, in any manner, in the quadrant. Therefore any quasi-monotone function, of whatever type, has a definite limit in the quadrant, whichever quadrant be taken.

Page 401. Lines 5 and 7, *for* "$F(h, k)$" *read* "$F(h, k)/hk$."

Page 402. Line 4 from the foot, *for* "h/k" *read* "k/h." Line 1 from the foot, *for* "k/h" *read* "h/k."

Page 408. Line 15, *for* "$y<0$" *read* "$y<\beta$." Line 16, *for* "$0<y$" *read* "$\beta<y$."

Page 435. Line 12, *for* "$\Sigma \geqq \Sigma$" *read* "$\Sigma \geqq \underline{\Sigma}$." *For* "$\Sigma$ and Σ," *read* "Σ and $\overline{\Sigma}$."

Page 450. Line 14, *for* "the set" *read* "the measure (J) of the set."

Page 456. Lines 10, 11 from the foot. In the formulae x should take the place of b.

Page 458. Line 15. In the formulae x should take the place of b.

Page 508. Lines 3, 4. The formula should *read*

$$-[f(a)\{\phi(\xi_1)-\phi(a)\}+\ldots+f(x_{m-1})\{\phi(\xi_m)-\phi(\xi_{m-1})\}$$
$$+f(b)\{\phi(b)-\phi(\xi_m)\}]+[f(x)\phi(x)]_a^b.$$

At the foot of the page, in the expressions for $\bar{S}$ and $\underline{S}$ the second Σ should be $\sum_{r=0}^{r=m}$, and in the next line "$\phi(x_0-a)=\phi(a)$" should be added.

It has been pointed out by Prof. D. G. Gillespie of Cornell University that the definition here given of the upper and lower RS-integrals is not always equivalent to that on p. 507. For example, let $\phi(x)=0$, for $0 \leqq x<\frac{1}{2}$; $\phi(x)=1$, for $\frac{1}{2} \leqq x \leq 1$; $f(x)=1$, for $x \neq \frac{1}{2}$, $f(\frac{1}{2})=0$. If $\frac{1}{2}$ be taken as a point of division in forming the upper sum of p. 508, this upper sum is zero, and the upper and lower integrals are both equal to zero, but the integral as defined on p. 507 does not exist, since $f(x)$ and $\phi(x)$ have a common point of discontinuity. In order to amend the definition on p. 508, so as to get rid of this discrepancy, we should take

$$\bar{S}=\sum_{r=1}^{r=m} U_{x_{r-1}+0}^{x_r-0}(f(x))\{\phi(x_r-0)-\phi(x_{r-1}+0)\}+\sum_{r=0}^{r=m}\bar{f}(x_r)\{\phi(x_r+0)-\phi(x_r-0)\},$$

$$\underline{S}=\sum_{r=1}^{r=m} L_{x_{r-1}+0}^{x_r-0}(f(x))\{\phi(x_r-0)-\phi(x_{r-1}+0)\}+\sum_{r=0}^{r=m}\underline{f}(x_r)\{\phi(x_r+0)-\phi(x_r-0)\},$$

where $\bar{f}(x_r)$, $\underline{f}(x_r)$ are the maxima and minima of $f(x)$ at x_r, $\phi(x_m+0)=\phi(b)$, $\phi(a-0)=\phi(a)$. On p. 510, $f(x')$ must be taken instead of $f(x')$.

The definitions and properties of Stieltjes' integrals have been treated in detail by Pollard (*Quarterly Journal*, vol. XLIX (1923), p. 73).

Page 511. Line 4 should *read* "$\phi(x)$ over the set of discontinuities of $f(x)$ should be zero is satisfied." Line 13 should *read* "To shew that the condition concerning the variation of $\phi(x)$...."

Page 512. Line 11 from the foot, *for* "convergence" *read* "continuity."

Page 520. Line 10 from the foot should *read* "If the interval (L, U) be successively sub-divided by introducing further points of division, such that the corresponding values of η form a sequence $\{\eta_m\}$, of...."

Page 533. Line 10, *for* "$\geqq a$" *read* "$\geqq a\epsilon$." Line 11, *for* "$<\beta+1$" *read* "$<(\beta+1)\epsilon$."

Page 539. Line 3, *for* "convergence" *read* "continuity." Line 5, *for* "$I(x, h')$" *read* "$I(x, x+h')$." Line 10, *for* "$I(x, h)$" *read* "$I(x, x+h)$."

Page 542. Line 2, *for* "in (a, b)" *read* "in e."

Page 553. Line 12 from the foot, *for* "$\Sigma\Delta x < K\Delta\xi$" *read* "$\Sigma\Delta x < K\Sigma\Delta\xi$."

Page 563. Line 4 from the foot. It is tacitly assumed that uv is an integral. To prove that the product of two indefinite integrals $u(x)$, $v(x)$ is an indefinite integral, we have

$$|u(x_2)v(x_2) - u(x_1)v(x_1)| \leq |u(x_1)||v(x_2) - v(x_1)| + |v(x_2)||u(x_2) - u(x_1)|$$
$$\leq A|v(x_2) - v(x_1)| + B|u(x_2) - u(x_1)|;$$

where A and B are fixed numbers. Hence the sum of the absolute variations of $u(x)v(x)$ over a set of intervals so chosen that the sums of the absolute variations of $u(x)$, $v(x)$ are both $<\eta$ is less than a fixed multiple of η. It follows that the condition of absolute continuity of $u(x)v(x)$ is satisfied.

Page 577. Line 10 from the foot, *for* "*If* $f(x^{(1)}, x^{(2)})$ *be summable*" *read* "*If* $f(x^{(1)}, x^{(2)})$ *be measurable.*"

Page 589. Line 8, *for* "$D(e_r, e_s)$" *read* "$D(e_r, e_s')$; *for* "$e_s =$" *read* "$e_s' =$."

Page 605. Line 5 should *read*, "Accordingly, an absolutely convergent integral of a function such that the points of infinite discontinuity form a set of measure zero, which exists...." This restriction was introduced on p. 599, and should be taken to govern the whole section on Harnack's definition.

Page 617. Line 11 from the foot, *for* "G" *read* "H." Line 2 from the foot, *for* "(α, β')" *read* "(α', β')."

Page 621. Line 4, *for* "G" *read* "H."

Page 630. Line 4 of (4), *for* "G" *read* "H."

Page 641. Line 7, *for* "x" *read* "ξ."

Page 653. Line 4 of (4)′, *for* "G" *read* "H."

Page 655. Line 17 from the foot, *for* "is satisfied" *read* "should be continuous is satisfied."

Page 658. Line 12, *for* "$g(\alpha)$, $g(\beta)$" *read* "$\phi(\alpha)$, $\phi(\beta)$." Line 3 from the foot, *for* "$\int_{a_n}^{\beta_n} F(x)\phi(x)\,dx$" *read* "$\int_{a_n}^{\beta_n} f(x)\phi(x)\,dx$."

LIST OF AUTHORS QUOTED IN VOLUME II

[*The numbers refer to pages*]

GENERAL INDEX TO VOLUME II

[*The numbers refer to pages*]

CORRECTIONS AND ADDITIONS TO VOLUME II (1926)

Page 8. Ex. (7). This example should be stated as follows: If $\{\beta_n\}$ denote a monotone sequence of decreasing numbers which converges to 0, and $\{a_n\}$ be a sequence which converges to 0, prove, by a method similar to that employed in the proof of the general theorem, that, if $\lim_{n\sim\infty} \frac{a_n - a_{n+1}}{\beta_n - \beta_{n+1}}$ exists, then $\lim_{n\sim\infty} \frac{a_n}{\beta_n}$ exists, and has the same value.

Page 249. A proof of the following more general theorem has been given by the author (*Journal of Lond. Math. Soc.* vol. I (1926), p. 211):

Let E be a measurable set of points (x), in any number of dimensions, and of either finite or infinite measure, and let $\{f_n(x)\}$ be a sequence of functions defined in E, and such that, for every value of n, $|f_n(x)|^k$ is summable over E, where k is some number greater than zero. If the condition

$$\lim_{n\sim\infty,\, n'\sim\infty} \int_{(E)} |f_n(x) - f_{n'}(x)|^k \, dx = 0$$

be satisfied, then there exists a function $f(x)$, defined at almost all points of E, such that $|f(x)|^k$ is summable over E, and satisfies the conditions

$$\lim_{n\sim\infty} \int_{(E)} |f(x) - f_n(x)|^k \, dx = 0,$$

$$\int_{(E)} |f(x)|^k \, dx = \lim_{n\sim\infty} \int_{(E)} |f_n(x)|^k \, dx.$$

Page 290. It has been tacitly assumed here that the function $s(x)$ and the sets of points e_n are measurable. In order to justify these assumptions, we observe that, in accordance with I, § 400, $s(x)$ is a measurable function, and consequently (see I, § 384) the functions $s(x) - s_{n+m}(x)$ are also measurable; hence the set of points at which $|s(x) - s_{n+m}(x)| \leq \epsilon$ is measurable, for each value of $n+m$. The set e_n is the set of points common to all the sets for which

$$|s(x) - s_n(x)| \leq \epsilon, \quad |s(x) - s_{n+1}(x)| \leq \epsilon, \;\ldots\; |s(x) - s_{n+m}(x)| \leq \epsilon, \;\ldots$$

are measurable. In accordance with the second theorem in I, § 131, the set e_n is measurable.

Page 323. It is here assumed that the sets e_h, e_A are measurable. It has however been pointed out to the author by R. L. Jeffery, of Acadia University, Nova Scotia, that $f(x, y)$ can be so defined that this is not the case. For example, let E denote the set of all points of the interval $(0, 1)$, and let L denote a non-measurable linear set of points, to which the point 0 may be taken to belong, which is non-measurable in each interval $(0, h)$. Such a set has been constructed by Van Vleck (*Trans. Amer. Math. Soc.*, vol. IX (1908), p. 237). Let $f(x, y)$ be defined in the rectangle $(0, 0;\ 1, 1)$ to have the value 0 at every point in the rectangle, except those points on the diagonal (x, x) which have as their projections on the side $(0, 1)$ points of $C(L)$, at which $f(x, x) = 1$. For all points x on $(0, 1)$, we have $\lim_{y\sim 0} f(x, y) = 0$; moreover, $f(x, y)$ is measurable for each value of y. The set e_h consists of the interval $(h + 0, 1)$ together with the points of L in the interval $(0, h)$; thus e_h is not measurable in the interval $(0, 1)$.

Whenever this possibility is realized it is necessary in § 225 to restrict h to have the values in an enumerable sequence $\{h_n\}$ which converges to zero, the values of y being also restricted to have the enumerable set of values of $y_0 + h_n$.

The sets $e_{h_1}, e_{h_2}, \ldots$ are then measurable, as has been shewn in the last note. Since then the theorem holds for all such sequences $\{h_n\}$, it holds in accordance with Heine's definition of the limit, and consequently in accordance with the definition of Cauchy, provided the equivalence of the two definitions be assumed (see I, § 211); this assumption requires the employment of the multiplicative axiom. The case of the sets e_A can be treated similarly. The argument of the preceding note is not applicable to shew that the set of points common to an unenumerable family of measurable sets is measurable.

Line 1. *For* $y + h$ *read* $y_0 + h$.

Line 7. *For* $\lim\limits_{h \sim 0} m(E - e_h)$ *read* $\overline{\lim\limits_{h \sim 0}}\, m(E - e_h)$.

Line 11. *For* $|f(\xi, y_0 + h_n) - f(x, y_0 + 0)|$ *read* $|f(\xi, y_0 + h_n) - f(\xi, y_0 + 0)|$.

Line 12. *For* $f(x, y_0 + 0)$ *read* $f(\xi, y_0 + 0)$.

Page 444. Insert, after line 4, the following:

Further, $\int_a^{x-\mu} F(x', x, n)\, dx'$ *converges to zero, as* $n \sim \infty$, *uniformly for all those values of* x *(in* G*) that are* $> a + \mu$, *and* $\int_{x+\mu}^{\beta} F(x', x, n)\, dx'$ *converges to zero, uniformly for all values of* x *(in* G*)* $< \beta - \mu$.

The following theorem, which is an immediate corollary of Theorem I, is usually sufficient for application:

If $\Phi(x', x, n)$ *has the value* $F(x', x, n)$ *for all values of* x *such that* $|(x' - x)| \geqq \mu$, *and is zero whenever* $|x' - x| < \mu$, *and if* $\Phi(x', x, n)$ *so defined, satisfies the conditions* (1) *and* (2), *of Theorem I, then*

$$\int_a^{x-\mu} F(x', x, n)\, dx' + \int_{x+\mu}^{b} F(x', x, n)\, dx'$$

converges to zero, uniformly for all values of x *(in* G*).*

Page 453. Line 2. *For* "the integral is" *read* "the integral differs from

$$\frac{\chi(a_n)}{a_n} \int_{a_n}^{\mu} t^2 F_1(t, n)\, dt \text{ by.}"$$

Line 6. *For* $N(a_n)\, V_{a_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\}$ *read* $N(a_n) \left[V_{a_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\} + \frac{\chi_1(a_n)}{a_n} \right]$.

Line 7 from the foot. *For* $a_n V_{a_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\}$ *read* $a_n V_{a_n}^{\mu} \left\{\frac{\chi(t)}{t}\right\} + \chi_1(a_n)$.

Page 558. Line 13 from the foot. *For*

$$\lim_{n\sim\infty} \int_0^{\frac{1}{2}\pi} 2\left[\tfrac{1}{2} + \cos 2t + \cos 4t + \ldots + \cos 2nt\right] dt \textit{ read } \lim_{n\sim\infty} \int_0^{\frac{1}{2}\pi} 2\left(\frac{1}{2} + \sum_{r=1}^{n-1} \frac{n-r}{n} \cos 2rt\right) dt.$$

Page 626. The theorem in § 409 has been extended by Kolmogoroff and Seliverstoff to the case $\epsilon = 0$ (*Atti dei Lincei, Rendiconti* (6), vol. II (1926), p. 307). Thus, *if* $\sum\limits_{n=2} (a_n^2 + b_n^2) \log n$ *is convergent, then* $\sum\limits_{n=1} (a_n \cos nx + b_n \sin nx)$ *converges almost everywhere.* An independent proof of this result has been given by Plessner (*Crelle's Journal*, vol. CLV (1926), p. 15).

Page 700. Delete lines 1–11. It is not correct that $\frac{1}{t^{(1)} t^{(2)}} - \frac{1}{\sin t^{(1)} \sin t^{(2)}}$ is summable over the cell $(0, 0; \frac{1}{2}\pi, \frac{1}{2}\pi)$. The two following corrections are requisite in consequence of this.

Page 705. § 464, line 5. *For* $\int F(t^{(1)}, t^{(2)}) \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}}\, d(t^{(1)}, t^{(2)})$ *read*

$$\int \left[\frac{t^{(1)} t^{(2)}}{\sin t^{(1)} \sin t^{(2)}} F(t^{(1)}, t^{(2)})\right] \frac{\sin m^{(1)} t^{(1)}}{t^{(1)}} \frac{\sin m^{(2)} t^{(2)}}{t^{(2)}}\, d(t^{(1)}, t^{(2)}).$$

Corrections and Additions to Volume II

Page 709. Delete the last two lines, and substitute the following:

The functions F_1, F_2 can be so chosen as to be not only non-increasing, but also non-negative. The function $\dfrac{t^{(1)}t^{(2)}}{\sin t^{(1)}\sin t^{(2)}}$ can, by proper choice of A, be expressed as the difference of A and $A - \dfrac{t^{(1)}t^{(2)}}{\sin t^{(1)}\sin t^{(2)}}$, both of which are non-negative and non-increasing. The function $\psi(t^{(1)}, t^{(2)})$ can be made identical successively with each of the four functions

$$AF_1(t^{(1)}, t^{(2)}),\quad \left(A - \frac{t^{(1)}t^{(2)}}{\sin t^{(1)}\sin t^{(2)}}\right)F_1(t^{(1)}, t^{(2)}),$$

$$AF_2(t^{(1)}, t^{(2)}),\quad \left(A - \frac{t^{(1)}t^{(2)}}{\sin t^{(1)}\sin t^{(2)}}\right)F_2(t^{(1)}, t^{(2)}),$$

all of which are monotone non-increasing.

The limits of the four corresponding integrals, as $m^{(1)} \sim \infty$, $m^{(2)} \sim \infty$, are $\frac{1}{4}\pi^2 AF_1(+0,+0)$, $\frac{1}{4}\pi^2(A-1)F_1(+0,+0)$, $\frac{1}{4}\pi^2 AF_2(+0,+0)$, $\frac{1}{4}\pi^2(A-1)F_2(+0,+0)$ respectively.

Thus the integral

$$\frac{1}{\pi^2}\int_{(0,0)}^{(\frac{1}{2}\pi,\,\frac{1}{2}\pi)} F(t^{(1)}, t^{(2)})\frac{\sin m^{(1)}t^{(1)}\sin m^{(2)}t^{(2)}}{\sin t^{(1)}\quad \sin t^{(2)}}\, d(t^{(1)}, t^{(2)})$$

converges to $\frac{1}{4}F(+0,+0)$, as $m^{(1)} \sim \infty$, $m^{(2)} \sim \infty$.

Page 768. The first theorem in § 495 holds good, as asserted by Menchoff, when $2 > \lambda > 0$. It is correct that the condition $2 > \lambda > \frac{4}{3}$ is necessary in order that the first theorem in § 495 may be deducible from the second. But, for the second theorem, the following more general one may be stated; it is proved on pp. 769, 770. From this more general theorem the more general form of the first can be deduced:

If $\omega(u)$ is a positive increasing function, such that the series $\sum\limits_{p=1}^{\infty}\dfrac{1}{\omega(u)}$ is convergent, and if $a_n = o(1)$, the convergence of the series

$$\sum_{n=1}^{\infty}\left[\omega\left(\log\log\frac{1}{|a_n|}\right)\right]^2\left(\log\frac{1}{|a_n|}\right)^2 a_n^2, \quad\text{.........................(A)}$$

entails the convergence, almost everywhere in the interval $(0, 1)$, *of the series*

$$\sum_{n=1} a_n\phi_n(x).$$

For, let $\omega(u) = u^2$, then $\sum\limits_{p=1}^{\infty}\dfrac{1}{\omega(p)}$ is convergent; also from the convergence of $|a_n|^{2-\lambda}$, we have $a_n = o(1)$.

We have also

$$\lim_{u\sim\infty}\frac{[\omega(\log\log u)]^2(\log u)^2}{u^\lambda} = \lim_{u\sim\infty}\frac{(\log\log u)^4(\log u)^2}{u^\lambda} = 0,$$

for $\lambda > 0$. Hence, for all sufficiently large values of u,

$$\left[\omega\left(\log\log\frac{1}{|a_n|}\right)\right]^2\left(\log\frac{1}{|a_n|}\right)^2 < C\frac{1}{|a_n|^\lambda},$$

where C is a positive constant; thus, when $\lambda > 0$

$$\left[\omega\left(\log\log\frac{1}{|a_n|}\right)\right]^2\left(\log\frac{1}{|a_n|}\right)^2 a_n^2 < C\,|a_n|^{2-\lambda}.$$

Therefore the convergence of the series $\Sigma\,|a_n|^{2-\lambda}$ entails the convergence of the series (A), and consequently the convergence almost everywhere, of the series $\Sigma\, a_n\phi_n(x)$.

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